MAT 1348
SOLUTIONS TO SUPPLEMENTAL EXERCISES

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1 Propositional Logic

1.

p	q	P_1	P_2	P_3	P_4	P_5	P_6
T	T	T	T	T	F	F	F
T	F	T	F	F	T	F	F
F	T	T	F	T	F	T	F
F	F	F	T	T	T	F	T

(a) From the table, the corresponding DNFs are

$$\begin{aligned}
P_1 &\equiv (p \wedge q) \vee (p \wedge \neg q) \vee (\neg p \wedge q) \\
P_2 &\equiv (p \wedge q) \vee (\neg p \wedge \neg q) \\
P_3 &\equiv (p \wedge q) \vee (\neg p \wedge q) \vee (\neg p \wedge \neg q) \\
P_4 &\equiv (p \wedge \neg q) \vee (\neg p \wedge \neg q) \\
P_5 &\equiv \neg p \wedge q \\
P_6 &\equiv \neg p \wedge \neg q
\end{aligned}$$

(b) Analyzing each compound proposition and then using equivalence laws we have

$$\begin{aligned}
P_1 &\equiv p \vee q \equiv \neg(\neg p \wedge \neg q) \\
P_2 &\equiv p \leftrightarrow q \equiv (p \to q) \wedge (q \to p) \equiv (\neg p \vee q) \wedge (\neg q \vee p) \equiv \neg(p \wedge \neg q) \wedge \neg(q \wedge \neg p) \\
P_3 &\equiv p \to q \equiv \neg p \vee q \equiv \neg(p \wedge \neg q) \\
P_4 &\equiv \neg q \\
P_5 &\equiv \neg p \wedge q \\
P_6 &\equiv \neg p \wedge \neg q.
\end{aligned}$$

(c) Using the form of each compound proposition and then equivalence laws,

$$\begin{aligned}
P_1 &\equiv p \vee q \equiv \neg\neg p \vee q \equiv \neg p \to q \\
P_2 &\equiv p \leftrightarrow q \equiv (p \to q) \wedge (q \to p) \equiv \neg\neg(p \to q) \wedge (q \to p) \\
&\equiv \neg(\neg(p \to q) \vee \neg(q \to p)) \equiv \neg((p \to q) \to \neg(q \to p)) \\
P_3 &\equiv p \to q \\
P_4 &\equiv \neg q \\
P_5 &\equiv \neg p \wedge q \equiv \neg(p \vee \neg q) \equiv \neg(q \to p) \\
P_6 &\equiv \neg(\neg p \to q).
\end{aligned}$$

2. (a)

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

(b) The truth table

p	q	r	$p \oplus q$	$(p \oplus q) \oplus r$	$q \oplus r$	$p \oplus (q \oplus r)$
T	T	T	F	T	F	T
T	T	F	F	F	T	F
T	F	T	T	F	T	F
T	F	F	T	T	F	T
F	T	T	T	F	F	F
F	T	F	T	T	T	T
F	F	T	F	T	T	T
F	F	F	F	F	F	F

Since the fifth and the seventh columns are the same, we conclude that the corresponding propositions, $(p \oplus q) \oplus r$ and $p \oplus (q \oplus r)$, are equivalent.

(c)

p	q	$p \oplus q$	$p \leftrightarrow q$	$\neg(p \leftrightarrow q)$
T	T	F	T	F
T	F	T	F	T
F	T	T	F	T
F	F	F	T	F

Since the third and the fifth columns are the same, we conclude that $p \oplus q \equiv \neg(p \leftrightarrow q)$.

3. Using equivalence laws,

$$\begin{aligned} \neg(a \to b) \to c &\equiv \neg(\neg(a \to b)) \vee c \\ &\equiv (a \to b) \vee c \\ &\equiv (\neg a \vee b) \vee c. \end{aligned}$$

4. (a) Truth tables.

(i) Denote $P_1 = (x \vee y) \wedge (\neg x \vee z)$ and note that the compound proposition $((x \vee y) \wedge (\neg x \vee z) \wedge (y \to z)) \to z$ can be written as $(P_1 \wedge (y \to z)) \to z$.

x	y	z	$x \vee y$	$\neg x \vee z$	P_1	$y \to z$	$P_1 \wedge (y \to z)$	$(P_1 \wedge (y \to z)) \to z$
T	T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	F	T
T	F	T	T	T	T	T	T	T
T	F	F	T	F	F	F	F	T
F	T	T	T	T	T	T	T	T
F	T	F	T	T	T	F	F	T
F	F	T	F	T	F	T	F	T
F	F	F	F	T	F	T	F	T

Hence, $((x \vee y) \wedge (\neg x \vee z) \wedge (y \to z)) \to z$ is a tautology.

(ii) Denote $P_2 = (x \to y) \wedge (x \to z)$ and note that the proposition $(x \to (y \vee z)) \to ((x \to y) \wedge (x \to z))$ can be written as $(x \to (y \vee z)) \to P_2$.

x	y	z	$y \vee z$	$x \to (y \vee z)$	$x \to y$	$x \to z$	P_2	$(x \to (y \vee z)) \to P_2$
T	T	T	T	T	T	T	T	T
T	T	F	T	T	T	F	F	F
T	F	T	T	T	F	T	F	F
T	F	F	F	F	F	F	F	T
F	T	T	T	T	T	T	T	T
F	T	F	T	T	T	T	T	T
F	F	T	T	T	T	T	T	T
F	F	F	F	T	T	T	T	T

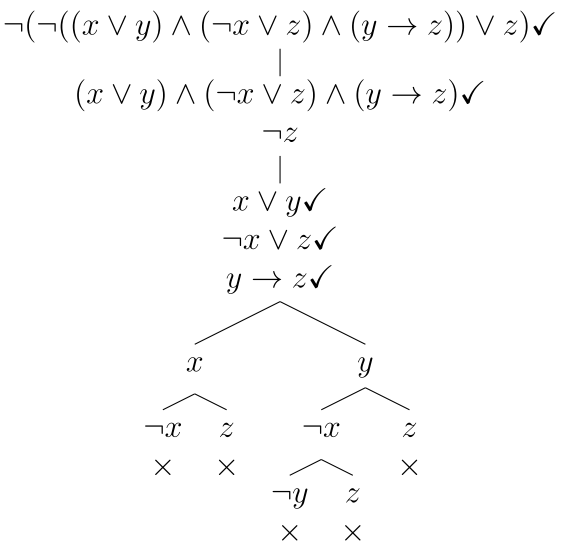
Hence, $(x \to (y \vee z)) \to ((x \to y) \wedge (x \to z))$ is not a tautology. If either (x is T, y is T and z is F) or (x is T, y is F and z is T), then the proposition $(x \to (y \vee z)) \to ((x \to y) \wedge (x \to z))$ is F.

(b) Truth trees.

(i) Consider the negation of the given compound proposition,

$$\neg(((x \vee y) \wedge (\neg x \vee z) \wedge (y \to z)) \to z) \equiv \neg(\neg((x \vee y) \wedge (\neg x \vee z) \wedge (y \to z)) \vee z),$$

and apply the truth trees method.

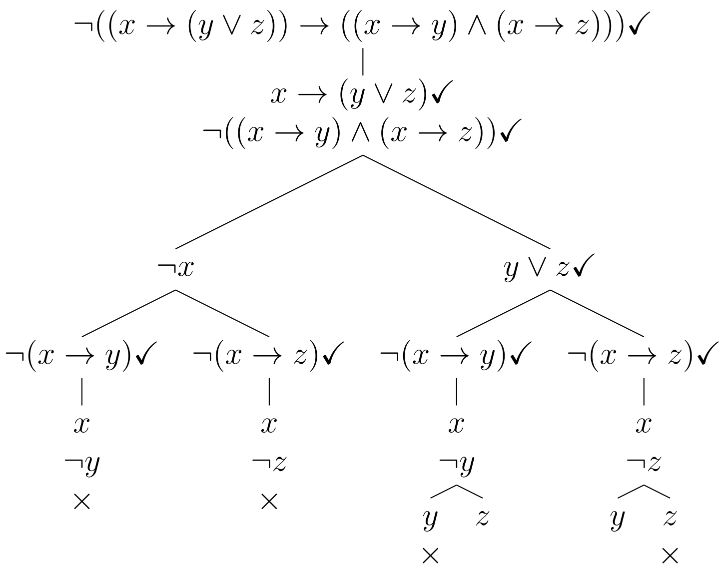


Since all the paths are inactive, we conclude that the negation proposition considered is a contradiction. This implies that the initial given compound proposition is a tautology.

(ii) Consider the negation of the given compound proposition is

$$\neg((x \to (y \vee z)) \to ((x \to y) \wedge (x \to z))),$$

and apply the method of truth trees.



Since there are complete active paths, we conclude that the negation proposition is not a contradiction and therefore, the initial proposition is *not* a tautology. The active paths provide the two counterexamples, i.e. truth values of x, y and z for which the proposition $(x \to (y \vee z)) \to ((x \to y) \wedge (x \to z))$ is false:

x	y	z
T	T	F
T	F	T

.

5. The argument can be written as

Hypothesis 1: $(P \to J) \to (\neg C \to M)$
Hypothesis 2: $\neg J \to \neg P$
Hypothesis 3: $\neg J \wedge E \to \neg C$
Hypothesis 4: $\neg M \to P$

Conclusion: $\neg(J \wedge \neg P) \to C$

Using truth tables or truth trees the above argument can be shown to be invalid. The following is a counterexample. If

P	M	J	E	C
F	T	F	T	F

, then each hypothesis is true whereas the conclusion is false.

6. The argument can be written as

Hypothesis 1: $B \to (D \to S)$
Hypothesis 2: $\neg D \to P$
Hypothesis 3: $(D \vee S) \to B$
Hypothesis 4: $P \to (\neg D \wedge \neg S)$

Conclusion: $B \wedge D$

Using truth tables or truth trees the above argument can be shown to be invalid. The following is a counterexample. If

B	D	P	S
F	F	T	F

, then each hypothesis is true whereas the conclusion is false.

7. For each compound proposition, we construct its truth table and also find its DNF by algebraic manipulations. For the truth trees method - see other file.

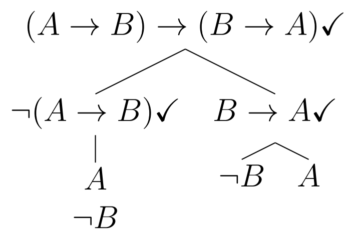
(i)

A	B	$A \to B$	$B \to A$	$(A \to B) \to (B \to A)$
T	T	F	T	T
T	F	F	T	T
F	T	T	F	F
F	F	T	T	T

Hence, a DNF for $(A \to B) \to (B \to A)$ is $(A \wedge B) \vee (A \wedge \neg B) \vee (\neg A \wedge \neg B)$. Algebraically,

$$\begin{aligned}(A \to B) \to (B \to A) &\equiv \neg(A \to B) \vee (B \to A)\\ &\equiv \neg(\neg A \vee B) \vee (\neg B \vee A)\\ &\equiv (A \wedge \neg B) \vee \neg B \vee A.\end{aligned}$$

Apply the truth trees method.



Therefore, a DNF of the compound proposition $(A \to B) \to (B \to A)$ is $(A \wedge \neg B) \vee \neg B \vee A$.

(ii)

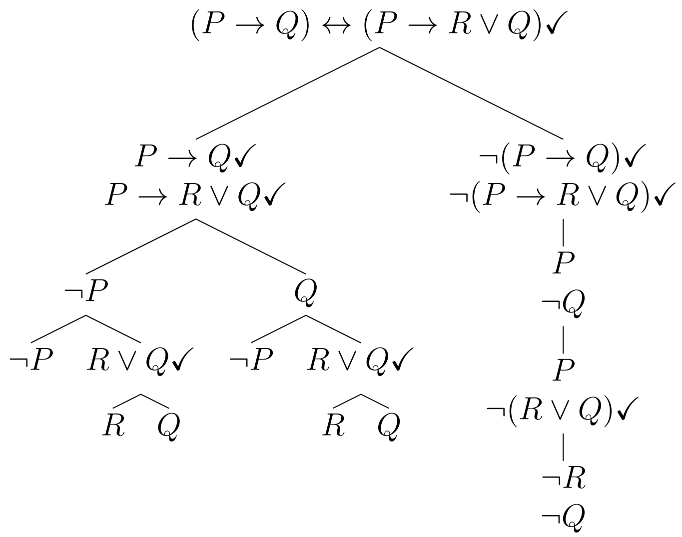
P	Q	R	$P \to Q$	$R \vee Q$	$P \to (R \vee Q)$	$(P \to Q) \leftrightarrow (P \to (R \vee Q))$
T	T	T	T	T	T	T
T	T	F	T	T	T	T
T	F	T	F	T	T	F
T	F	F	F	F	F	T
F	T	T	T	T	T	T
F	T	F	T	T	T	T
F	F	T	T	T	T	T
F	F	F	T	F	T	T

Then, a DNF for $(P \to Q) \leftrightarrow (P \to R \vee Q)$ is $(P \wedge Q \wedge R) \vee (P \wedge Q \wedge \neg R) \vee (P \wedge \neg Q \wedge \neg R) \vee (\neg P \wedge Q \wedge R) \vee (\neg P \wedge Q \wedge \neg R) \vee (\neg P \wedge \neg Q \wedge R) \vee (\neg P \wedge \neg Q \wedge \neg R)$.

Algebraically,

$$
\begin{aligned}
&(P \to Q) \leftrightarrow (P \to R \lor Q) \\
&\equiv ((P \to Q) \land (P \to R \lor Q)) \lor (\neg(P \to Q) \land \neg(P \to R \lor Q)) \\
&\equiv ((\neg P \lor Q) \land (\neg P \lor (R \lor Q))) \lor (\neg(\neg P \lor Q) \land \neg(\neg P \lor (R \lor Q))) \\
&\equiv ((\neg P \lor Q) \land (\neg P \lor R \lor Q)) \lor ((P \land \neg Q) \land (P \land \neg R \land \neg Q)) \\
&\equiv ((\neg P \lor Q) \land ((\neg P \lor Q) \lor R)) \lor ((P \land \neg Q) \land ((P \land \neg Q) \land \neg R)) \\
&\equiv ((\neg P \lor Q) \land ((\neg P \lor Q) \lor R)) \lor ((P \land \neg Q) \land ((P \land \neg Q) \land \neg R)) \\
&\equiv (\neg P \lor Q) \lor ((P \land \neg Q) \land \neg R) \\
&\equiv \neg P \lor Q \lor (P \land \neg Q \land \neg R).
\end{aligned}
$$

Apply the truth tree method.



It follows that a DNF of the compound proposition $(P \to Q) \leftrightarrow (P \to R \lor Q)$ is $\neg P \lor (\neg P \land R) \lor (\neg P \land Q) \lor (Q \land R) \lor Q \lor (P \land \neg R \land \neg Q)$.

(iii) Let P denote the proposition $\neg A \to (B \to (A \to (B \land C)))$.

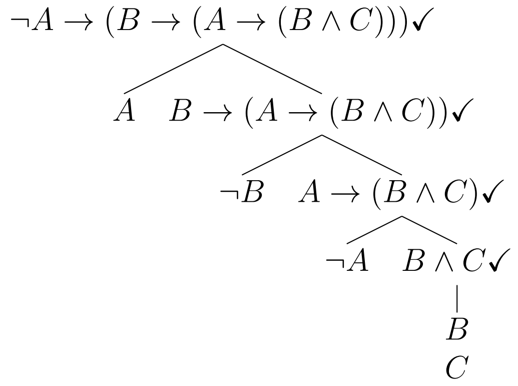
A	B	C	$B \land C$	$A \to (B \land C)$	$B \to (A \to (B \land C))$	P
T	T	T	T	T	T	T
T	T	F	F	F	F	T
T	F	T	F	F	T	T
T	F	F	F	F	T	T
F	T	T	T	T	T	T
F	T	F	F	T	T	T
F	F	T	F	T	T	T
F	F	F	F	T	T	T

Since P is a tautology, a DNF of it is T.

Algebraically,

$$\begin{aligned}
\neg A \to (B \to (A \to (B \wedge C))) &\equiv \neg\neg A \vee (B \to (A \to (B \wedge C))) \\
&\equiv A \vee (\neg B \vee (A \to (B \wedge C))) \\
&\equiv A \vee (\neg B \vee (\neg A \vee (B \wedge C))) \\
&\equiv A \vee \neg B \vee \neg A \vee (B \wedge C) \\
&\equiv T.
\end{aligned}$$

Apply the truth tree method.



It follows that a DNF of $\neg A \to (B \to (A \to (B \wedge C)))$ is $A \vee \neg B \vee \neg A \vee (B \wedge C) \equiv T$.

8. From the truth table, the compound proposition F is T only in the following cases.

x	y	z	F
T	T	T	T
T	F	F	T
F	F	T	T

Hence, a DNF for F is $(x \wedge y \wedge z) \vee (x \wedge \neg y \wedge \neg z) \vee (\neg x \wedge \neg y \wedge z)$.

2 Knights and Knaves

Unless otherwise stated, the following propositional variables are used:

a: "Person A is a knight."
b: "Person B is a knight."
c: "Person C is a knight."

1. A says $\neg a \wedge \neg b \wedge \neg c$, and B says $(\neg a \wedge \neg b \wedge c) \vee (a \wedge \neg b \wedge c) \vee (a \wedge b \wedge \neg c)$. For a and A's statement, as well as b and B's statement to be logically equivalent, the only possibility is that A is a knave and C is a knight, while B can be either.

2. See the handout *A Method of Solving Knights-And-Knaves Questions*.

3. A says $\neg b$, and B says $a \leftrightarrow c$. For a and A's statement, as well as b and B's statement to be logically equivalent, the only possibility is that C is a knave, and A and B are of opposite types.

4. A says $b \leftrightarrow c$, and C says either $a \leftrightarrow b$ or $\neg(a \leftrightarrow b)$. A truth table shows that a and A's statement, as well as c and C statement, can be simultaneously logically equivalent only in the first case, that is, if C says $a \leftrightarrow b$. In other words, C answers yes.

5. See the handout *A Method of Solving Knights-And-Knaves Questions.*

6. Let h be the proposition "A eats his hat." Then A says $a \to h$. For a and A's statement to have the same truth value, we must have a and h both true, that is, A eats his hat.

7. A says $a \to \mathbb{T}$. For a and A's statement to have the same truth value, a must be true.

8. A says $a \to \mathbb{F}$. For a and A's statement can not have the same truth value; A can not be an Islander.

9. See the handout *A Method of Solving Knights-And-Knaves Questions.*

10. Let x be the proposition "X is innocent," and y be the proposition "Y is innocent." Then A says $\neg x \to \neg y$ and B says $x \oplus \neg y$. Assuming A and B are of distinct types, proposition a and A's statement, as well as proposition b and B's statement, will be simultaneously equivalent precisely when a and x are true, and b and y are false. Hence A and B need not be of the same type.

11. A says b and B says $a \to c$. This is possible only when A, B, C are all knights.

12. Let b be the proposition "I love Betty," and j be the proposition "I love Jane." It is given that $b \vee j$ and $b \to j$ are both true propositions. From a truth table we see that j must be true, while b can be either.

13. The speaker makes the statement $(b \to j) \to b$. Assuming this is true, b must be true, while j can be either.

14. Let l be the proposition "I love Linda," and k be the proposition "I love Kathy." The speaker (say A) says l as well as $l \to k$. For a and his two statements to all be logically equivalent, the propositions a, l, k must all be true. That is, the speaker is a knight.

15. Let g be the proposition "There is gold on the island." Then A says $a \leftrightarrow g$. This is equivalent to a if and only if g is true and a is either. Hence there is gold on the island.

16. Now A says $\neg(a \leftrightarrow g)$. This is equivalent to a if and only if g is false and a is either. Hence there is no gold on the island.

17. Let m be the proposition "This is the island of Maya." Then A said $b \wedge m$ and B said $\neg a \wedge m$. Since a must be logically equivalent to A's statement, and at the same time, b logically equivalent to B's statement, we find that a, b, m must all be false. Thus it is not Maya.

18. Now A and B both say $\neg a \wedge \neg b \wedge m$. As above, we find that a, b, m must all be false. Thus it is not Maya.

19. Now A and B both say $\neg(a \wedge b) \wedge m$. As above, we find that a, b, m must all be false. Thus it is not Maya.

20. Omitted (question unclear).

3 Proofs

1. The statement is equivalent with the following conditional: "If r is a rational number and x is an irrational number, then $r + x$ is an irrational number."

   Let
   p : "r is a rational number and x is an irrational number"
   q : "$r + x$ is an irrational number".

   We need to prove $p \to q$. To construct a proof by contradiction we assume that the negation, $\neg(p \to q)$ is true. Note that

   $$\neg(p \to q) \equiv \neg(\neg p \vee q) \equiv p \wedge \neg q.$$

   Hence, we assume p is true and $\neg q$ is true. Since p is true, r and $r + x$ are rational numbers. However, the difference between two rational numbers is always rational. Hence, $(r + x) - r$ is a rational number (or x is a rational number). This statement gives a contradiction because p is true (x is irrational).

   Therefore, $\neg(p \to q)$ is false and hence $p \to q$ is true.

2. Define the propositional variable p : "$\sqrt[3]{3}$ is irrational". To show p is true we construct a proof by contradiction.

   Assume $\neg p$ is true. Then, there exist integers m and n (that have only 1 as a common divisor), with $n \neq 0$ such that $\sqrt[3]{3} = \dfrac{m}{n}$. This implies that $m^3 = 3n^3$ from where it follows that 3 is a divisor of m^3. Hence, 3 is a divisor of m: there exists an integer k such that $m = 3k$.

   We have $27k^3 = 3n^3$, or $9k^3 = n^3$. Then, 3 is a divisor of n^3 and so 3 is a divisor of n. It follows that 3 is a divisor of m and n which gives a contradiction with the fact that m and n have only 1 as a common divisor.

   Therefore, p is true.

3. Note that

   - $\max(x,\ y) = x$, when $x \geq y$ and $\max(x,\ y) = y$, when $y > x$
   - $\min(x,\ y) = x$, when $x \leq y$ and $\max(x,\ y) = y$, when $y < x$.

   Hence, we need to consider two cases $x < y$ and $x \geq y$. Define the propositional variables

p_1 : "$x < y$"
p_2 : "$x \geq y$"
q : $\max(x,\ y) + \min(x,\ y) = x + y$.

We need to prove that q is true in each of the two cases (note that the two cases cover all possible situations). Hence, we prove: $p_1 \to q$ and $p_2 \to q$.

Case (i). Assume p_1 is true. Then, $\max(x,\ y) = y$ and $\min(x,\ y) = x$. This implies that q is true.
Case (ii). Assume p_2 is true. Then, $\max(x,\ y) = x$ and $\min(x,\ y) = y$. This implies that q is true.

4. We construct a proof by cases.
   Define the propositional variables
   p_1 : "a^a is rational"
   p_2 : "a^a is irrational".

   Case (i). Assume p_1 is true, i.e. a^a is rational. Then the statement: "at least one of the numbers a^a and $(a^a)^a$ is rational" is true.

   Case (ii). Assume that p_2 is true, i.e. a^a is irrational. Note that $(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} = \sqrt{2}^2 = 2$ is rational. Hence, the statement: "at least one of the numbers a^a and $(a^a)^a$ is rational" is true.

5. We construct a proof by contradiction. Assume that the statement is not true, i.e. each number $a_1,\ a_2,\ ...,\ a_n$ is smaller than the average. Denote this average by $m = \dfrac{a_1 + \ldots a_n}{n}$. Hence, we assume that $a_1 < m$ and $a_2 < m, \ldots,$ and $a_n < m$.

   It follows that $a_1 + \ldots a_n < m + \ldots + m = n \cdot m$. This implies that $\dfrac{a_1 + \ldots a_n}{n} < m$, which means $m < m$. This is a contradiction.

   Therefore, at least one number must be grater or equal than the average.

6. Let m and n (with $m < n$) be two distinct rational numbers. To construct a proof by contradiction, we consider the negation of the given proposition and assume it is true. This means we assume that between m and n there is a finite number (p) of distinct rational numbers $q_1, \ldots, q_p$.

   Consider the number $\dfrac{m + q_1}{2}$. Clearly, this is a rational number and $m < \dfrac{m + q_1}{2} < q_1$. Hence, we found another distinct rational that is between m and n (and the procedure can be repeated). This contradicts the fact that we assumed there were only p distinct rationals between m and n.

7. Define the propositional variables
   p : "$0 < a < 1$"
   q : "$a > a^2$".

   We need to prove that $p \to q$. Remember that the implication is equivalent to $\neg q \to \neg p$. To construct an indirect proof, we assume that $\neg q$ is true and prove that $\neg p$ is true.

If $\neg q$ is true then $a \leq a^2$. This implies that $a^2 - a \geq 0$, or $a(a-1) \geq 0$. The solution of the inequality is $(-\infty,\ 0] \cup [1,\ \infty)$. Hence, $\neg p$ is true.

8. Define the propositional variables
p : "$n^5 + 7$ is even"
q : "n is odd".

To prove $p \to q$ using an indirect proof, we assume $\neg q$ is true and show that $\neg p$ is true.

If $\neg q$ is true, then n is even: there exists an integer k such that $n = 2k$. Hence, $n^5 + 7 = (2k)^5 + 7 = 32k^5 + 7 = 2(16k^5 + 3) + 1$. If we denote by $m = 16k^5 + 3$, we can write $n^5 + 7 = 2m + 1$, with m an integer. Therefore, $n^5 + 7$ is odd, and consequently, $\neg p$ is true.

9. Note that $|x| = x$, when $x \geq 0$ and $|x| = -x$, when $x < 0$.
We have $-|x| \leq x \leq |x|$ and similarly, $-|y| \leq y \leq |y|$. These imply

$$-|x| - |y| \leq x + y \leq |x| + |y|. \tag{1}$$

We consider two cases corresponding to the sign of the sum $x + y$. Note that these cases cover all possible situations. Define the propositional variables
p_1 : "$x + y \geq 0$"
p_2 : "$x + y < 0$"
q : "$|x + y| \leq |x| + |y|$".

To prove that q is true, we use a proof by cases and show that $p_1 \to q$ and $p_2 \to q$.
Case (i). Assume p_1 is true, $x + y \geq 0$. Then $|x + y| = x + y$ which together with the right hand side of (1) implies that q is true.
Case (ii). Assume p_2 is true, $x + y < 0$. Then $|x + y| = -(x + y)$ which together with the left hand side of (1) implies that q is true.

10. Define the propositional variables
p : 3 divides n^2
q : 3 divides n.

We need to prove that $p \to q$. We use the method of contradiction: we assume that the negation $\neg(p \to q)$ is true. Since $\neg(p \to q) \equiv p \wedge \neg q$ we assume p is true and $\neg q$ is true. This implies that 3 is not a divisor of n. Hence, there exists an integer m such that $n = 3m + 1$ or $n = 3m + 2$. We continue using the method of proof by cases.

Case (i). Assume $n = 3m + 1$ is true. Then $n^2 = 9m^2 + 6m + 1 = 3(3m^2 + 2m) + 1 = 3k + 1$, with $k = 3m^2 + 2m$ - integer. This gives a contradiction with the fact that p is true (3 divides n^2).

Case (ii). Assume $n = 3m+2$ is true. Then $n^2 = 9m^2+12m+4 = 3(3m^2+4m+1)+1 = 3l + 1$, with $l = 3m^2 + 4m + 1$ - integer. This gives a contradiction with the fact that p is true.

Since each case leads to a contradiction we conclude that $\neg(p \to q)$ is false (or $p \to q$ is true).

11. The statement of the theorem is of type $p \to q$, where
p : "x^2 does not divide $a^2 + b^2$"
q : "x does not divide a or x does not divide b".

To construct an indirect proof, we assume that $\neg q$ is true: x divides a and x divides b. We need to prove that $\neg p$ is true: "x^2 divides $a^2 + b^2$.

Since x divides a and x divides b, there exist integers k and l such that $a = kx$ and $b = lx$. It follows that $a^2 + b^2 = k^2x^2 + l^2x^2 = (k^2 + l^2)x^2$, where $(k^2 + l^2)$ is an integer. This proves that x^2 divides $a^2 + b^2$.

12. Define p : "$x^3 + 3x + 5 = 0$ has no rational roots".

The proof can be done by contradiction and then use a proof by cases.

Assume the $\neg p$ is true, i.e. the equation $x^3 + 3x + 5 = 0$ has at least one rational root. Denote it by $r = \frac{m}{n}$, where m and n are integers, $n \neq 0$ and m and n have no common divisors other than 1. Then, the initial cubic equation can be written as

$$m^3 + 3mn^2 + 5n^3 = 0. \quad (2)$$

For the rest of the proof we consider 4 cases, corresponding to the parity of m and n.

Define the propositional variables
p_1 : "m and n are both even"
p_2 : "m is even and n is odd"
p_3 : "m is odd and n is even"
p_4 : "m and n are both odd".

Note that the 4 cases cover all possible situations. To prove that $\neg p$ leads to a contradiction, we prove that each case gives a contradiction.

Case (i). Assume p_1 is true. This implies that 2 is a divisor of m and also 2 is a divisor of n. This is a contradiction with the fact that m and n have no other common divisor but 1.

Case (ii). Assume p_2 is true. Since n is odd, $5n^3$ is odd. Since m is even, it follows that m^3 and $3mn^2$ are even and therefore, their sum is even. Hence, the left hand side of (2) is odd. This is a contradiction since 0 is even.

Case (iii). Assume p_3 is true. Since m is odd, m^3 is odd. Since n is even, it follows that $3mn^2$, $5n^3$ are even and therefore their sum is even. These imply that the left hand side of (2) is odd. This is a contradiction since 0 is even.

Case (iv). Assume p_4 is true. Then, each term of the sum appearing at the left hand side of (2) is odd. It follows that the sum is odd which contradicts the fact that 0 is even.

Therefore, $\neg p$ is false (and p is true).

4 Sets

1. (a) $A \cap B = \{1,\ 3,\ 4\}$
   (b) $A \cup B = \{1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 9\}$
   (c) $A - B = \{2,\ 5\}$
   (d) $B - A = \{6,\ 9\}$
   (e) $A \oplus B = (A - B) \cup (B - A) = \{2,\ 5,\ 6,\ 9\}$
   (f) $(A \oplus B) \cap A = \{2,\ 5\}$
   (g) $(A \oplus B) \cup B = \{1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 9\}$

2. (a) $\emptyset \subseteq A$ is true
   (b) $\emptyset \in A$ is true
   (c) $\{\emptyset\} \in A$ is true
   (d) $\{\emptyset\} \subseteq A$ is true
   (e) $\{\emptyset,\ \{\emptyset\}\} \in A$ is true
   (f) $\{\{\emptyset,\ \{\emptyset\}\}\} \in A$ is false
   (g) $\{\{\emptyset\}\} \in A$ is false
   (h) $\{\{\emptyset\}\} \subseteq A$ is true
   (i) $\{\{\emptyset\},\ \{\emptyset,\ \{\emptyset\}\}\} \subseteq A$ is true
   (j) $\{\emptyset,\ \{\emptyset\},\ \{\{\{\emptyset\}\}\}\} \subseteq A$ is false
   (k) $\{\emptyset,\ \{\{\emptyset\}\}\} \in \mathcal{P}(A)$ is false
   (l) $\{\{\{\{\emptyset\}\}\}\} \subseteq \mathcal{P}(A)$ is false

3. (a) (i) $|A| = 3$,
   $\mathcal{P}(A) = \{\emptyset,\ \{a\},\ \{b\},\ \{\{a,\ b\}\},\ \{a,\ b\},\ \{a,\ \{a,\ b\}\},\ \{b,\ \{a,\ b\}\}, \{a,\ b,\ \{a,\ b\}\}\}$,
   $|\mathcal{P}(A)| = 2^3 = 8$
   (b) (ii) $|B| = 2$, $\mathcal{P}(B) = \{\emptyset,\ \{\emptyset\},\ \{\{\emptyset\}\},\ \{\emptyset,\ \{\emptyset\}\}\}$ and $|\mathcal{P}(B)| = 2^2 = 4$

4. To prove that $B = C$, we need to show $B \subseteq C$ and $C \subseteq B$.

   Proof $B \subseteq C$. Let $x \in B$.
   Remember that $A \oplus B = (A - B) \cup (B - A) = (A \cup B) - (A \cap B)$. There are two cases.

   Case 1. Assume $x \in A$. Since $x \in B$, it follows that $x \in A \cap B$ and hence, $x \notin A \oplus B$.

   Since $A \oplus B = A \oplus C$, we obtain $x \notin A \oplus C$ and hence, it follows that $x \notin A - C$ and $x \notin C - A$.

   From $x \notin C - A$ and $x \in A$, we obtain $x \in A \cap C$. This concludes the proof of $x \in C$.

   Case 2. Assume $x \notin A$. Since $x \in B$, this implies $x \in B - A$ and hence $x \in A \oplus B$.
   From the last relation it follows that $x \in A \oplus C$ and hence $x \in (A - C) \cup (C - A)$.

Therefore, we have $x \in (A - C)$ or $x \in (C - A)$ and since $x \notin A$ we conclude that $x \in C - A$ and so $x \in C$.

Proof $C \subseteq B$ follows similarly if we replace B by C in the previous proof.

5. (a)

$$\begin{aligned}
\overline{(\overline{A} - \overline{B})} \cap B &= \overline{(\overline{A} \cap B)} \cap B \\
&= (A \cup \overline{B}) \cap B \\
&= (A \cap B) \cup (\overline{B} \cap B) \\
&= (A \cap B) \cup \emptyset \\
&= (A \cap B) \\
&= (A - \overline{B}).
\end{aligned}$$

(b)

$$\begin{aligned}
\overline{(\overline{A} - B)} \cap C &= \overline{(\overline{A} \cap \overline{B})} \cap C \\
&= (A \cup B) \cap C \\
&= (A \cap C) \cup (B \cap C)
\end{aligned}$$

(c)

$$\begin{aligned}
(A - C) - (B - C) &= (A \cap \overline{C}) \cap \overline{(B \cap \overline{C})} \\
&= (A \cap \overline{C}) \cap (\overline{B} \cup C) \\
&= (A \cap \overline{B} \cap \overline{C}) \cup (A \cap \overline{C} \cap C) \\
&= ((A - B) \cap \overline{C}) \cap \emptyset \\
&= (A - B) - C
\end{aligned}$$

(d)

$$\begin{aligned}
A \cap (B - A) &= A \cap (B \cap \overline{A}) \\
&= A \cap B \cap \overline{A} \\
&= \emptyset
\end{aligned}$$

(e)

$$\begin{aligned}
(B - A) \cup (C - A) &= (B \cap \overline{A}) \cup (C \cap \overline{A}) \\
&= (B \cup C) \cap \overline{A} \\
&= (B \cup C) - A
\end{aligned}$$

6. To prove that the two sets are equal, we first prove that $A \times (B \cap C) \subseteq (A \times B) \cap (A \times C)$.

   Let $(x, y) \in A \times (B \cap C)$. Then $x \in A$ and $y \in B \cap C$. Since $y \in B \cap C$, it follows that $y \in B$ and $y \in C$. Therefore, $(x, y) \in A \times B$ and $(x, y) \in A \times C$. These imply $(x, y) \in (A \times B) \cap (A \times C)$.

   A similar argument can be used to prove $(A \times B) \cap (A \times C) \subseteq A \times (B \cap C)$.

7. (a) $[-3,\ 6] \cap (-2,\ 7] = (-2,\ 6]$
   (b) $(-5, 7] \cap \mathbb{Z} = \{-4,\ -3,\ -2,\ -1,\ 0,\ 1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7\}$

8. We construct proof by contradiction. Assume that there exist sets $A \subseteq \mathbb{N}$ and $B \subseteq \mathbb{N}$ such that $A \times B = \{(0, 0),\ (1, 1)\}$. It follows that $(0, 0) \in A \times B$ and $(1, 1) \in A \times B$. Hence, $\{0, 1\} \subseteq A$ and $\{0, 1\} \subseteq B$.

   Take $x = 0$ and $y = 1$. We know that $x \in A$ and $y \in B$. However, $(x, y) = (0, 1) \notin A \times B$ which is a contradiction.

9. The left hand side of the set identity is $(A - B) - C = (A \cap \overline{B}) \cap \overline{C} = A \cap (\overline{B} \cap \overline{C})$.

   The right hand side is $A - (B - C) = A \cap \overline{(B - C)} = A \cap \overline{(B - C)} = A \cap \overline{(B \cap \overline{C})} = A \cap (\overline{B} \cup C)$.

   In general, the identity is not true. The following is a counterexample. Let $U = \{1,\ 2,\ 3,\ 4\}$, $A = \{4\}$, $B = \{1,\ 2,\ 3,\ 4\}$ and $C = \{3,\ 4\}$.

   Then, $A - B = \emptyset$ and so $(A - B) - C = \emptyset$. However, $B - C = \{1,\ 2\}$ and so $A - (B - C) = \{4\}$.

10. We have $|A \cup B| = |A| + |B| - |A \cap B| = 6 + 8 - 3 = 11$. Hence, $|\mathcal{P}(A \cup B)| = 2^{11}$.

11. (a) The statement is false since $A \cap B = \emptyset$ implies that they have no common elements. E.g. Take $A = \{1\}$ and $B = \{2\}$. Here, $A \cap B = \emptyset$, however, $A \neq B$.
    (b) The statement is false since $A - B = \emptyset$ implies that $A \cap \overline{B} = \emptyset$. E.g. Take $A = \{1\}$ and $B = \{1,\ 2\}$. Here, $A - B = \emptyset$, however, $A \neq B$.
    (c) The statement is true.

        To prove this let $x \in A$. We need to prove that $x \in B$.

        We construct a proof by contradiction. Assume that $x \in A$ but $x \notin B$. Then $x \in A$ and $x \in \overline{B}$ and hence $x \in A - B$. This is a contradiction with the fact that $A - B = \emptyset$.

        Hence, $A \subseteq B$.
    (d) The statement is true. If $A \cup B = \emptyset$, then each set A and B is empty. Hence, $A = B = \emptyset$.
    (e) The statement is true. Note that since $A \oplus B = (A - B) \cup (B - A)$, then $A \oplus B = \emptyset$ implies (see part (d)) that
        i. $A - B = \emptyset$. This gives $A \subseteq B$ (see part (c)).
        ii. $B - A = \emptyset$. This gives $B \subseteq A$ (see part (c)).

        Hence, $A = B$.
    (f) The statement is true. Since $A \times B = \emptyset$, then $A = B = \emptyset$.
    (g) The statement is false. If $\overline{A} - \overline{B} = \emptyset$, then (see part (c)) $\overline{A} \subseteq \overline{B}$ and therefore $B \subseteq A$. The following is a counterexample.

        Let $U = \{1,\ 2,\ 3\}$, $A = \{1,\ 2\}$ and $B = \{2\}$. Then, $\overline{A} = \{3\}$ and therefore, $\overline{A} - \overline{B} = \overline{A} \cap B = \emptyset$. However, $A \neq B$.

(h) The statement is false. Note that $A - B = A$ implies $A \cap \overline{B} = A$ and hence, $A \subseteq \overline{B}$. The following is a counterexample.

Let $A = \{1\}$, $B = \emptyset$ and $U = \{1,\ 2\}$. Then $A - B = A \cap \overline{B} = A \cap U = A$. However, A is not a subset of B.

(i) The statement is false. $A \cup B = A$ implies that $B \subset A$. Let $A = \{1,\ 2\}$, $B = \{1\}$ and $U = \{1,\ 2,\ 3\}$. Note that $A \cup B = \{1,\ 2\} = A$ but $B \neq \emptyset$.

(j) The statement is true. Let $x \in A$. We prove that $x \in B$.

Assume, by contradiction that $x \in A$ but $x \notin B$. Then $x \in \overline{B}$ and since $\overline{B} \subseteq \overline{A}$ then $x \in \overline{A}$. This is a contradiction with the fact that $x \in A$.

Hence, $A \subseteq B$.

12. Note that $-x^2+x+2 = -(x^2-x-2) = -(x-2)(x+1)$. The solutions of the equation $-x^2 + x + 2 = 0$ are $x = -1$ and $x = 2$. The quadratic function $-x^2 + x + 2$ has a positive sign between its roots. Therefore, $S = [-1,\ 2]$. Then, $S \cap \mathbb{Z} = \{-1,\ 0,\ 1,\ 2\}$ and so $|S \cap \mathbb{Z}| = 4$.

13. (a) $A_{100} = \{-100, \ldots, -1, 0, 1, 2, 3, \ldots\}$ and $A_{96} = \{-96, \ldots, -1, 0, 1, 2, 3, \ldots\}$ and so $A_{100} - A_{96} = \{-100,\ -99,\ -98,\ -97\}$.

(b) $\cup_{n=1}^{\infty} A_n = \mathbb{Z}$. To see this, note that since each set $A_n \subseteq \mathbb{Z}$, then $\cup_{n=1}^{\infty} A_n \subseteq \mathbb{Z}$.

It is left to prove that $\mathbb{Z} \subseteq \cup_{n=1}^{\infty} A_n$. Let $n \in \mathbb{Z}$. We split the rest of the proof into three cases, according to the sign of n.

Case 1. Assume that $n > 0$ then $n \in A_n$ and so $n \in \cup_{n=1}^{\infty} A_n$.

Case 2. Assume that $n < 0$ so $n \in A_{-n}$. Hence, $n \in \cup_{n=1}^{\infty} A_n$.

Case 3. If $n = 0$, then $n \in A_1$ and therefore $n \in \cup_{n=1}^{\infty} A_n$.

This concludes the proof of $\mathbb{Z} \subseteq \cup_{n=1}^{\infty} A_n$, and so $\mathbb{Z} = \cup_{n=1}^{\infty} A_n$.

(c) We claim that $\cap_{n=1}^{\infty} A_n = A_1$, with $A_1 = \{-1,\ 0,\ 1, \ldots\}$. To prove it we should prove both inclusions.

Clearly, $\cap_{n=1}^{\infty} A_n \subseteq A_1$ (if $x \in \cap_{n=1}^{\infty} A_n$, then $x \in A_n$, for every $n \geq 1$ and so $x \in A_1$).

We need to prove that $A_1 \subseteq \cap_{n=1}^{\infty} A_n$. Let $x \in A_1$. Then, $x \in A_n$, for each $n \geq 2$ (in addition, note that $A_1 \subseteq A_2 \subseteq \ldots A_n \subseteq \ldots$). Hence, $x \in \cap_{n=1}^{\infty} A_n$.

5 Functions

1. (a) $f(n) = n + 5$ is not onto. To see this, take $m = 1 \in \mathbb{N}$. There is no $n \in \mathbb{N}$ such that $f(n) = 1$. However, the function is one-to-one. If $f(n_1) = f(n_2)$, i.e. $n_1 + 5 = n_2 + 5$ then $n_1 = n_2$.

(b) $g(n) = \left[\dfrac{n}{2}\right]$ is not one-to-one. Take $n_1 = 0$ and $n_2 = 1$. Then $g(n_1) = [0] = 0$ and $g(n_2) = \left[\dfrac{1}{2}\right] = 0$. But g is onto. Let $m \in \mathbb{N}$. Then there exists $n \in \mathbb{N}$ such that $g(n) = m$. (You may take $n = 2m$, for example).

(c) Let $f : \mathbb{N} \longrightarrow \mathbb{N}$ be given by

$$f(n) = \begin{cases} n+1, & \text{if } n \text{ is even} \\ n-1, & \text{if } n \text{ is odd} \end{cases}$$

To show that f is one-to-one we consider $n_1 \in \mathbb{N}$ and $n_2 \in \mathbb{N}$ such that $f(n_1) = f(n_2)$. Then, we consider the following cases.

Case 1. Assume that $n_1 + 1 = n_2 + 1$. Then $n_1 = n_2$.

Case 2. Assume that $n_1 - 1 = n_2 - 1$. Then $n_1 = n_2$.

To show that f is onto, take any $m \in \mathbb{N}$. There are two cases.

Case 1. If m is even, then $m+1$ is odd and $f(m+1) = (m+1) - 1 = m$. Hence, there exists $n \in \mathbb{N}$, $(n = m+1)$ such that $f(n) = m$.

Case 2. If m is odd, then $m-1$ is even and $f(m-1) = (m-1) + 1 = m$. Hence, there exists $n \in \mathbb{N}$, $(n = m-1)$ such that $f(n) = m$.

Hence, f is bijective.

(d) Let $f : \mathbb{N} \longrightarrow \mathbb{N}$ be given by $f(n) = 1$. Clearly, f is not bijective (is neither a one-to-one function nor onto.)

2. Let $g : A \longrightarrow B$ and $f : B \longrightarrow C$ be two functions.

   (a) Let $x_1 \in A$ and $x_2 \in A$ such that $g(x_1) = g(x_2)$. Then $f(g(x_1)) = f(g(x_2))$, or $(f \circ g)(x_1) = (f \circ g)(x_2)$. Since $f \circ g$ is one-to-one, it follows that $x_1 = x_2$. Hence, g is one-to-one.

   (b) The statement is not true, in general. The following provides a counterexample. Let $A = \{a\}$, $B = \{b_1,\ b_2\}$ and $C = \{c\}$. We define the functions as follows. Let $g(a) = b_1$, $f(b_1) = f(b_2) = c$. Then $(f \circ g)(a) = c$ and f are clearly onto. However, g is not onto since there is no element in the domain that is mapped into b_2 (or $b_2 \notin Image(g)$).

3. Let $f : A \longrightarrow B$ be a function, and S and T two subsets of A. Show that:

   (a) The equality of the two sets can be proved using the definition.
   Let $y \in f(S \cup T)$. Then there exists $x \in S \cup T$ such that $f(x) = y$. It follows that $x \in S$ or $x \in T$, and $f(x) = y$. This gives $y \in f(S)$ or $y \in f(T)$ which proves $f(S \cup T) \subseteq f(S) \cup f(T)$.
   The proof of $f(S) \cup f(T) \subseteq f(S \cup T)$ can be done similarly.

   (b) Let $y \in f(S \cap T)$. Then, there exists $x \in S \cap T$ such that $f(x) = y$. This implies that $x \in S$ and $x \in T$, and $f(x) = y$. Hence, $y \in f(S) \cap f(T)$.

4. We show that f is not one-to-one. Take $m_1 = 1$, $n_1 = 1$, $m_2 = 2$ and $n_2 = 2$. Clearly, $(m_1,\ n_1) \neq (m_2,\ n_2)$ but $f(m_1,\ n_1) = 1$ and $f(m_2,\ n_2) = 1$.

We show that f is onto. Let $q \in \mathbb{Q}$. From the definition of a rational number, there exist integers m and n, with $n \neq 0$ such that $q = \dfrac{m}{n}$. This proves that for any $q \in \mathbb{Q}$, there exist $(m,\ n) \in \mathbb{Z} \times (\mathbb{Z} - \{0\})$ such that $f(m,\ n) = \dfrac{m}{n}$.

5. For each of the following assignments, determine whether it is a function or not. If it is a function, is it one-to-one? Is it onto?

   (a) $f_1 : \mathbb{R} \longrightarrow \mathbb{R}, \quad f_1(x) = -x^3 + 1$ is a bijective function (one-to-one since $-x_1^3 + 1 = -x_2^3 + 1$ implies $x_1 = x_2$ and onto since for any $y \in \mathbb{R}$, there exist $x \in \mathbb{R}$: $x = \sqrt[3]{1-y}$ such that $f(x) = y$).

   (b) $f_2 : \mathbb{N} \longrightarrow \mathbb{Z}, \quad f_2(n) = n^2 + 3$ is one-to-one ($n_1^2 + 3 = n_2^2 + 3$ implies $n_1 = n_2$ since they are both natural numbers) but not onto (If $z = 0 \in \mathbb{Z}$, there is no $n \in \mathbb{N}$ such that $f(n) = 0$).

   (c) $f_3 : \mathbb{R} \longrightarrow [0,\ +\infty), \quad f_3(x) = 2^x$ is one-to-one ($2^{x_1} = 2^{x_2}$ implies $x_1 = x_2$) but not onto (for $y = 0$, there does not exist $x \in \mathbb{R}$ such that $2^x = 0$).

   (d) $f_4 : \mathbb{N} \longrightarrow \mathbb{N}, \quad f_4(n) = \sqrt{n} + 1$ is not a function since $\sqrt{n} + 1 \notin \mathbb{N}$.

   (e) $f_5 : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}, \quad f_5(x, y) = x + y$ is not one-to-one (if $(x_1,\ y_1) = (0,\ 0)$ and $(x_2,\ y_2) = (1,\ -1)$, $f(x_1,\ y_1) = f(x_2,\ y_2) = 0$) but f is onto (if $y \in \mathbb{R}$, there exists a pair, e.g. $(0,\ y) \in \mathbb{R} \times \mathbb{R}$, with $f(0,\ y) = y$).

   (f) $f_5 : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R} \times \mathbb{N}, \quad f_6(x, y) = (2x, 0)$ is not on-to-one ($(1,\ 2)$ and $(1,\ 3)$ are both mapped into $(2,\ 0)$) and not onto (for $(1,\ 5) \in \mathbb{R} \times \mathbb{N}$ there does not exist a pair $(x,\ y) \in \mathbb{R} \times \mathbb{R}$ such that $f(x,\ y) = (1,\ 5)$).

6. Let $(a_1, b_1) \in A \times B$ and $(a_2, b_2) \in A \times B$ with $\lambda(a_1, b_1) = \lambda(a_2, b_2)$. Then, $2^{\phi(a_1)}3^{\psi(b_1)} = 2^{\phi(a_2)}3^{\psi(b_2)}$ and so $2^{\phi(a_1)-\phi(a_2)} = 3^{\psi(b_1)-\psi(b_2)}$. It follows that $\phi(a_1) - \phi(a_2) = \psi(b_1) - \psi(b_2) = 0$. Since ϕ and ψ are one-to-one functions, we obtain $a_1 = a_2$ and $b_1 = b_2$ and so $(a_1, b_1) = (a_2, b_2)$.

7. (a) Let $x_1 \in (0, +\infty)$ and $x_2 \in (0, +\infty)$ be such that $f(x_1) = f(x_2)$. Then, $2x_1^2 + 3 = 2x_2^2 + 3$. This implies $x_1^2 = x_2^2$ and since they are positive numbers, we can conclude that $x_1 = x_2$. Hence, f is one-to-one.

   (b) For any $x \in (0, +\infty)$, we have $x^2 > 0$ and so $2x^2 + 3 > 3$. Hence, $Image(f) = (3,\ \infty)$. Since $Image(f) \neq (0,\ \infty)$, we conclude that f is not onto.

8. The function $f \circ g : \mathbb{R} \longrightarrow \mathbb{R}$ is given by $(f \circ g)(x) = f(g(x)) = f(x^3 - 2) = \sqrt[3]{(x^3 - 2) + 2} = x$, for any $x \in \mathbb{R}$.

   The function $g \circ f : \mathbb{R} \longrightarrow \mathbb{R}$ is given by $(g \circ f)(x) = g(f(x)) = g(\sqrt[3]{x + 2}) = (\sqrt[3]{x + 2})^3 - 2 = x$, for any $x \in \mathbb{R}$.

   Since $f \circ g = \mathbf{1}_{\mathbb{R}}$ and $g \circ f = \mathbf{1}_{\mathbb{R}}$ we conclude that $f^{-1} = g$. This implies that f is a bijection and henceg is also a bijection.

9. (a) Since $|A| = 4 < |B| = 6$ the statement is true.

(b) Since $|\mathcal{P}(A)| = 2^4 < 2^6 = |\mathcal{P}(B)|$ the statement is false.

(c) Since $|\mathcal{P}(A)| = 2^4 > 6 = |B|$ the statement is true.

(d) Since $|A \times B| = |A| \cdot |B| = 4 \cdot 6 = 24 < 2^6 |\mathcal{P}(B)|$ the statement is true.

(e) Since $|\mathcal{P}(\mathcal{P}(A))| = 2^{2^4} = 2^{16} < 2^{24} = |\mathcal{P}(A \times B)|$ the statement is false.

10. The functions are not one-to-one.

If $(x_1 = 1.2$ and $x_2 = 1.5)$ then $(f(x_1) = \lfloor 1.2 \rfloor = 1$ and $f(x_2) = \lfloor 1.5 \rfloor = 1)$ and $(g(x_1) = \lceil 1.2 \rceil = 2$ and $g(x_2) = \lceil 1.5 \rceil = 2)$.

However, the functions are onto.

Let $m \in \mathbb{Z}$ be arbitrary.

Then there exists $x \in \mathbb{R}$ such that the equation $f(x) = m$ has at least one solution in $\mathbb{R}$ ($x = m$ is one solution). Similarly, there exists $x \in \mathbb{R}$ such that the equation $g(x) = m$ has at least one solution in $\mathbb{R}$ ($x = m$ is one solution).

11. (a) The conclusion follows once we prove that $g \circ f$ and $(g \circ f)^{-1}$ are invertible. We show this by actually identifying their inverses.

Since $(g \circ f) \circ (f^{-1} \circ g^{-1}) = g \circ f \circ f^{-1} \circ g^{-1} = g \circ \mathbf{1}_B \circ g^{-1} = g \circ g^{-1} = \mathbf{1}_C$, and similarly, $(f^{-1} \circ g^{-1}) \circ (g \circ f) = \mathbf{1}_A$, it follows that $(g \circ f)^{-1} = (f^{-1} \circ g^{-1})$ and hence, $g \circ f$ and $f^{-1} \circ g^{-1}$ are invertible (f^{-1} and g^{-1} exist since f and g are assumed to be bijective).

(b) Let $f : [0,\ \infty) \longrightarrow \mathbb{R}$, $f(x) = \sqrt{x}$ and $g : \mathbb{R} \longrightarrow [0,\ \infty)$, $g(x) = x^2$.

The functions f and g are not bijective since f is not onto (for $y = -1 \in \mathbb{R}$ the equation $f(x) = -1$ has no solution in $[0,\ \infty)$) and g is not one-to-one $(g(-1) = g(1) = 1)$.

However, $(g \circ f)(x) = g(f(x)) = g(\sqrt{x}) = (\sqrt{x})^2 = x$ is bijective. (Note that in this case, $(f \circ g)(x) = |x|$, for $x \in \mathbb{R}$.)

6 Relations

1. R_1 is reflexive since $(x,\ x) \in R_1$, for any $x \in A$

R_1 is symmetric since $(x,\ y) \in R_1$ implies $(y,\ x) \in R_1$, for any $x \in A$ and $y \in A$ (or equivalently, there is no pair $(x,\ y) \in A \times A$ such that $(x,\ y) \in R_1$ and $(y,\ x) \notin R_1$) (details are omitted)

R_1 is transitive since $(x,\ y) \in R_1$ and $(y,\ z) \in R_1$ implies $(x,\ z) \in R_1$ (details are omitted)

R_1 is not antisymmetric since $(1,\ 2) \in R_1$ and $(2,\ 2) \in R_1$ but $1 \neq 2$

R_2 is reflexive since $(x,\ x) \in R_2$, for any $x \in A$

R_2 is symmetric since $(x,\ y) \in R_2$ implies $(y,\ x) \in R_2$, for any $x \in A$ and $y \in A$ (details are omitted)

R_2 is transitive since $(x,\ y) \in R_2$ and $(y,\ z) \in R_2$ implies $(x,\ z) \in R_2$

R_2 is antisymmetric since $(x,\ y) \in R_2$ and $(y,\ x) \in R_2$ implies $x = y$

R_3 is not reflexive since $(1,\ 1) \notin R_3$, and $1 \in A$

R_3 is not symmetric since $(1,\ 4) \in R_3$ but $(4,\ 1) \in R_3$

R_3 is not transitive since $(1,\ 3) \in R_3$ and $(3,\ 1) \in R_3$ but $(1,\ 1) \notin R_3$

R_3 is not antisymmetric since $(1,\ 3) \in R_3$ and $(3,\ 1) \in R_3$ but $1 \neq 3$

2. R_1 is not reflexive since $(a,\ a) \notin R_1$

   R_1 is symmetric: if $(x,\ y) \in R_1$ then $(y,\ x) \in R_1$ is true vacuously

   R_1 is transitive: if $(x,\ y) \in R_1$ and $(y,\ z) \in R_1$ then $(x,\ z) \in R_1$ is true vacuously

   R_1 is antisymmetric: if $(x,\ y) \in R_1$ and $(y,\ x) \in R_1$ then $x = y$ is true vacuously

   R_2 is not reflexive, not symmetric, transitive and antisymmetric

   R_3 is not reflexive, symmetric, not transitive and not antisymmetric

   R_4 is reflexive, not symmetric, transitive and antisymmetric

3. R_1 is not reflexive since e.g. $(1,\ 1) \notin R_1$

   R_1 is not symmetric since e.g. $(1,\ 2) \in R_1$ but $(2,\ 1) \notin R_1$

   R_1 is transitive since $(a,\ b) \in R_1$ and $(b,\ c) \in R_1$ implies $(a,\ c) \in R_1$ (that is if $a < b$ and $b < c$ then $a < c$)

   R_1 is antisymmetric since $(a,\ b) \in R_1$ and $(b,\ a) \in R_1$ implies $a = b$ vacuously

   R_2 is not reflexive since e.g. $(4,\ 4) \notin R_2$

   R_2 is not symmetric since e.g. $(4,\ 2) \in R_2$ but $(2,\ 4) \notin R_2$

   R_2 is not transitive since e.g. $(-3,\ -2) \in R_2$ and $(-2,\ 0) \in R_2$ but $(-3, 0) \notin R_2$

   R_2 is not antisymmetric since e.g. $(-1,\ -2) \in R_2$ and $(-2,\ -1) \in R_2$ but $-1 \neq -2$

4. (a) The statement is true since $|\mathcal{P}(A \times B)| = 2^{15} = 32768$

   (b) The statement is false. The number of binary relations from B to A equals to the number of binary relations from A to B

   (c) The statement is false. A binary relation from $\mathcal{P}(A)$ to $\mathcal{P}(B)$ is a subset of $\mathcal{P}(\mathcal{P}(A) \times \mathcal{P}(B))$ which is not equal to $\mathcal{P}(A \times B)$.

   (d) The statement is true since the number of binary relation from A to B that contain the subset $\{(x,\ 0);\ x \in A\}$ is equal to the number of binary relations that can be defined from A to the set $\{1,\ 2,\ 3,\ 4\}$, which is $2^{3 \cdot 5} = 4096$.

(e) The statement is false. The number of binary relation from A to B that contain the subset $\{(a,\ y);\ y \in B\}$ is equal to the number of binary relations that can be defined from the set $\{b,\ c\}$ to B, which is $2^{2\cdot 5} \neq 4096$.

5. R_1 is not a function since $(1,\ a) \in R_1$ and $(1,\ b) \in R_1$ (the element 1 would be mapped into two different values).

   R_1 is a function. It is not one-to-one since $(0,\ c) \in R_2$ and $(1,\ c) \in R_2$ (two different inputs have the same output). It is not onto since there is no $x \in A$ such that $(x,\ b) \in R_2$ (or $(x,\ e) \in R_2$)

   R_3 is a function. It is not one-to-one since e.g. $(0,\ c) \in R_3$ and $(1,\ c) \in R_3$. It is not onto since there is no $x \in A$ such that $(x,\ b) \in R_3$ (or $(x,\ e) \in R_3$)

   R_4 is a function. It is not one-to-one since $(1,\ a) \in R_4$ and $(3,\ a) \in R_4$. It is not onto since there is no $x \in A$ such that $(x,\ e) \in R_4$.

   R_5 is not a function since it does not contain an element of the form $(1,\ y)$ (the element 1 is not mapped into an element on B).

   R_6 is a function. It is one-to-one and onto (and therefore bijective).

6. (a) R_1 is not reflexive since e.g. $(1,\ 1) \notin R_1$.
   R_1 is symmetric
   R_1 is not transitive e.g. $(1,\ 2) \in R_1$ and $(2,\ 3) \in R_1$ but $(1,\ 3) \notin R_1$

   (b) R_1 is not reflexive since e.g. $(2,\ 2) \notin R_2$.
   R_2 is symmetric
   R_2 is transitive $(x,\ y) \in R_2$ and $(y,\ z) \in R_2$ implies $(x,\ z) \in R_2$

   (c) R_3 is reflexive since for any $x \in \mathbb{Z}$ we have $xR_3x = x + x^2 = x(x+1)$ which is an even integer
   R_3 is not symmetric. Let $(2,\ 1) \in \mathbb{Z} \times \mathbb{Z}$. Since $2 + 2 \cdot 1 = 4$ is even, $2R_31$. However, $(1,\ 2) \notin R$ since $1 + 1 \cdot 2 = 3$ is not even
   R_3 is transitive. Let $(x,\ y) \in R_3$ and $(y,\ z) \in R_3$. Then $x + xy$ is even and $y + yz$ is even. The proof continues by considering two cases, according to the parity of y.
   Case 1. Assume that y is even. Then xy is even and since $x + xy$ is even, it follows that x is even. Hence, $x + xz$ is even.
   Case 2. Assume that y is odd. Then, since $y + yz$ is even it follows that z is odd and so $x + xz$ is even.

7. (a) R_1 is reflexive ($A \subseteq A$, for any $A \in \mathcal{P}(U)$), not symmetric ($A \subseteq B$ does not imply $B \subseteq A$), transitive ($A \subseteq B$ and $B \subseteq C$ implies $A \subseteq C$) and antisymmetric ($A \subseteq B$ and $B \subseteq A$ implies $A = B$)

   (b) R_2 is not reflexive ($A \cap A = A \neq \emptyset$ not true for any $A \in \mathcal{P}(U)$), symmetric ($A \cap B = \emptyset$ implies $B \cap A = \emptyset$), not transitive (e.g. if $A = \{1,\ 2\}$, $B = \{4\}$ and

$C = \{2,\ 3\}$, then $A \cap B = \emptyset$, $B \cap C = \emptyset$ but $A \cap C = \{2\}$) and not antisymmetric (e.g. A $=\{1\}$, $B = \{2\}$ and $A \cap B = B \cap A = \emptyset$)

(c) First show that $A - B = \emptyset$ if and only if $A \subseteq B$.

To prove the equivalence, we first assume $A - B = \emptyset$ is true and show that $A \subseteq B$ holds (see also Exercise 11, part (c) in Section 4).

Let $x \in A$. Then, since $A - B = A \cap \overline{B} = \emptyset$, it follows that $x \notin \overline{B}$ and hence $x \in B$. This proves $A \subseteq B$.

Now we assume that $A \subseteq B$ holds and show $A - B = \emptyset$.

Case 1. If $A = \emptyset$, then $A - B = \emptyset$ is true.

Case 2. Assume $A \neq \emptyset$. Let $x \in A$. Then, since $A \subseteq B$, we have $x \in B$ and so $x \notin \overline{B}$. This implies $A - B = A \cap \overline{B} = \emptyset$.

Therefore, $A\,R_3\,B \Leftrightarrow A \subseteq B$.

Therefore, R_3 is equivalent to R_1 and so it has the same properties.

(d) Using Exercise 11, part (e) in Section 4, we conclude that $A \oplus B = \emptyset$ if and only if $A = B$.

Therefore, R_4 becomes $A\,R_4\,B \Leftrightarrow A = B$.

R_4 is reflexive ($A = A$ for any $A \in \mathcal{P}(U)$), symmetric, transitive ($A = B$ and $B = C$ implies $A = C$) and antisymmetric (if $A = B$ and $B = A$, then $A = B$)

8. (a) We first show that R is symmetric.

Let $(a,\ b) \in R$. Since R is reflexive, $(b,\ b) \in R$. Now, since R is cyclic, we apply the definition (with $c = b$) and obtain that $(b,\ a) \in R$ which proves that R is symmetric.

We now prove that R is transitive.

Let $(a,\ b) \in R$ and $(b,\ c) \in R$. Since R is cyclic we obtain $(c,\ a) \in R$ which together with the fact that R is symmetric (proved before) proves that $(a,\ c) \in R$.

(b) Let $(a,\ b) \in R$ and $(b,\ c) \in R$. Then, since R is transitive, $(a,\ c) \in R$. This together with the fact that R is symmetric implies that $(c,\ a) \in R$ which proves that R is cyclic.

9. (a) Let $R = \{(2,\ 2),\ (2,\ 3),\ (3,\ 2),\ (3,\ 3)\}$. R is symmetric and transitive but not reflexive since $(1,\ 1) \notin R$.

(b) The proof in not correct since the following assumption is used at the beginning of the argument: for any $a \in A$, there exists $b \in A$ such that $(a,\ b) \in R$. This statement is not true in general (see example given at the previous part: for $a = 1 \in A$ there is no element $b \in A$ such that $(1,\ b) \in R$).

7 Equivalence Relations

1. (a) The relation R is reflexive (for any $x \in A$, xRx, since $\dfrac{x}{x} = 3^0$), symmetric (for any $x, y \in A$, xRy implies yRx, since $\dfrac{x}{y} = 3^k$ implies $\dfrac{y}{x} = 3^{-k}$ for some $k \in \mathbb{Z}$)

and transitive (for any $x, y, z \in A$, xRy and yRz imply xRz, since $\dfrac{x}{y} = 3^k$ and $\dfrac{y}{z} = 3^l$ imply $\dfrac{x}{z} = 3^{k+l}$ for some $k \in \mathbb{Z}$ and $l \in \mathbb{Z}$).

(b) The partition is $\{1\}$, $\left\{\dfrac{1}{3}, \dfrac{1}{27}, 3\right\}$, $\left\{\dfrac{1}{4}\right\}$, $\left\{\dfrac{1}{36}, \dfrac{9}{4}\right\}$, $\{2\}$, $\left\{\dfrac{2}{9}\right\}$, $\{5\}$.

2. (a) The relation R is reflexive (for any $f \in A$, fRf, since $f(x)-f(x) = 0$), symmetric (for any $f, g \in A$, fRg implies gRf, since $f(x)-g(x) = c$ implies $g(x)-f(x) = -c$ for some constant $c \in \mathbb{Z}$) and transitive (for any $f, g, h \in A$, fRg and gRh imply fRh, since $f(x) - g(x) = c$ and $g(x) - h(x) = d$ imply $f(x) - h(x) = c + d$ for some $c \in \mathbb{Z}$ and $d \in \mathbb{Z}$).

(b) $[f(x)]_R = \{g : \mathbb{Z} \to \mathbb{R},\ g$ is a function such that $f(x)-g(x) = c,$ for some constant c $\in \mathbb{Z}\} = \{g : \mathbb{Z} \to \mathbb{R},\ g$ is a function such that $g(x) = 2x + c,$ for some constant c $\in \mathbb{Z}\}$. Hence, f_1,$f_5 \in [f(x)]_R$

3. (a) The fact that R is an equivalence relation on $\mathbb{W}$ follows from properties of real numbers. Note that two pairs are included in R if and only if their components have same sign, respectively, i.e. $(x,\ y)R(a,\ b)$ if and only if (x and a have same sign and y and b have same sign)

(b) There are four equivalence classes on $\mathbb{W}$ (corresponding to the four quadrants in the plane):
$[(1,\ 1)]_R$, $[(1,\ -1)]_R$, $[(-1,\ 1)]_R$, $[(-1,\ -1)]_R$.

4. (a) Follows easily from properties of real numbers.

(b) The following two sets form the partition on $\mathcal{A}$ defined by R.

$$\left\{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}\right\},$$

$$\left\{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 4 \\ 3 & 6 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\right\}.$$

5. The given binary relation on $\mathbb{R}^2$ can be equivalently defined as

$$(x,\ y)R\,(x',\ y') \Leftrightarrow x - y = x' - y'.$$

(a) Follows easily from properties of real numbers.

(b) $[(1,\ 1)]_R = \{(x,\ y) \in \mathbb{R}^2,\ x - y = 0\} = \{(x,\ x) \in \mathbb{R}^2\}$. The equivalence class of the vector $(1,\ 1)$ represents the points located on the first diagonal, i.e the points located on the line $y = x$.

6. (a) R is an equivalence relation since it is:
- reflexive ($a = a \cdot 1$)

- symmetric (if $b = a \cdot c$ then $a = \frac{1}{c}b$, for some $c > 0$)
- transitive (if $b = a \cdot c$ and $d = b \cdot e$, for some $c > 0$ and $e > 0$, then $d = a \cdot (ce)$, with $ce > 0$)

(b) The partition consists of $[1]_R = (0,\ \infty)$, $[-1]_R = (-\infty, 0)$ and $[0]_R = \{0\}$.

7. (a) Note that $\kappa(a) \in \mathbb{N}^*$, for any $a \in \mathbb{N}^*$. R is an equivalence relation since it is:
   - reflexive ($\kappa(a) = \kappa(a)$)
   - symmetric (if $\kappa(a) = \kappa(b)$ then $\kappa(b) = \kappa(a)$)
   - transitive (if $\kappa(a) = \kappa(b)$ and $\kappa(b) = \kappa(c)$, then $\kappa(a) = \kappa(c)$)

   (b) $[2]_R = \{n \in \mathbb{N}^*,\ \kappa(2) = \kappa(n)\}$. Since $\kappa(2) = 2$, $[2]_R = \{n \in \mathbb{N}^*,\ n \text{ is even}\}$.

8. (a) omitted

   (b) The partition is the following sequence of sets.
   $\{011,\ 111\}, \{010,\ 110\}, \{001,\ 101\}, \{000,\ 100\}$

9. (a) omitted

   (b) $[(0,\ 2)]_R = \{(x,\ y) \in \mathbb{R}^2,\ x^2 + y^2 = 4\}$. The sets consists of the points located on the circle of radius 2 and origin $(0,\ 0)$.
   Other elements in the equivalence class of $(0,\ 2)$ are e.g. $(2,\ 0), (-2,\ 0), (0,\ -2)$.

10. (a) R is an equivalence relation since it is
    - reflexive: ARA since $A = 1 \cdot A$
    - symmetric: ARB implies BRA, since $A = \lambda B$ implies $B = \frac{1}{\lambda}A$, for some $\lambda \in \mathbb{R}^*$
    - transitive: ARB and BRC imply ARC, since $A = \lambda B$ and $B = \gamma C$ imply $A = \delta C$, where $\delta = \lambda \cdot \gamma \in \mathbb{R}^*$

    (b) The partition is $\{A_1,\ A_5,\ A_8\}$, $\{A_2,\ A_6\}$, $\{A_3,\ A_7\}$ and $\{A_4\}$.

11. Let $a, b \in \mathbb{R}^*$ such that aRb. Then $|a|b = a|b|$, or $\frac{a}{|a|} = \frac{b}{|b|}$. Hence aRb if a and b have the same sign, so the given relation can be equivalently written as

$$aR\,b \Leftrightarrow a \text{ and } b \text{ have the same sign.}$$

    (a) omitted

    (b) $[1]_R = (0,\ \infty)$.

8 Basic Counting Techniques

1. Note that in this question we assume that a license plate has six characters.

   Let A be the set of license plates that can be made with three digits followed by three letters and B be the set of license plates that can be made with three letters followed by three digits. We need to compute the cardinality of $A \cup B$.

Since $A \cap B = \emptyset$, using the principle of exclusion-inclusion, we have $|A \cup B| = |A| + |B|$.

Using the product rule, $|A| = 10 \cdot 10 \cdot 10 \cdot 26 \cdot 26 \cdot 26 = 260^3$, $|B| = 260^3$ and hence $|A \cup B| = 2 \cdot 260^3 = 35152000$.

2. Assume that n is even. Then, there exists $k \in \mathbb{N}^*$ such that $n = 2k$. Hence, we only need to count the number of binary strings of length k that can be formed. This number is equal to 2^k, using the product rule. Therefore, there are $2^{\frac{n}{2}}$ palindromes of length n, if n is even.

   Assume that n is odd. Then, there exists $k \in \mathbb{N}^k$ such that $n = 2k + 1$. The number of binary strings of length k that can be formed is 2^k, but since n is odd, we need to multiply by 2 (at position $k + 1$ we can have either 0 or 1). Therefore, there are $2^{\frac{n-1}{2}+1} = 2^{\frac{n+1}{2}}$ palindromes of length n, if n is odd.

3. There are 5 odd digits and 26 letters in total. Using the product rule and excluding repetitions, the number of postal codes of the given form is $5 \cdot 26 \cdot 4 \cdot 25 \cdot 3 \cdot 24 = 936000$.

4. Assume that $A = \{x_1, \dots, x_7\}$ and $B = \{y_1, \dots, y_4\}$.

   (a) There are 4 ways in which $f(x_1)$ can be defined, 4 ways in which $f(x_2)$ can be defined, etc. In total, using the product rule, there are $4^7 = 16384$ ways in which a function from A to B can be defined.

   (b) There are 7 ways in which $f(y_1)$ can be defined, 7 ways in which $f(y_2)$ can be defined, etc. In total, using the product rule, there are $7^4 = 2401$ ways in which a function from B to A can be defined.

   (c) Since $|A| > |B|$, we cannot define an one-to-one function from A to B.

   (d) There are 7 ways in which $f(x_1)$ can be defined. Since the function must be one-to-one, there are 6 ways in which $f(x_2)$ can be defined (after choosing the value of $f(x_1)$). After specifying the values of $f(x_1)$ and $f(x_2)$, there are 5 ways of defining $f(x_3)$ and finally, using the same procedure, there are 4 ways of defining $f(x_4)$. In total, using the product rule, there are $7 \cdot 6 \cdot 5 \cdot 4 = 840$ ways in which a one-to-one function from B to A can be defined.

   (e) We first compute the number of functions that can be defined from A to B that are **not** onto.

   For every $1 \leq i \leq 4$, we define the set $F_i = \{f : A \to B,\ y_i \notin \text{Image}(f)\}$. The number of functions that can be defined from A to B that are not onto is

$$\begin{aligned}
|F_1 \cup F_2 \cup F_3 \cup F_4| &= |F_1| + |F_2| + |F_3| + |F_4| - |F_1 \cap F_2| - |F_1 \cap F_3| \\
&- |F_1 \cap F_4| - |F_2 \cap F_3| - |F_2 \cap F_4| - |F_3 \cap F_4| \\
&+ |F_1 \cap F_2 \cap F_3| + |F_1 \cap F_2 \cap F_4| + |F_1 \cap F_3 \cap F_4| + |F_2 \cap F_3 \cap F_4| \\
&- |F_1 \cap F_2 \cap F_3 \cap F_4| \\
&= 3^7 + 3^7 + 3^7 + 3^7 - 2^7 - 2^7 - 2^7 - 2^7 - 2^7 - 2^7 \\
&+ 1^7 + 1^7 + 1^7 + 1^7 - 0 \\
&= 7984.
\end{aligned}$$

Now, the number of onto functions that can be defined from A to B is equal to $16384 - 7984 = 8400$.

(f) Since $|B| < |A|$, we cannot define an onto function from B to A.

5. Note that there are 9 digits that can be placed on the first position of the number (corresponding to the cardinality of $\{1, \dots 9\}$), 9 digits that can be placed on the second position, after the first position was occupied (that is $10 - 1$), etc. The number of integers with the given property is $9 \cdot 9 \cdot 8 \cdot 7 \cdot 6 = 27216$.

6. Let $A = \{n \in \mathbb{N},\ 1 \le n \le 2125,\ n \text{ is divisible by } 3\}$ and $B = \{n \in \mathbb{N},\ 1 \le n \le 2125,\ n \text{ is divisible by } 11\}$

(a) The number of positive integers less or equal to 2125 that are divisible by 3 or 7 is $|A \cup B|$. To compute it we need $|A| = \left[\frac{2125-1}{3}\right] = 708$, $|B| = \left[\frac{2125-1}{11}\right] = 193$ and $|A \cap B| = \left[\frac{2125-1}{33}\right] = 64$. Using the principle of inclusion-exclusion, $|A \cup B| = |A| + |B| - |A \cap B| = 708 + 193 - 64 = 837$.

From here it follows that the number of integers between 7 and 2125 that are divisible by 3 or 11 is $837 - 2 = 835$, i.e. subtract two numbers: 3 and 6 since they are less than 7 and are divisible by 3.

(b) Note that an integer is said to be prime with 11 if it and 11 have no other common divisors than 1. Since 11 is prime, we need to find the number of positive integers that are not divisible by 11. From part (a) there are 193 positive integers less or equal than 2125 that are divisible by 11. So, $|\overline{B}| = 2125 - 193 = 1932$.

Since the first 6 of these numbers are also prime with 11, we conclude that the number of integers between 7 and 2125 that are prime with 11 is $1932 - 6 = 1926$.

(c) We need to compute $|A - B|$. We write $A = (A - B) \cup (A \cap B)$, and since $(A - B) \cap (A \cap B) = \emptyset$, using the principle of inclusion-exclusion we obtain $|A| = |A - B| + |A \cap B|$. From here, we conclude that $|A - B| = 708 - 64 = 644$, i.e. the number of positive integers less than 2125 that are divisible by 3 and not by 11 is 644.

Hence, the number of integers between 7 and 2125 that are divisible with 3 but not with 11 is $644 - 2 = 642$.

7. Let A-be the set of length 13 binary strings that begin with 0110 and B-be the set of length 13 binary strings that end with 1000. We need $|A \cup B|$. Using the principle of inclusion-exclusion, $|A \cup B| = |A| + |B| + |A \cap B|$.

We have $|A| = 2^9$, $|B| = 2^9$ and $|A \cap B| = 2^5$ and so $|A \cup B| = 2^9 + 2^9 - 2^5 = 992$.

8. Let $A = \{n \in \mathbb{N},\ 1 \le n \le 250,\ n \text{ is divisible by } 4\}$ and $B = \{n \in \mathbb{N},\ 1 \le n \le 250,\ n \text{ is divisible by } 6\}$. Then $A \cap B = \{n \in \mathbb{N},\ 1 \le n \le 250,\ n \text{ is divisible by } 12\}$. We need $|A \cup B|$.

We have $|A| = \left[\dfrac{250-1}{4}\right] = 62$, $|B| = \left[\dfrac{250-1}{6}\right] = 41$ and $|A \cap B| = \left[\dfrac{250-1}{12}\right] = 10$. It follows that $|A \cup B| = 62 + 41 - 20 = 83$.

9. For every $0 \leq i \leq n$ we define $A_i = \{$the set of binary strings of length $i\}$. Note that the binary string of length 0 is the empty string so $A_0 = \{$empty string$\}$ and $|A_0| = 1$. Since the sets A_i are mutually disjoint, the number of binary strings of length at most n is

$$\begin{aligned} |A_0 \cup \ldots A_n| &= |A_0| + |A_1| + \ldots + |A_n| = 1 + 2 + 2^2 + 2^3 \ldots + 2^n = \sum_{i=0}^{n} 2^i \\ &= \frac{2^{n+1} - 1}{2 - 1} = 2^{n+1} - 1. \end{aligned}$$

10. In this question it is assumed that the plates can have either 4, 5 or 6 characters.

    Let
    A be the set consisting of all 4 characters license plates that are formed using 2 letters followed by 2 digits
    B be the set consisting of all 5 characters license plates that are formed using 2 letters followed by 3 digits
    C be the set consisting of all 5 characters license plates that are formed using 3 letters followed by 2 digits
    D be the set consisting of all 6 characters license plates that are formed using 3 letters followed by 3 digits.

    We need to find $A \cup B \cup C \cup D$. Using the product rule, $|A| = 26^2 \cdot 10^2 = 67600$, $|B| = 26^2 \cdot 10^3 = 676000$, $|C| = 26^3 \cdot 10^2 = 1757600$ and $|D| = 26^3 \cdot 10^3 = 17576000$. Since the sets are mutually disjoint, it follows that $A \cup B \cup C \cup D = |A| + |B| + |C| + |D| = 20077200$.

11. For $6 \leq i \leq 9$, we define

    $$P_i = \{\text{i characters passwords that contain at least two distinct characters}\}.$$

    We need to find $|P_6 \cup P_7 \cup P_8 \cup P_9|$.

    For $6 \leq i \leq 9$, since the length i password can contain characters that are either a lower letter, an upper case or a digit, it follows that there are $2 \cdot 26 + 10$ options for a certain position. Because a password cannot have identical characters it follows that $|P_i| = 62^i - 62$.

    Therefore, since P_i are mutually disjoint, for $6 \leq i \leq 9$, we have $|P_6 \cup P_7 \cup P_8 \cup P_9| = |P_6| + |P_7| + |P_8| + |P_9| = (62^6 - 62) + (62^7 - 62) + (62^8 - 62) + (62^9 - 62) =$.

12. (a) Since the bride should stand next to the groom (i.e. either to the left or to the right), the number of ways the group of 6 people can be arranged equals to twice the number of ways in which we can arrange a group of 5 people. Hence, there are $2 \cdot 5! = 240$ ways in which a group with the given property can be arranged.

    (b) There are 6! ways in which a group of 6 can be arranged and so, using part (a), there are $6! - 2 \cdot 5! = 480$ ways to arrange the group so that the bride does not stand next to the groom.

(c) The arranging procedure can be divided into the following steps.

T_1 : arrange the 4 people in the group that are neither the bride neither the groom nor the bride

T_2 : arrange the groom in one of the 5 gaps between the guests and the bride to the left of groom

There are 4! ways to perform T_1.

According to the position of the groom, there are different ways in which the bride can be placed. Using the summation rule, the number of ways T_2 can be performed is $1 + 2 + 3 + 4 + 5$. The first term corresponds to the situation when the groom is placed on the fifth position and the bride is placed on the sixth, the second term corresponds to the situation when the groom is placed on the fourth position, and hence the bride can be placed either on the fifth or on the sixth position, etc.

Using the product rule, there are $15 \cdot 24 = 360$ ways to arrange people in a picture such that the bride is situated to left of the groom.

9 The Pigeonhole Principle

1. Draw the three diagonals to divide the regular hexagon into 5 equilateral triangles, each with side of length 1. In each triangle, any two points in the interior or on the perimeter are at distance at most one. Since there are 6 triangles and 7 points, by PP, two points will have to lie in the same triangle, and hence be at distance at most 1.

2. For each computer, the number of computers it is directly linked to (call them neighbours) is in the set $\{1, 2, 3, 4, 5\}$. There are 6 computers and 5 possible numbers of neighbours; hence by PP, at least two computers must have the same number of neighbours.

3. There are $7 \cdot 12 = 84$ possible pairs (day of the week, month). Hence by PP, we need at least 85 people to guarantee that at least two among them were born on the same day of the week and in the same month.

4. Assuming that every package contains 20 distinct cards, we need at least 28 packages, since $27 \cdot 20 < 551 \leq 28 \cdot 20$.

5. There are 14 possible remainders $(0, 1, 2, \ldots, 13)$ when dividing by 14. Hence any set of 15 integers will contain two that give the same remainder when divided by 14. The difference of these numbers will be then divisible by 14.

6. There are 9 possible remainders $(0, 1, 2, \ldots, 8)$ when dividing by 9. The "worst-case scenario" is to have $9 \cdot 5 = 45$ integers such that no 6 have the same remainder. However, by PP, any set of 46 integers will contain 6 that have the same remainder.

7. The number of such birth certificate codes is $n = 10^4 \cdot 26^3$. By PP, to guarantee that at least 26 certificates carry the same code, we need at least $25n+1$ certificates. Hence the number of people (certificates) is at least $25 \cdot 10^4 \cdot 26^3 + 1 = 4,394,000,001$.

8. The number of subsets of A of cardinality at most 3 is $\binom{9}{0} + \binom{9}{1} + \binom{9}{2} + \binom{9}{3} = 1 + 9 + 36 + 84 = 130$. The sum of the elements that such a set can have is in the set $\{0, 1, 2, \ldots, 24\}$. Since $130 > 24 \cdot 5$, any there exist 6 subsets of cardinality at most 3 that have the same sum of elements.

9. Draw a grid that divides the rectangle into squares of side length 10cm. There will be 200 such squares. Any three points within (or on the perimeter) of one of these squares define a triangle of area at most half the area of the square, that is, at most 50cm^2. Since there are 500 points and 200 squares, by PP, some square indeed contains at least 3 points (otherwise there would be at most 400 points).

10. Consider the integers 1, 11, 111, ..., up to the integer with 7778 repeated ones, and their remainders when divided by 7777. Since there are only 7777 possible remainders when divided by 7777, two of these 7778 integers (say x with a ones and y with b ones, for $x < y$) have the same remainder. Hence their difference $y - x$ is divisible by 7777. But $y - x$ has $b - a$ ones followed by a zeros. Let z be the number with $b - a$ repeated ones. Then $y - x = z \cdot 10^a$. So 7777 divides $z \cdot 10^a$, but since 7777and 10^a have no common divisors, 7777 must in fact divide z. Hence z is the integer with repeated ones that is divisible by 7777.

11. The possible numbers of mistakes are $0, 1, 2, \ldots, 12$ (boxes). Placing the 30 objects (students) into 12 boxes, by PP, we'll end up with at least one box with at least 3 objects.

12. For each pair $i, j \in \{1, 2, \ldots, 9\}$, $i < j$, let $M_{ij} = (\frac{x_i+x_j}{2}, \frac{y_i+y_j}{2}, \frac{z_i+z_j}{2})$ denote the midpoint of the line segment joining points $P_i = (x_i, y_i, z_i)$ and $P_j = (x_j, y_j, z_j)$. M_{ij} will have integer coordinates if and only if x_i and x_j, y_i and $_j$, and z_i and z_j have equal parity. There are 8 possible triples (q_1, q_2, q_3) where each q_i is either "odd" or "even"; these are our boxes. Placing the 9 points into these 8 boxes we'll end up with a box with at least two points, say P_i and P_j. Then their midpoint M_{ij} will have integer coordinates as explained above.

10 Permutations and Combinations

1. $2 \cdot (n!)^2$

2. $\binom{40}{17}$

3. The coefficient of x^k is 0 if k is odd, and $\binom{100}{50-\frac{k}{2}}$ if k is even.

4. Essentially, we have 2 symbols, five 011s and nine 1s. These can be arranged in $\binom{14}{5}$ ways.

5. (a) $-\binom{200}{101}2^{99}3^{101}$
   (b) $\binom{24}{2}2^2$
   (c) $\binom{8}{4}2^4$ and 0, respectively.

6. (a) $\binom{9}{4}$
   (b) $\binom{9}{0}+\binom{9}{1}+\binom{9}{2}+\binom{9}{3}+\binom{9}{4}$
   (c) $2^9-\left(\binom{9}{0}+\binom{9}{1}+\binom{9}{2}+\binom{9}{3}\right)$

7. (a) 2^{44}
   (b) $\binom{44}{14}$
   (c) $\binom{6}{3}\binom{44}{9}$

8. (a) $\binom{12}{3}\binom{13}{2}$ (since S contains 12 even and 13 odd elements)

   (b) $\sum_{i=1}^{9}\binom{10}{i}\binom{9}{i}$

9. (a) $\binom{13}{4}$
   (b) $\binom{12}{4}$
   (c) $\sum_{i=0}^{3}\binom{5}{i}\binom{8}{5-i}$
   (d) $\binom{10}{4}$

10. (a) $\binom{38}{6}-\sum_{i=0}^{2}\binom{10}{i}\binom{28}{6-i}$
    (b) $\binom{8}{3}\binom{30}{3}$
    (c) $\binom{10}{4}\binom{20}{2}$
    (d) $\binom{28}{2}\binom{5}{2}\binom{15}{2}-\binom{28}{2}\binom{4}{1}\binom{14}{1}$

11. $P(26,3)P(10,3)=26\cdot 25\cdot 24\cdot 10\cdot 9\cdot 8$

12. omitted

13. $\binom{52}{5}\binom{47}{5}\binom{42}{5}\binom{37}{5}\binom{32}{5}\binom{27}{5}$

11 Mathematical Induction

1. For all $n\in\mathbb{Z}^+$, define the proposition $P(n)$ as

$$1^2+2^2+\ldots+n^2=\frac{n(n+1)(2n+1)}{6}. \qquad (*)$$

   BI: to prove $P(1)$. The LHS of (*) is 1 and the RHS is $\frac{1(1+1)(2\cdot 1+1)}{6}=1$ Hence $P(1)$ holds.

   IS: We must show $P(k)\to P(k+1)$ for all $k\in\mathbb{Z}^+$. Take any $k\in\mathbb{Z}^+$ and assume $P(k)$ holds; that is, $1^2+2^2+\ldots+k^2=\frac{k(k+1)(2k+1)}{6}$. Now consider $P(k+1)$. The LHS is

(using IH)

$$\begin{aligned} 1^2 + 2^2 + \ldots + k^2 + (k+1)^2 &= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \\ &= \frac{1}{6}(k+1)(2k^2 + k + 6k + 6) \\ &= \frac{1}{6}(k+1)(k+2)(2(k+1)+1), \end{aligned}$$

which is the RHS of $P(k+1)$. Thus $P(k) \to P(k+1)$ holds.

By Mathematical induction, since $P(1)$ holds and $P(k) \to P(k+1)$ holds for all $k \in \mathbb{Z}^+$, we conclude that $P(n)$ holds for all $n \in \mathbb{Z}^+$.

2. omitted

3. Let $n = 2m - 1$. We thus have to show that

$$P(n) : (2m-1)^2 - 1 \text{ is divisible by } 8$$

for all $m \in \mathbb{Z}^+$.

BI: to prove $P(1)$. Now $(2 \cdot 1 - 1)^2 - 1 = 0$, which is divisible by 8. Hence $P(1)$ holds.

IS: We must show $P(k) \to P(k+1)$ for all $k \in \mathbb{Z}^+$. Take any $k \in \mathbb{Z}^+$ and assume $P(k)$ holds; that is, $(2k-1)^2 - 1$ is divisible by 8. Consider $P(k+1)$.

$$\begin{aligned} (2(k+1)-1)^2 - 1 &= ((2k-1)+2)^2 - 1 = (2k-1)^2 + 4(2k-1) + 4 - 1 \\ &= ((2k-1)^2 - 1) + 8k \end{aligned}$$

Since $(2k-1)^2-1$ is divisible by 8 by IH, and $8k$ is clearly divisible by 8, $(2(k+1)-1)^2-1$ is divisible by 8. Thus $P(k) \to P(k+1)$ holds.

By Mathematical induction, since $P(1)$ holds and $P(k) \to P(k+1)$ holds for all $k \in \mathbb{Z}^+$, we conclude that $P(m)$ holds for all $m \in \mathbb{Z}^+$.

4. We thus have to show that

$$P(n) : 4^{n+1} + 5^{2n-1} \text{ is divisible by } 21$$

for all $m \in \mathbb{Z}^+$.

BI: to prove $P(1)$. Now $4^{1+1} + 5^{2 \cdot 1 - 1} = 4^2 + 5^1 = 21$, which is divisible by 21. Hence $P(1)$ holds.

IS: We must show $P(k) \to P(k+1)$ for all $k \in \mathbb{Z}^+$. Take any $k \in \mathbb{Z}^+$ and assume $P(k)$ holds; that is, $4^{k+1} + 5^{2k-1}$ is divisible by 21. Consider $P(k+1)$.

$$\begin{aligned} 4^{(k+1)+1} + 5^{2(k+1)-1} &= 4^{k+2} + 5^{2k+1} = 4 \cdot 4^{k+1} + 25 \cdot 5^{2k-1} \\ &= 4(4^{k+1} + 5^{2k-1}) + 21 \cdot 5^{2k-1} \end{aligned}$$

Since $4^{k+1}+5^{2k-1}$ is divisible by 21 by IH, and $21 \cdot 5^{2k-1}$ is clearly divisible by 21, $4^{(k+1)+1}+5^{2(k+1)-1}$ is divisible by 21. Thus $P(k) \to P(k+1)$ holds.

By Mathematical induction, since $P(1)$ holds and $P(k) \to P(k+1)$ holds for all $k \in \mathbb{Z}^+$, we conclude that $P(n)$ holds for all $n \in \mathbb{Z}^+$.

5. It can be checked that $2^n < n^3$ for $n \in \{1, 2, \ldots, 9\}$. We shall prove

$$P(n) : 2^n > n^3$$

for all $n \in \mathbb{Z}$, $n \geq 10$.

BI: to prove $P(10)$. Now $2^{10} = 1024 > 1000 = 10^3$. Hence $P(10)$ holds.

IS: We must show $P(k) \to P(k+1)$ for all $k \in \mathbb{Z}$, $k \geq 10$. Take any $k \in \mathbb{Z}$, $k \geq 10$ and assume $P(k)$ holds; that is, $2^k > k^3$. Consider $P(k+1)$.

$$\begin{aligned} 2^{k+1} &= 2 \cdot 2^k > 2 \cdot k^3 = k^3 + k^3 = k^3 + 3 \cdot \frac{k^3}{3} = k^3 + \left(\frac{k^3}{3} + \frac{k^3}{3} + \frac{k^3}{3}\right) \\ &> k^3 + \left(\frac{k^3}{3} + \frac{k^2}{3} + \frac{k}{3}\right) > k^3 + \left(9\frac{k^2}{3} + 9\frac{k}{3} + 1\right) = (k+1)^3 \end{aligned}$$

Thus $P(k) \to P(k+1)$ holds.

By Mathematical Induction, since $P(1)$ holds and $P(k) \to P(k+1)$ holds for all $k \in \mathbb{Z}$, $k \geq 10$, we conclude that $P(n)$ holds for all $n \in \mathbb{Z}$, $n \geq 10$.

6. We must prove

$$P(n) : f_n > \left(\frac{1+\sqrt{5}}{2}\right)^{n-2}$$

for all $n \in \mathbb{Z}$, $n \geq 3$. We shall use Strong Induction.

BI: to prove $P(3)$. Note $f_3 = f_2 + f_1 = f_1 + f_0 + f_1 = 3$ and $\left(\frac{1+\sqrt{5}}{2}\right)^{3-2} = \frac{1+\sqrt{5}}{2} < \frac{1+\sqrt{9}}{2} = 2$. Hence $P(3)$ holds.

IS: We must show $(P(3) \wedge P(4) \wedge \ldots \wedge P(k)) \to P(k+1)$ for all $k \in \mathbb{Z}$, $k \geq 3$. Take any $k \in \mathbb{Z}$, $k \geq 3$ and assume $P(3), P(4), \ldots, P(k)$ all hold; that is, $f_i > \left(\frac{1+\sqrt{5}}{2}\right)^{i-2}$ for all $i \in \{3, 4, \ldots, k\}$. Examine f_{k+1}. If $k = 3$, then $f_{k+1} = f_4 = f_3 + f_2 = 3 + 2 = 5$ and $\left(\frac{1+\sqrt{5}}{2}\right)^{4-2} = \left(\frac{1+\sqrt{5}}{2}\right)^2 = \frac{3+\sqrt{5}}{2} < \frac{3+\sqrt{9}}{2} = 3$. Hence $P(4)$ holds.

Otherwise, $k \geq 4$ and by IH we obtain

$$\begin{aligned} f_{k+1} &= f_k + f_{k-1} > \left(\frac{1+\sqrt{5}}{2}\right)^{k-2} + \left(\frac{1+\sqrt{5}}{2}\right)^{k-3} \\ &= \left(\frac{1+\sqrt{5}}{2}\right)^{k-3}\left(\frac{1+\sqrt{5}}{2} + 1\right) = \left(\frac{1+\sqrt{5}}{2}\right)^{k-3}\frac{3+\sqrt{5}}{2} = \left(\frac{1+\sqrt{5}}{2}\right)^{k-1} \end{aligned}$$

Thus $(P(3) \wedge P(4) \wedge \ldots \wedge P(k)) \to P(k+1)$ holds.

By Strong Induction, since $P(3)$ holds and $(P(3) \wedge P(4) \wedge \ldots \wedge P(k)) \to P(k+1)$ holds for all $k \in \mathbb{Z}$, $k \geq 3$, we conclude that $P(n)$ holds for all $n \in \mathbb{Z}$, $n \geq 3$.

7. We shall prove

$$P(n) : n^2 \geq 2n + 3$$

for all $n \in \mathbb{Z}$, $n \geq 3$.

BI: to prove $P(3)$. Now $3^2 = 9 \geq 2 \cdot 3 + 3$. Hence $P(3)$ holds.

IS: We must show $P(k) \to P(k+1)$ for all $k \in \mathbb{Z}$, $k \geq 3$. Take any $k \in \mathbb{Z}$, $k \geq 3$ and assume $P(k)$ holds; that is, $k^2 \geq 2k + 3$. Consider $P(k+1)$. Using the IH we obtain

$$\begin{aligned} (k+1)^2 &= k^2 + 2k + 1 \\ &\geq (2k+3) + (2k+1) = 4k + 4 = 2(k+1) + 3 + (2k-1) > 2(k+1) + 3 \end{aligned}$$

Thus $P(k) \to P(k+1)$ holds.

By Mathematical Induction, since $P(1)$ holds and $P(k) \to P(k+1)$ holds for all $k \in \mathbb{Z}$, $k \geq 3$, we conclude that $P(n)$ holds for all $n \in \mathbb{Z}$, $n \geq 3$.

8. We shall prove

$$P(n) : 7^n - 2^n \text{ is divisible by } 5$$

for all $n \in \mathbb{N}$.

BI: to prove $P(0)$. Now $7^0 - 2^0 = 0$, which is divisible by 5. Hence $P(0)$ holds.

IS: We must show $P(k) \to P(k+1)$ for all $k \in \mathbb{N}$. Take any $k \in \mathbb{N}$ and assume $P(k)$ holds; that is, $7^k - 2^k$ is divisible by 5. Consider $P(k+1)$. Using the IH we obtain

$$7^{k+1} - 2^{k+1} = 7 \cdot 7^k - 2 \cdot 2^k = 7(7^k - 2^k) + 5 \cdot 2^k$$

Since $7^k - 2^k$ is divisible by 5 by IH, and $5 \cdot 2^k$ is clearly divisible by 5, we conclude that $7^{k+1} - 2^{k+1}$ is divisible by 5. Thus $P(k) \to P(k+1)$ holds.

By Mathematical Induction, since $P(0)$ holds and $P(k) \to P(k+1)$ holds for all $k \in \mathbb{N}$, we conclude that $P(n)$ holds for all $n \in \mathbb{N}$.

9. omitted (this is very similar to Question 1).

10. We shall prove

$$P(n) : 1 \cdot 1! + 2 \cdot 2! + \ldots + n \cdot n! = (n+1)! - 1$$

for all $n \in \mathbb{Z}^+$.

BI: to prove $P(1)$. Now $1 \cdot 1! = 1 = (1+1)! - 1$. Hence $P(1)$ holds.

IS: We must show $P(k) \to P(k+1)$ for all $k \in \mathbb{Z}^+$. Take any $k \in \mathbb{Z}^+$ and assume $P(k)$ holds; that is, $1 \cdot 1! + 2 \cdot 2! + \ldots + k \cdot k! = (k+1)! - 1$. Consider $P(k+1)$. Using the IH we obtain

$$\begin{aligned} 1 \cdot 1! + 2 \cdot 2! + \ldots + k \cdot k! + (k+1) \cdot (k+1)! &= (k+1)! - 1 + (k+1) \cdot (k+1)! \\ &= (1 + k + 1) \cdot (k+1)! - 1 \\ &= (k+2)! - 1 \end{aligned}$$

Thus $P(k) \to P(k+1)$ holds.

By Mathematical Induction, since $P(1)$ holds and $P(k) \to P(k+1)$ holds for all $k \in \mathbb{Z}^+$, we conclude that $P(n)$ holds for all $n \in \mathbb{Z}^+$.

11. We must prove

$$P(n) : a_n = 2^n + (-1)^n$$

for all $n \in \mathbb{N}$. We shall use Strong Induction.

BI: to prove $P(0)$. Note $a_0 = 2 = 2^0 + (-1)^0$. Hence $P(0)$ holds.

IS: We must show $(P(0) \wedge P(1) \wedge \ldots \wedge P(k)) \to P(k+1)$ for all $k \in \mathbb{N}$. Take any $k \in \mathbb{N}$ and assume $P(0), P(1), \ldots, P(k)$ all hold; that is, $a_i = 2^i + (-1)^i$ for all $i \in \{0, 1, \ldots, k\}$. Examine a_{k+1}. If $k = 0$, then $a_{k+1} = a_1 = 1$ by definition, and $2^1 + (-1)^1 = 1$, so $P(1)$ holds.

Otherwise, $k \geq 1$ and by IH we obtain

$$\begin{aligned} a_{k+1} &= a_k + 2a_{k-1} = 2^k + (-1)^k + 2(2^{k-1} + (-1)^{k-1}) \\ &= 2^k + (-1)^k + 2^k - 2 \cdot (-1)^k = 2^{k+1} + (-1)^{k+1} \end{aligned}$$

Thus $(P(0) \wedge P(1) \wedge \ldots \wedge P(k)) \to P(k+1)$ holds.

By Strong Induction, since $P(0)$ holds and $(P(0) \wedge P(1) \wedge \ldots \wedge P(k)) \to P(k+1)$ holds for all $k \in \mathbb{N}$, we conclude that $P(n)$ holds for all $n \in \mathbb{N}$.

12. (1) $a_1 = a_3 = a_5 = a + 7 = 2$, $a_2 = 4$, $a_4 = 16$, $a_6 = 4$, $a_8 = 256$

(2) We must prove

$$P(n) : a_n \leq 2^n$$

for all $n \in \mathbb{Z}^+$. We shall use Strong Induction.

BI: to prove $P(1)$. Clearly $a_1 = 2 \leq 2^1$. Hence $P(1)$ holds.

IS: We must show $(P(1) \wedge P(2) \wedge \ldots \wedge P(k)) \to P(k+1)$ for all $k \in \mathbb{Z}^+$. Take any $k \in \mathbb{Z}^+$ and assume $P(1), P(2), \ldots, P(k)$ all hold; that is, $a_i \leq 2^i$ for all $i \in \{1, 2, \ldots, k\}$. Examine a_{k+1}. If k is even, then $a_{k+1} = 2 \leq 2^{k+1}$ and so $P(k+1)$ holds.

Otherwise, k is odd and $k \geq 1$, and by IH we obtain

$$a_{k+1} = a^2_{\frac{k+1}{2}} \leq \left(2^{\frac{k+1}{2}}\right)^2 = 2^{k+1}$$

Thus $(P(1) \wedge P(2) \wedge \ldots \wedge P(k)) \rightarrow P(k+1)$ holds.

By Strong Induction, since $P(1)$ holds and $(P(1) \wedge P(2) \wedge \ldots \wedge P(k)) \rightarrow P(k+1)$ holds for all $k \in \mathbb{Z}^+$, we conclude that $P(n)$ holds for all $n \in \mathbb{Z}^+$.

13. omitted

14. omitted

12 Graphs

1. (a) Does not exist, as every graph has an even number of vertices of odd degree.
   (b) Does not exist. Such a graph would have a vertex of degree 5, but there are only 4 other vertices it can be adjacent to.
   (c) Exists: this is a star on 5 vertices.
   (d) Does not exist: if u and v are 2 vertices of degree 3, then each must be adjacent to the remaining two vertices. Hence there can not be a vertex of degree 1.
   (e) Exists.
   (f) Does not exist: if u and v are 2 vertices of degree 6, then each must be adjacent to the remaining five vertices. Hence there can not be a vertex of degree 1.
   (g) Exists: C_3.
   (h) Does not exist: the maximum degree a simple graph with 6 vertices can have is 5.

2. (i) $\binom{n}{2} = \frac{n(n-1)}{2}$. The complete graph K_n achieves this upper bound.
   (ii) $\frac{1}{2}(2+2+3+3+4) = 7$.
   (iii) $|E| = \frac{1}{2}\sum_{v \in V} \deg(v) \geq \frac{1}{2}\sum_{v \in V} 3 = \frac{3}{2}n$.

3. Let G be a simple graph with $n \geq 2$ vertices. The possible degrees for a vertex in G are $0, 1, 2, \ldots, n-1$. However, if there is a vertex of degree $n-1$, there can not be a vertex of degree 0. Hence there are at most $n-1$ possible degrees. By the Pigeonhole Principle, at least two of the n vertices must hence have the same degree.

4. (1) $|V(G)| = \binom{9}{4} = 126$.
   (2) Let A be a vertex in G. To construct a subset B of S such that $|A \cap B| = 1$, we choose 1 vertex from A for the intersection, and then 3 out of $S - A$ for $B - A$. There are $4 \cdot \binom{5}{3} = 40$ such subsets. Hence A as a vertex in G has degree 40, and G is regular of degree 40. By the Handshaking Theorem, $|E| = \frac{1}{2}\sum_{v \in V} \deg(v) = \frac{1}{2} 126 \cdot 40 = 2520$.

5. For all $n \in \mathbb{Z}^+$, define a proposition $P(n)$: "K_n has $\frac{n(n-1)}{2}$ edges."
   BI: $n = 1$. Clearly K_1 has 0 edges, and $\frac{1(1-1)}{2} = 0$. Hence BI holds.
   IS: We must show $P(k) \rightarrow P(k+1)$ for all $k \in \mathbb{Z}^+$. Take any $k \in \mathbb{Z}^+$ and assume $P(k)$ holds; that is, K_k has $\frac{k(k-1)}{2}$ edges. Now consider K_{k+1}. Take any vertex in K_{k+1}. Removing this vertex and all edges incident with it (observe there are k of them) we obtain a graph H isomorphic to K_k. By IH, H has $\frac{k(k-1)}{2}$ edges. Hence G has $\frac{k(k-1)}{2} + k = \frac{(k+1)k}{2}$ edges, and $P(k+1)$ holds. Thus $P(k) \rightarrow P(k+1)$ holds.

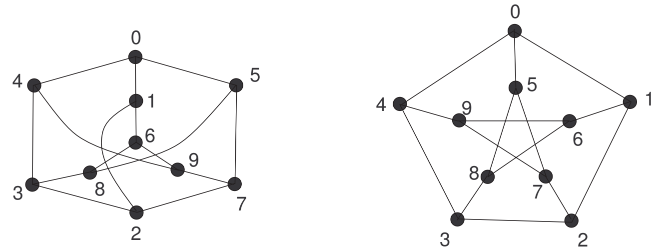
By Mathematical induction, since $P(1)$ holds and $P(k) \rightarrow P(k+1)$ holds for all $k \in \mathbb{Z}^+$, we conclude that $P(n)$ holds for all $n \in \mathbb{Z}^+$.

6. (1) Let $G = (V, E)$ be a graph with n vertices that is regular of odd degree k. We have $|E| = \frac{1}{2}\sum_{v \in V} \deg(v) = \frac{1}{2}\sum_{v \in V} k = \frac{1}{2}nk$. Since k is odd, and $|E|$ is an integer, n must be even.
(2) Since n must be even from (1), $n = 2m$ for an integer m. Then $|E| = mk$ as seen above, and E is a multiple of k.

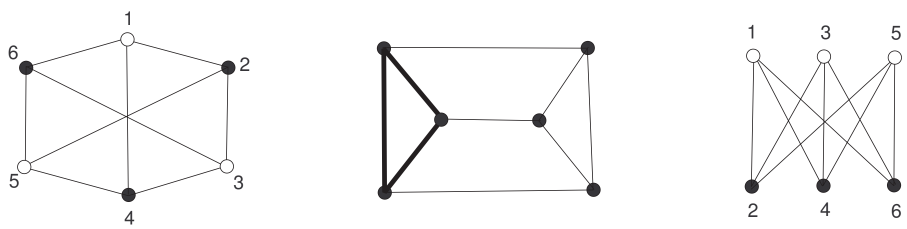
7. omitted

8. omitted

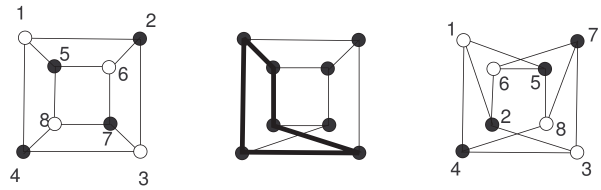
9. Isomorphic with the indicated isomorphism.



10. The first and third graph are isomorphic (with the indicated isomorphism), and they are bipartite graphs. The second graph is not bipartite (it contains C_3 as a subgraph) and hence not isomorphic to the other two.

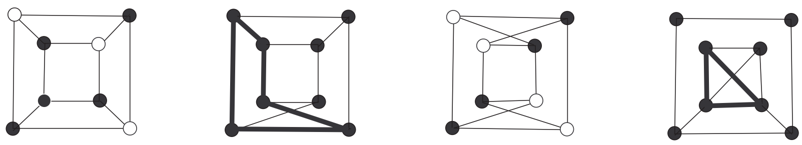


11. The first and third graph are isomorphic, and they are bipartite graphs. The second graph is not bipartite (it contains C_5 as a subgraph) and hence not isomorphic to the other two.



12. Bipartite. Not bipartite (it contains C_5 as a subgraph). Bipartite. Not bipartite (it contains C_3 as a subgraph).

13. (a) No as it has vertices of odd degree.



(b) Yes since it has exactly two vertices of odd degree, namely, u and v. An open Euler trail has to start at u and end at v or vice-versa. Example of an open Euler tour: $uacdabcvdubv$.

(c) In $G + uv$, all vertices have even degree. Hence, $G + uv$ has an Euler tour, but no open Euler trail. Example of an Euler tour: $uacdabcvdubvu$.

14. Consider the graph G whose vertices are the five quarters and he edges are the fourteen bridges.

(a) Here we are looking for an Euler tour in G. Since G has vertices of odd degree (A and B), it has no Euler tour.

(b) Yes, an open Euler tour exists since G has exactly two vertices of odd degree, A and B. An open Euler tour must have these two vertices and endpoints. One possible open Euler tour: $b_2b_1b_5b_6b_7b_8b_{12}b_4b_3b_{14}b_{13}b_{10}b_9b_{11}$.

15. omitted

16. omitted

17. (a) Let n and e be the number of vertices and edges, respectively, in a full 5-ary tree T with 101 leaves. Then $n = 5i + 1 = i + 101$, where i is the number of internal vertices. Solving for i we obtain $i = 25$. Now $e = n - 1 = (i + 101) - 1 = 125$.

(b) Since in a full m-ary tree of height h the number of leaves is at most m^h, we have $51 \leq m^3$. That gives $m \geq 4$. On the other hand, $n = \ell + i = mi + 1$ (where n is the number of vertices, ℓ the number of leaves, an di the number of internal vertices). From here we obtain $\ell - 1 = (m - 1)i$. In our case, $50 = (m - 1)i$ or $2 \cdot 5^2 = (m - 1)i$. Since $m - 1$ is an integer, $m - 1 \in \{1, 2, 5, 10, 25, 50\}$. Since $m \geq 4$ from above, we have $m \in \{6, 11, 26, 51\}$. It can be verified that $m = 51$ and $m = 26$ can not give a tree of height 3, while trees with $m = 6$ and $m = 11$ can be easily constructed.