MAT2377A - Probability and Statistic for Engineers

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Created January 7th, 2014, Updated February 11th, 2014

Chapter 1 & 2: Introduction to Probability

Sets

Set A set is a collection of items.

$A = \{2, 3, 4\}$
$A = B$ implies that any $x \in A \rightarrow x \in B$

Operations on Sets

Let A and B be two sets.
$A \cap B = A$ intersects $B = A$ and $B = \{x : x \in A$ and $x \in B\}$
$A \cup B = A$ union $B = A$ or $B = \{x : x \in A$ or $x \in B\}$

(1) Example:

\$ = {2,3,4}\$ and $B = \{3, 4, 5\} \rightarrow A \cap B = \{3, 4\}, A \cup B = \{2, 3, 4, 5\}$

Sample Space

Sample Space All the possible outcomes in a random experiment.

$S =$ the set of outcomes in a random experiment.

(2) Example: Flip a coin

$S = \{H, T\}$

(3) Example: Roll a die

$S = \{1, 2, 3, 4, 5, 6\}$

(4) Example: Flip a coin until a heads appears.

$S = \{H, TH, TTH, \cdots\}$

(5) Example: For a circle of radius 1, any point within the circle is what set?

$S = \{(x \times y) : x^2 + y^2 \leq 1\}$

Countable Set A set in which you can list the entries, and an uncountable set is one which will not let you list the entries.

Event

Event Any subset of S (sample space) is called an event.

(6) Example:

Coin is heads: $E = \{H\} \subset \{H, T\}$

So E is an event.

(7) Example:

Die is on even number: $E = \{2, 4, 6\}$

(8) Another example:

$E = \{(x, y) : x^2 + y^2 \leq \frac{1}{4}\}$

Enumeration (Counting)

1. Multiplication Rule.

   If task 1 can be completed in n_1 different ways and task 2 can be completed in n_2 different ways, then both tasks can be completed by $n_1 \times n_2$ ways.

   (9) Example: How many 5 digit numbers are there?

   Task 1: 1,2,3,4,5,6,7,8,9. $n_1 = 9$.

   Task 2: 0-9. $n_2 = 10$.

   ...

   Task 5: 0-9. $n_5 = 10$.

   $9 \times 10^4 = 90,000$

   (10) Example: How many 5 digit numbers with no repeats do we have?

   $n_1 = 9, n_2 = 9, n_3 = 8, n_4 = 7, n_5 = 6$

   $= 9 * 9 * 8 * 7$

   (11) Example: In how many different ways can we write 1,2,3,4,5 in unique order?

   $5 * 4 * 3 * 2 * 1 = 5!$

   (12) Example: How many different orders of these letters is possible, where repeat letters are to be treated as non-unique: INDEPENDENT?

   $11!/(3! * 2! * 3!)$

2. Permutation

   (13) Example: With letters a, b, c, d, write all possible permutations with 2 letters.

   ab, ac, ad,
   ba, bc, bd,
   ca, cb, cd,
   da, db, dc.

   $P_2^4 = 4 \times 3 = 12$

   $P_r^n = ?$

   n letters, $\{a_r \cdots a_n\}$, $n \leq r$

   r letters.

   n = number of ways for completing task 1.

n-1 = number of ways for completing task 2.

$\cdots$ until task r:

$n-(r-1)$ = number of ways for completing task r.

Rule of Permutations $P_r^n = n(n-1) \times \cdots \times (n-r+1)$

Permutations of r letters from n letters. ($r \leq n$)

$P_r^n = n(n-1) \ldots (n-r+1)$

(14) Example

$P_2^4 = 4 \times 3 = 12 \to \{a, b, c, d\}$

In general for Square Permutations

$P_n^n = n(n-1) \ldots (3)(2)(1) = n!$

In general for Any Permutations

$P_r^n = \frac{n(n-1) \ldots (n-r+1)(n-r)(n-r-1) \ldots 1}{(n-r)(n-r-1) \ldots 1} = \frac{n!}{(n-r)!}$

Combinations

$C_r^n = \binom{n}{r}$

Combination # of combination of r letters from n leters.

(15) Example: Let $n = 4, r = 2$

$a, b, c, d \to ab, ac, ad, bc, bd, cd$

$\binom{4}{2} = 6$

$P_2^4 = 2\binom{4}{2}$

(16) Example

$\binom{4}{3} =?$

$S = \{a, b, c, d\}$

$abc, abd, acd, bcd \to \binom{4}{3} = 4$

$P_3^4 = 3!\binom{4}{3}$

General Combination Formula

$\therefore P_r^n = r!\binom{n}{r} \rightarrow \binom{n}{r} = \frac{P_r^n}{r!} = \frac{\frac{n!}{(n-r)!}}{r!} = \frac{n!}{r!(n-r)!}$

$\binom{n}{r} = \frac{n!}{r!(n-r)!}$

$P_r^n = \frac{n!}{(n-r)!}$

(17) Example: How many hands of 5 cards do we have?
$\binom{52}{5} = \frac{52!}{5!47!} = \frac{52\times51\times50\times49\times48}{120}$

(18) Example: How many diagonals do we have in a polygone with n sides?
$\binom{5}{2} - 5 = 5$
In general, the solution is:
$\binom{n}{2} - n = \frac{n(n-1)}{2} - n$
$= \frac{n(n-1)-2n}{n} = \frac{n(n-1-2)}{2} = \frac{n(n-3)}{2}$

(19) Example: A city has m by n streets in a grid. How many ways can someone traverse from one corner of the city to the other?
$\frac{(m+n)!}{m!n!} = \binom{m+n}{m} = \binom{m+n}{n}$

(20) Example: How many ways can we write out $(a+b)^5$ expanded?
$(a+b)^5 = a^5 + \frac{5!}{4!1!}a^4b + \frac{5!}{3!2!}a^3b^2 + \frac{5!}{2!3!}a^2b^3 + \frac{5!}{1!4!}ab^4 + b^5$

A General Polynomial Combinational Formula

$$(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^i b^{n-i}$$

$(a+b+c)^n = \sum\sum \binom{n}{i,j} a^i b^j c^{n-i-j}$

Properties of Combinations

(i) $\binom{n}{r} = \binom{n}{n-r}$
$\frac{n!}{r!(n-r)!} = \frac{n!}{r!(n-r)!(n-(n-r))!}$

(ii) $\binom{n}{r-1} + \binom{n}{r} = \binom{n+1}{r}$
$\frac{n!}{(r-1)!(n-r+1)1} + \frac{n!}{r!(n-r)!}$
$= \frac{rn!+n!(n-r+1)}{r!(n-r+1)!}$
$= \frac{(n+1)!}{r!(n-r+1)!}$
$= \binom{n+1}{r}$

(21) Example: How many subsets do we have if the set has n elements?

$S = \{a, b, c\}$

$\{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}$

You are effectively doing, in this example, n choose 0, n choose 1, n choose etc until n.

$\binom{n}{0} + \binom{n}{1} + \ldots + \binom{n}{n} = 2^n$

To explain this, we can use the multiplication rule and say, "For task i, choose the i^{th} letter or do not choose it."

Defining Probability

(22) Example: Flip 2 coins.

$S = \{HH, HT, TH, TT\}$

Define E = having at least 1 head.

$E = \{TH, HT, HH\}$

$P(E) = \frac{3}{4}$

Define F = exactly one head = $\{TH, HT\}$

$P(F) = \frac{2}{4}$

(23) Example: Flip a coin until you see the first head.

$S = \{H, TH, TTH, \ldots\}$

Event E; Stop before or at 4^{th} trial = $\{H, TH, TTH, TTTH\}$

$P(E) = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16}$

(24) Example: A point is picked at random from a circle with radius r

$S = \{(x, y) : x^2 + y^2 \leq R^2\}$

Let $E = \{(x, y) : x^2 + y^2 \leq \frac{R^2}{4}\}$

$\frac{R^2}{4} = (\frac{R}{2})^2$

$P(E) = \frac{Area(E)}{Area(S)} = \frac{\pi \times (\frac{R}{2})^2}{\pi \times R^2} = \frac{1}{4}$

Probability

Let S be the sample space of a random experiment. Let P be a set function with the following probabilities:

Assumptions:

i) $P(E) \geq 0$

ii) If E_1, E_2, ... are disjoint ($E_i \cap E_j = 0; i \neq j$) then $P(E_1 \cup E_2 \ldots) = \sum_i P(E_i)$

iii) $P(S) = 1$

Properties of P

1. $P(\emptyset) = 0$
2. $P(A - B) = P(A) - P(A \cap B)$
   Note: $A - B = \{x | x \in A, x \notin B\}$
   $P((A - B) \cup (A \cap B)) = P(A)$
   $P(A - B) + P(A \cap B) = P(A)$
   $P(A - B) = P(A) - P(A \cap B)$
3. If $A \subset B \rightarrow P(A) \leq P(B)$
   $B = A \cup (B - A)$
   $P(B) = P(A) + P(B - A) \geq 0$
   $\therefore P(A) \leq P(B)$
4. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
   Proof:
   $A \cup B = A \cup (B - A)$
   $P(A \cup B) = P(A) + P(B - A) = P(A) + P(B) - P(A \cap B)$
5. $0 \leq P(E) \leq 1$
   If $\emptyset \subset E \subset S$
   $0 = P(\emptyset) \leq P(E) \leq P(S) = 1$
   $P(E') = 1 - P(E)$

(25) Example: Union of 3 sets
Let $B \cup C = D$
$P(A \cup B \cup C) = P(A \cup D) = P(A) + P(D) - P(A \cap D)$
$A \cap D = A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
$P(A \cap D) = P(A \cap B) + P(A \cap C) - P(A \cap B \cap C)$
$\therefore P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(B \cap C) - P(A \cap B) - P(A \cap C) + P(A \cap B \cap C)$

(26) Example: Union of 4 sets

$P(A \cup B \cup C \cup D)$
$= P(A) + P(B) + P(C) + P(D) - P(A \cap B) - P(A \cap C) - P(B \cap C) - P(C \cap D) - P(A \cap D) - P(B \cap D) + P(A \cap B \cap C) + P(A \cap B \cap D) + P(A \cap C \cap D) + P(B \cap C \cap D) - P(A \cap B \cap C \cap D)$

(27) Example: 100 different letters are sent to 100 different addresses at random. What is the probability that *at least* 1 letter reaches its designate address?

Hint: In questions, you can look for key phrases: *at least* suggests union, and *all* suggests intersection.

$E_i = i^{th}$ letter goes to the right address.

$P(E_1 \cup E_2 \cup E_3 \cup \ldots \cup E_1 00) = P(E_1) + P(E_2) + \ldots + P(E_1 00) - P(E_1 \cap E_2) - P(E_1 \cap E_3) - \ldots - P(E_9 9 \cap E_1 00) + P(E_1 \cap E_2 \cap E_3) + \ldots\ldots - P(E_1 \cap \ldots \cap E_1 00)$
$P(E_i) = \frac{1}{100}$
$i \neq j,\ P(E_i \cap E_j) = \frac{1}{100} \times \frac{1}{99}$
$P(E_i \cap E_j \cap E_k) = \frac{1}{100} \times \frac{1}{99} \times \frac{1}{98}$
$\rightarrow 100 * (\frac{1}{100}) - \binom{100}{2} * \frac{1}{100*99} + \binom{100}{3} * \frac{1}{100*99*98} - \ldots$
$= 1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \ldots - \frac{1}{100!}$
$= 1 - e^{-1}$ is the probability that 1 letter will reach the correct address.

Conditional Probability

Let S be the sample space for a random experiment.

Let $B \subset S$ (B is an event)

Such that $P(B) > 0$

For any event A we define $P(A|B) = \frac{P(A \cap B)}{P(B)}$

(28) Example: A die is rolled. Let A = result is more than 4, let B = outcome is even.

Find $P(A|B)$

Solution:

- $A = \{5, 6\}$
- $B = \{2, 4, 6\}$

Given that you have a result in B, what are the chances that it is a result in A? Since B has 2, 4, and 6, one third of the time, A will be met as well.

$A \cap B = \{6\}$

$P(A \cap B) = \frac{1}{6}$

$P(B) = \frac{3}{6}$

$P(A|B) = \frac{\frac{1}{6}}{\frac{3}{6}}$

$= \frac{1}{3}$

Total Probability Rule

Total Probability Rule Let S be a sample space in a random experiment. Let A be any event $(A \subset S)$

Also we assume $S = E_1 \cup E_2 \cup \ldots \cup E_k$, such that E_i's are disjoint.

Then $P(A) = \sum_{i=1}^{k} P(A|E_i)P(E_i)$

Proof $A \subset S$

$A \cap S = A$

$A = A \cap (E_1 \cup E_2 \cup \ldots \cup E_k)$

$= (A \cap E_1) \cup (A \cap E_2) \cup \ldots \cup (A \cap E_k)$

$\therefore P(A) = P(A \cap E_1) + P(A \cap E_2) + \ldots + P(A \cap E_k)$

$= \frac{P(A \cap E_1)}{P(E_1)} P(E_1) + \ldots + \frac{P(A \cap E_k)}{P(E_k)} P(E_k)$

$= P(A|E_1)P(E_1) + \ldots + P(A|E_k)P(E_k)$

(29) Example (Polya): There are m white chips and n black chips in the box. You remove one chip from the box without looking at it, and it is put inside. A second chip is then drawn. What is the probability that the second chip is white?

Let $P(A)$ represent the probability that the 2^{nd} chip is white.

Solution:

- E_1 = 1^{st} chip is white
- E_2 = 1^{st} chip is black.

$P(A) = P(A|E_1)P(E_1) + P(A|E_2)P(E_2)$

$= \frac{m-1}{m+n-1} \times \frac{m}{m+n} + \frac{m}{m+n-1} \times \frac{n}{m+n}$

$= \dfrac{m}{m+n}$

Slightly more explanation: Probability doesn't measure the physical characteristics of the box, it measures based on information that you have. Not knowing the first chip, the second chip, or the k chip, will not change the chance of the k+1's chip.

(30) Example: You have two boxes, both with m white chips and n black chips, and you take one random chip from the first box and place it in the second box. Now you take a random chip from the second box. What is the probability that this chip is white?

$E_1 = 1^{\text{st}}$ chip is white

$E_2 = 2^{\text{nd}}$ chip is black

$P($ 2^{nd} chip is white$) = P(A|E_1)P(E_1) + P(A|E_2)P(E_2)$

$= \frac{m+1}{m+n+1} \times \frac{m}{m+n} + \frac{m}{m+n+1} \times \frac{n}{m+n}$

$= \dfrac{m}{m+n}$

The total probability rule is useful for tangled experiments, it can help untangle them and find a simple solution.

(31) Example: You roll a die. You now flip a coin the number of times given by the die. What is the probability that no heads were flipped?

$P(\text{no heads}) = ?$

- $E_1 =$ Die is 1
- $E_2 =$ Die is 2
- $E_3 =$ Die is 3
- $E_4 =$ Die is 4
- $E_5 =$ Die is 5
- $E_6 =$ Die is 6

A = no heads

$P(A) = P(A|E_1)P(E_1) + \ldots + P(A|E_6)P(E_6)$

$P(A|E_1) = \frac{1}{2}$

$P(A|E_2) = \frac{1}{4}$

$P(A|E_3) = \frac{1}{8}$

$P(A|E_4) = \frac{1}{16}$

$P(A|E_5) = \frac{1}{32}$

$P(A|E_6) = \frac{1}{64}$

$P(E_i) = \frac{1}{6}$ where $1 \leq i \leq 6$

$P(A) = \frac{1}{2} \times \frac{1}{6} + \ldots + \frac{1}{64} \times \frac{1}{6}$

Baye's Rule

Baye's Rule Let S be the sample space and $S = E_1 \cup \ldots \cup E_k; E_i$'s disjoint and $A \subset C$

$$P(E_1|A) = \frac{P(A|E_1)P(E_1)}{\sum_k P(A|E_k)P(E_k)}$$

Proof:

$$P(E_1|A) = \frac{(P(A \cap E_1)/P(E_1))P(E_1)}{P(A)}$$

$$P(E_1|A) = \frac{P(A|E_1)P(E_1)}{\sum_k P(A|E_k)P(E_k)}$$

(32) In a transmittion system, you send either 0 or 1. Let $P(\{0\}) = P(\{1\}) = 0.5$

Trans	Receive
0	0
1	1

$P[Rec0|Send0] = 0.99$

$P[Rec1|Send1] = 0.95$

$P[$1 was sent $|Rec1]$

$E_1 =$ send 1

$E_2 =$ send 0

$A =$ Receive 1

Applying the base rule:

$$P(E_1|A) = \frac{P(A|E_1)P(E_1)}{P(A|E_1)P(E_1) + P(A|E_2)P(E_2)}$$

$= \frac{.95 \times .5}{.95 \times .5 + .01 \times .5}$

$\therefore = \frac{95}{96}$

(33) Example: 0.01 of people in a population have a certain disease.

$P(+|sick) = 0.98$

$P(-|healthy) = 0.99$

$P(sick|+) =?$

$E_1 = sick$

$E_2 = healthy$

A =test is +

$P(E_1|A) = \frac{P(A|E_1)P(E_1)}{P(A|E_1)P(E_1)+P(A|E_2)P(E_2)}$

$\frac{.98\times.01}{.98\times.01+.01\times.99} - \frac{.98}{.98+.99}$

Practice Midterm Questions

(34) There are 5 components connected on a circuit. To get from component 1 to 5, you may take paths through components 2-3 or just component 4.

The components have a certain percentage of chance that they will work:

- 1: 0.5
- 2: 0.8
- 3: 0.7
- 4: 0.5
- 5: 0.5

What is the chance that the entire system functions?

Ai = i$^{\text{th}}$ component works, i=1,2,3,4,5

Independence Two events A and B are independent if

$P(A \cap B) = P(A)P(B)$

$P(A_1 \cap ((A_2 \cap A_3) \cup A_4) \cap A_5)$
$= P(A_1)P(A_5)P((A_2 \cap A_3) \cup A_4)$
$= .5 \times .5(P(A_2 \cap A_3) + P(A_4) - P(A_2 \cap A_3 \cap A_4))$
$= .5 \times .5((0.8)(0.7) + 0.5 - .8 \times .7 \times .5)$
$= 0.195$

To get the probability of none of them working, you'd take the complement of P(none of them work).

(35) Here is a table showing the income of 1000 families.

Income table	Husband (<=30,000)	Husband (> 30,000)	Wife total
Wife (<= $30,000)	425	400	825
Wife (> $30,000)	65	110	175
Husband Total:	490	510	1000 families

What is the probability that a husband makes >30,000 if his wife makes >30,000?

$P(husband > 30,000 | wife > 30,000)$

$= \frac{P(husband > 30k \cap wife > 30k)}{P(wife > 30k)}$

$= \frac{\frac{110}{1000}}{\frac{175}{1000}}$

$= \frac{110}{175}$

What is the probability that a husband makes more than 30k?

$P(husband > 30k) = \frac{510}{1000}$

(36) The probability of pipe welds being defective is 0.02. The probability of them being good is 0.98. A device for checking the weld defects has a probability of working on defective welds of 0.90. The device also sends a bad signal on a good weld 5% of the time.

- $P(Defective) = 0.02$
- $P(Good) = 0.98$
- $P(signal\ is\ sent | Defective) = 0.90$
- $P(signal\ is\ sent | Good) = 0.05$
- $P(Defective | Signal\ is\ received) =?$

This is using Baye's Rule.

- $E_1 = Defective$
- $E_2 = Good$
- $A = signalisreceived$
- $P(E_1|A)?$

Solution

$P(E_1|A) = \frac{P(A|E_1)P(E_1)}{P(A|E_1)P(E_1)+P(A|E_2)P(E_2)}$

$= \frac{.9\times.02}{0.9\times0.02+0.05\times0.98}$

$= 0.27$

(37) Let A and B two events.

- $P(A) = 0.3$
- $P(B) = 0.5$
- $P(A \cup B) = 0.65$

Is A independent from B?

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$0.65 = 0.3 + 0.5 - P(A \cap B)$

$P(A \cap B) = 0.15$

$P(A)P(B) = 0.15$

Since $P(A \cap B) = P(A)P(B)$**, A and B are independent**

Find $P(A' \cup B') = P(A') + P(B') - P(A' \cap B')$

$= 0.7 + 0.5 - (0.7)(0.5)$

$= 0.85$

DeMorgan's Law

$(A \cup B)' = A' \cap B'$

$(A \cap B)' = A' \cup B'$

$P(A' \cup B') = P[(A \cap B)'] = 1 - P(A \cap B) = 1 - 0.15 = 0.85$

Chapter 3 - Discrete Random Variables

Discrete Random Variables

Discrete Random Variables In a random experiment with a sample space S, we map members of S to real numbers.

$S \to R$

w = outcome in S

(38) Roll two dice. What is the probability of their added outcomes?

$S = \{(x, y) : x = 1, 2, \ldots, 6 y = 1, 2, \ldots, 6\}$

$\#(S) = 36$

$X(x, y) = x + y$

$X \in \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$

x	2	3	4	5	6	7	8	9	10	11	12
$\frac{P(X=x)}{36}$	1	2	3	4	5	6	5	4	3	2	1

(39) In a box containing 10 items there are two defective items.

#	Condition
8	Good
2	Defective

Draw two items at random from this box without replacement.

Let $x = \#$ of defective items in the sample of two items.

Therefore, $X \in \{0, 1, 2\}$

Find $P(x = 0), P(x = 1), P(x = 2)$

Solution:

$$P(x = 0) = \frac{\binom{8}{2}}{\binom{10}{2}} = \frac{28}{45}$$

$$P(x = 1) = \frac{\binom{8}{1} \times \binom{2}{1}}{\binom{10}{2}} = \frac{16}{45}$$

$$P(x = 2) = \frac{\binom{2}{2}}{\binom{10}{2}} = \frac{1}{45}$$

x	0	1	2
$f(x) = P(X = x)$	$\frac{28}{45}$	$\frac{16}{45}$	$\frac{1}{45}$

Note that all the probabilities add up to 1. $f(x)$ is called the probability mass function for the random variable X.

Expected Value (mean)

Expected Value (mean): $E(X) = \sum_{\imath} x\imath f(x\imath) = \mu$

$\mu = 0 * \frac{28}{45} + 1 * \frac{16}{45} + 2 * \frac{1}{45} = \frac{18}{45} = \frac{6}{15} = \frac{2}{5} = 0.4$

Variance and Standard Deviation

$E(X - \mu)^2 = \sigma^2$

$\sqrt{E(X - \mu)^2} = \sigma = S.D$

So using the numbers from example (39):

$\sigma^2 = E(X - \mu)^2 = \sum_{x} (x - \mu)^2 f(x)$

$= (0 - 0.4)^2(\frac{28}{45}) + (1 - 0.4)^2(\frac{16}{45}) + (2 - 0.4)^2(\frac{1}{45})$

(40) Example: Mean and Variance

x	$f(x)$
0	0.18
1	0.34
2	0.33
4	0.15

$E(X) = \mu = 0(0.18) + 1(0.34) + 2(0.33) + 4(0.15)$
$= 0.34 + 0.66 + 0.6 = 1.6$
$\sigma^2 = (0-1.45)^2(0.18) + (1-1.6)^2(0.34) + (2-1.6)^2(0.33) + (4-1.6)^2(0.15)$

Properties of Probability Mass Function

i) $f(x) \geq 0$
ii) $\sum_x f(x) = 1$

(41) Let X be a discrete random variable with the probability mass function $f(x)$ given below.

x	0	1	2
$f(x)$	C	2C	3C

Find $E(X)$, σ^2, $P(X \geq \mu + \sigma)$

Solution:

$C + 2C + 3C = 1$

$C = \frac{1}{6}$

x	0	1	2
$f(x)$	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{3}{6}$

$E(X) = 0(\frac{1}{6}) + 1(\frac{2}{6}) + 2(\frac{3}{6} = \frac{2}{6} + \frac{6}{6} = \frac{8}{6} = \frac{4}{3}$
$\sigma^2 = (0 - \frac{4}{3})^2 + (1 - \frac{4}{3})^2 + (2 - \frac{4}{3})^2(\frac{3}{6})$
$\frac{16}{9} \times \frac{1}{6} + (\frac{1}{9})(\frac{2}{6}) + (\frac{4}{9})(\frac{3}{6}) = \frac{16}{54} + \frac{2}{54} + \frac{12}{54} = \frac{30}{54} = \frac{5}{9}$
$\sigma = \frac{\sqrt{5}}{3}$

$P(x \geq \mu + \sigma) = P(x \geq \frac{4}{3} + \frac{\sqrt{5}}{3}) = 0$

Properties for mu (μ) and sigma squared (σ^2)

1) $\sigma^2 = E(x^2) - (E(x))^2$

   Proof:

   $E(X-\mu)^2 = E[x^2 + \mu^2 - 2\mu x]$

   $= \sum_x (x^2 + \mu^2 - 2\mu x) f(x)$

   $= \sum_x x^2 f(x) + \sum_x \mu^2 f(x) - \sum_x 2\mu x f(x)$

   $= E(x^2) + \mu^2 \sum_x f(x) - 2\mu \sum_x x f(x)$

   $= E(x^2) + \mu^2 - 2\mu * \mu$

   $= E(x^2) - \mu^2$

   (42) Flip a coin 3 times

   Let x=# of heads

   Find $E(x)$, σ^2, σ

   Solution:

   $\{HHH, HHT, HTH, THH, TTH, THT, HTT, TTT\}$

x	0	1	2	3
$P(x = x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

   $E(X) = \mu$

   $= 0\frac{1}{8} + 1\frac{3}{8} + 2\frac{3}{8} + 3\frac{1}{8}$

   $= 1.5$

   $E(x^2) = 0^2\frac{1}{8} + 1^2\frac{3}{8} + 2^2\frac{3}{8} + 3^2\frac{1}{8} = \frac{3}{8} + \frac{12}{8} + \frac{9}{8} = \frac{24}{8} = 3$

   $\sigma^2 = E(x^2) - \mu^2 = 3 - 1.5^2 = 3 - 2.25 = 0.75$

   $\sigma = \sqrt{0.75}$

Cumulative Distribution Function

Cumulative Distribution Function Let x be a random variable with probability mass function f(x),

$F(x) = P(X \leq x) = \sum_{t \leq x} f(t)$

Note, given F you can find f. There is a one-to-one relationship. $F \leftrightarrow f$

(43) In example (42), find the cumulative distribution function.

Solution:

$F(x) = P(X \leq x) =$

$P(X \leq x)$	x
0	$x < 0$
$\frac{1}{8}$	$0 \leq x < 1$
$\frac{4}{8}$	$1 \leq x < 2$
$\frac{7}{8}$	$2 \leq x < 3$
$\frac{8}{8}$	$3 \leq x$

(44) Find the relevant probabilities.

$$F_x(x) = \begin{cases} 0 \text{ if } x < 0 \\ 0.17 \text{ if } x \leq x < 1 \\ 0.4 \text{ if } 1 \leq < 2 \\ 0.59 \text{ if } 2 \leq x < 3 \\ 0.72 \text{ if } 3 \leq x < 4 \\ 0.8 \text{ if } 4 \leq x < 5 \\ 1 \text{ if } x \geq 5 \end{cases}$$

$F(3) = P(x \leq 3) = 0.72$

$$\begin{aligned} P(2 < x \leq 3) &= F(3) - F(2) \\ &= 0.72 - 0.59 \\ &= 0.13 \end{aligned}$$

$P(0 < x < 0.75) =$ total amount of jumps in$F = 0$

Solution: Write PMF:

$f(x) = P(X = x)$

x	0	1	2	3	4	5
$f(x)$	0.17	0.23	0.19	0.13	0.08	0.2

$E(x) = \sum_x xf(x) = 0\times0.17+1\times0.23+2\times0.19+3\times0.13+4\times0.08+5\times0.2$

$= 2.23$

Okay, what about σ^2?

$$
\begin{aligned}
E(x^2) &= \sum_x x^2 f(x) \\
&= 0^2 \times 0.17 + 1^2 \times 0.23 + 2^2 \times 0.19 + 3^2 \times 0.13 + 4^2 \times 0.08 + 5^2 \times 0.2 \\
&= 8.44
\end{aligned}
$$

Uniform Distance on Finite Sets

Uniform Distance on Finite Sets If x is a discrete random variable uniformly distributed on $\{1, 2, \ldots k\}$, a die with k sides is rolled.

$P(X = x) = \begin{cases} \frac{1}{k} & x = 1, 2, \ldots, k \\ 0 & \text{elsewhere} \end{cases}$

What is the average of x?

$E(x) = \frac{k+1}{2}$ which is equivalent to $E(x) = \sum_x x f(x)$

What is the variance of x?

$$
\begin{aligned}
\sigma^2 &= E(x^2) - (E(x))^2 \\
&= E(x^2) - (\tfrac{k+1}{2})^2
\end{aligned}
$$

$E(x^2) = \frac{1^2 + 2^2 + \cdots + k^2}{k}$

$$\sum_{i=1}^{n} [(i+1)^3 - i^3] = (2^3 - 1^3) + (3^3 - 2^3) + \cdots + ((n+1)^3 - n^3)$$

$$\sum_{i=1}^{n} [(i+1)^3 - i^3] = (n+1)^3 - 1$$

$$\sum_{i=1}^{n} [i^3 + 3i^2 + 3i + 1 - i^3] = 3\sum_{i=1}^{n} i^2 + 3\sum_{i=1}^{n} i$$

$$3\sum_{i=1}^{n} i^2 + 3\frac{n(n+1)}{2} + n$$

$$\sum_{i=1}^{n} = \frac{(n+1)^3 - 1 - \frac{3n(n+1)}{2} - n}{3} = \frac{n(n+1)(2n+1)}{6}$$

$E(x^2) = \frac{1^2 + \cdots + k^2}{k} = \frac{\frac{k(k+1)(2k+1)}{6}}{k} = \frac{(k+1)(2k+1)}{6}$

$\sigma^2 = \frac{(k+1)(2k+1)}{6} - \frac{(k+1)^2}{4} = \frac{k^2-1}{12} = \sigma^2$

Bernoulli's Trials

Bernoulli's Trials Let $S = \{0, 1\}$, where 0 is failure, and 1 is success.

$P(x = 1) = P, P(x = 0) = 1 - p = q$

Probability mass function for Bernoulli's random variable.

x	0	1
$f(x)$	1-p	p

$$f(x) = \begin{cases} p^x(1-p)1-x & x = 0,1 \\ 0 & \text{elsewhere} \end{cases}$$

We have $E(x) = 0(1-p) + 1 \times p = p$

$E(x^2) = 0^2 \times (1-p) + 1^2 \times p = p$

$\sigma^2 = p - p^2 = p(1-p)$

Binomial Distribution

Binomial Distribution Repeat Bernoulli's experiment independently n times.

Y = # of successes in n trials.

$$Y = \sum_{i=1}^{n} x_i$$

$E(Y) = E(x_1 + \cdots + x_n) = E(x_1) + \cdots + E(x_n) = np$

Probability mass function for Y:

P(y=k) = ?

k = 0,1,2,. . . ,n

What is the chance that you experience k successes?

That means that you have k successes, and n-k successes.

$$\begin{aligned} P(Y=k) &= \tfrac{n!}{k!(n-k)!} \\ &= \tbinom{n}{k} p^k (1-p)^{n-k} k = 0, 1, 2, \ldots, n \end{aligned}$$

(45) In a multiple choice test, each question has 5 possible answers, but only one is correct. If the test has 10 questions, what is the probability that a student who answers questions at random scores:

For this example, you may use the table at pages 739-741 in the textbook.

i) %80 or more?

n=10 (10 questions)

p=0.2 (chance of getting it right at random)

q=0.8 (chance of getting it wrong at random)

$P(Y \geq 8) = P(Y = 8) + P(Y = 9) + P(Y = 10)$

$= \binom{10}{8} 0.2^8 0.8^2 + \binom{10}{9} 0.2^9 0.8^1 + \binom{10}{10} 0.2^1 0$

$= 1 - P(Y \leq 7)$

$1 - 0.999 = 0.001$

ii) %50 or more?
$P(Y \geq 5) = 1 - P(Y \leq 4) = 1 - 0.9672 = 0.0328$

iii) Exactly %10?
$= \binom{10}{1} 0.2^1 0.8^9$

Result If X and Y are independent,

1) $Var(x \pm y) = Var(x) + Var(y)$
2) $Var(cX) = c^2 Var(x)$
$E[cX)^2] - (E(cX))^2 = c^2 E(x^2) - c^2 (E(x))^2 = c^2 \sigma^2$

Geometric Distribution

Geometric Distribution In a sequence of Bernoulli's trials, let X = # of trials we need to have to see the first success.

$P(\{S\}) = p, X \in \{1, 2, 3, \dots\}$ $P(\{F\}) = q = 1 - p, P(X = k) = P(FF \cdots FS) = pq^{k-1}$

k	1	2	3	4	...
$P(X = k)$	p	pq	pqˆ2	pqˆ3	...

Notice that if $S = 1 + q + q^2 + \cdots$

$qS = q + q^2 + q^3 + \cdots$

$S - qS = 1$

$S(1 - q) = 1$

$S = \frac{1}{1-q} = \frac{1}{p}$

(46) The probability of manufacturing a defective item is p = 0.05. Find:

i) The probability that the fifth manufactured item is the first defective item.
Solution:
- p = 0.05
- q = 0.95 $\therefore P(X = 5) = 0.95^4 \times 0.05$

ii) The probability that the first defective item appears at or after 10^{th} manufactured item.
Solution:
$P(X \geq 10) = 1 - P(X \leq 9) = pq^9 + pq^10 + pq^11 + \cdots$

$= pq^9(1 + q + q^2 + \cdots)$
$= \frac{pq^9}{p} = q^9 = 0.95^9$

iii) Given that the first ten produced items are not defective, what is the probability that the fifteenth manufactured item is defective?
Solution:
$P(x = 15|x \geq 11) = \frac{P(A \cap B)}{P(B)} = \frac{P(x=15)}{P(x \geq 11)} = \frac{pq^14}{q^10} = pq^4 = (0.05)(0.95)^4$
$\{15\} = \{15\} \cap \{11, 12, 13, \dots\}$
$\therefore It's the same as the solution to i), because the tests are independent.$

Finding Expected Value and Variance for Geometric Distance

We know that $P(X = k) = pq^{k-1} \quad k = 1, 2, \dots$

$$E(x) = \sum_{k=1}^{\infty} kpq^{k-1} = P[1 + 2q + 3q^2 + 4q^3 + \cdots]$$

$= p\frac{d}{dq}(\frac{1}{1-q}) = p\frac{1}{(1-q)^2} = \frac{1}{p}$

$$E[x(x-1)] = \sum_{k=1}^{\infty} k(k-1)pq^{k-1} = \sum_{k=2}^{\infty} k(k-1)pq^{k-1}$$

$$= pq\sum_{k=2}^{\infty} k(k-1)q^{k-1} = pq[2 \times 1 + 3 \times 2q + 4 \times 3 \times q^2 + \cdots]$$

$= pg\frac{d}{dq}\frac{1}{(1-q)^2}$

$= pq\frac{2(1-q)}{(1-q)^4} = 2pq\frac{1}{(1-q)^3} = \frac{2pq}{p^3} = \frac{2q}{p^2}$

$E(x) = \frac{1}{p}$

$E(x^2) - E(x) = \frac{2q}{p^2}$

$E(x^2) = \frac{2q}{p^2} + \frac{1}{p}$

$E(x^2) - (E(x))^2 = \frac{2q}{p^2} + \frac{1}{p} - \frac{1}{p^2} = \frac{2q+p-1}{p^2} = \frac{q+q+p-1}{p^2} = \frac{q}{p^2}$

Bertrand Paradox

Is this even a paradox?

Take a game, where you flip a coin and you get the number of dollars back, doubling for each successful tails, according to this table:

H	TH	TTH	TTTH	TTTTH	...
1	2	4	8	16	...

What is the expected pay for the game?

$\frac{1}{2} \times 1 + \frac{1}{4} \times 2 + \frac{1}{8} \times 4 + \frac{1}{16} \times 8 + \cdots = E(prize)$

$= \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots = \infty$

Idea: Let's say there's a millionaire who won't pay more than 2^{30} rolls.

H	TH	TTH	TTTH	TTTTH	$\cdots$	$T^{30}H$	$T^{30}H$	$\cdots$
1	2	4	2^3	2^4	$\cdots$	2^{30}	2^{30}	$\cdots$

$E(prize) = 1 \times \frac{1}{2} + 2 \times \frac{1}{4} + 4 \times \frac{1}{8} + \cdots + 2^30 \times \frac{1}{2^31} + 2^30(\frac{1}{2^32} + \frac{1}{33} + \cdots)$

$= 15 + 2^30 \times \frac{1}{2^32}(1 + \frac{1}{2} + \frac{1}{4} + \cdots)$

$15 + \frac{1}{4}(\frac{1}{1-\frac{1}{2}})$

$= 15.5$

Negative Binomial Distribution

In Bernoulli's trials, let x = # of trials to see the r$^{\text{th}}$ success.

k = r, r+1, r+2, ...

$P(x = k) = \binom{k-1}{r-1} p^{r-1} q^{k-1} p$

$= \binom{k-1}{r-1} p^r q^{k-1}$, where $k = r, r+1, \ldots$

Y_1 = # of trials to see the first success.

Y_2 = # of trials to see the second success after you see the first success.

...

Y_r

$x = Y_1 + Y_2 + \cdots + Y_r$

$E(x) = \frac{r}{p}$, $var(x) = \frac{r(q)}{p^2}$

Poisson Distribution

In Bernoulli's trials with n experiements P=P({S}), let $np = \lambda$ we can write:

$P(X = k) = \binom{n}{k} p^k (1-p)^{n-k} = \frac{n!}{k!(n-k)!}(\lambda/n)^k(1-(\lambda/n))^{n-k}$

There is a break here because I was late for class.

$P(X = k) \to \frac{e^{-\lambda}\lambda^k}{k!} = g(k)$

$k = 0, 1, 2, \ldots$

(47) Is $g(k) = \frac{e^{-\lambda}\lambda^k}{k!}$ a proper probability mass function?
$k = 0, 1, 2, \ldots$

(a) Notice that $e^{-\lambda}\frac{\lambda^k}{k!} \geq 0$.

(b) $1 = \sum_{k=0}^{\infty} e^{-\lambda}\frac{\lambda^k}{k!} = e^{-\lambda}(\sum_{k=0}^{\infty}\frac{\lambda^k}{k!})$
$= e^{-\lambda}e^{\lambda} = e^0 = 1$

Mean and Variance for Poisson Distribution

Mean: $E(x) = \mu = \sum_{k=0}^{\infty} k\frac{e^{\lambda}\lambda^k}{k!} = \sum_{k=1}^{\infty}\frac{e^{\lambda}\lambda^k}{(k-1)!}$

$$\lambda\sum_{k=1}^{\infty}\frac{e^{-\lambda}\lambda^{k-1}}{(k-1)!} = \lambda\sum_{k=0}^{\infty}k\frac{e^{\lambda}\lambda^k}{k!} = \lambda$$

Variance for a Poisson Distribution

$E[E(x-1)] = \sum_{k=0}^{\infty} k(k-1)\frac{e^{-\lambda}\lambda^k}{k!} = \sum_{k=2}^{\infty} k(k-1)\frac{e^{-\lambda}\lambda^k}{k!} = 1$

$E(x(x-1)) = \lambda^2 = E(x^2) - E(x) \to E(x^2) = \lambda^2 + \lambda$

$\sigma^2 = Var(x) = E(x^2) - (E(x))^2 = \lambda^2 + \lambda - \lambda^2 = \lambda$

Traffic

Between time 0 and time t, you want to know how many customers will arrive.

N(t) = #customers arrive in [o, t].

$P[N(t) = k] =?$

We are going to divide the time into sub-intervals that are very small, so that P becomes smaller.

Conversely, because there are so many sub intervals, n is very large.

$P[N(t) = k] = \frac{e^{-\lambda t}(\lambda t)^k}{k!}$

(48) A 911 agent receives 5 calls/hour.

i) Calculate the probability that she receives 2 calls or more in 30 minutes.
In 30 minutes, $\lambda = 2.5$ (average # calls per 30 minutes)
$P(x \geq 2) = 1 - P(x = 0) - P(x = 1) = 1 = \frac{e^{-\lambda}\lambda^0}{0!} - \frac{e^{-\lambda}\lambda^1}{1!}$

ii) Find the probability that the agent is occupied more than average in the next 3 hours.
$\lambda = 15$

$$P(x > 15) == 1 - P(x \leq 15) = 1 - \sum_{k=0}^{1} 5e^{-}15\frac{15^k}{k!}$$

Hypergeometric Distribution

(49) From a regular deck of cards (52 cards), 5 cards are drawn at random.

i) Find the probability that you have 3 aces.
How many ways can you draw 5 cards from 52 cards?
$\frac{\binom{4}{3}\binom{2}{48}}{\binom{52}{5}}$

ii) Find the probability that all of them are the same suit.
$\frac{\binom{13}{5} \times 4}{\binom{52}{5}}$

iii) Find the probability that you have 4 Kings.
$\frac{\binom{4}{4} \times \binom{48}{1}}{\binom{52}{5}}$

(50) You have a box with m white chips and n black chips. You pick out k. What is the probability that X = x?

$x = \#$ white chips

$P(X = x) = \frac{\binom{M}{x}\binom{N}{k-x}}{\binom{m+n}{k}}$

In the same model, if we replace each chip after noticing the color...

$P(X = x)\binom{k}{x}\frac{m}{m+n}$

Continuous Random Variables

Continuous Random Variables If a random experiment takes many values in an interval then we need to define continuous random variable.

Examples are:

1. Height of a randomly selected student.
2. Age of a person who is diagnosed with skin cancer.
3. The temperature of a certain futures date.

(51) A point is picked at random from a circle of radius R. Let D = the distance of the selected point to the center.

$S = (x, y) : x^2 + y^2 \leq R^2$

$D = \sqrt{x^2 + y^2}, D \in [0, R]$

$P[D \leq x] = F(x)$

F is the cumulative distribution function for random variable D.

$$P[D \leq x] = P[(x, y) : x^2 + y^2 \leq x^2] = \frac{\pi x^2}{\pi R^2} = F(x)$$

$$F(x) = \begin{cases} 0 \text{ if } x < 0 \\ \frac{x^2}{R^2} \text{ if } 0 \leq x \leq R \\ 1 \text{ if } x \geq R \end{cases}$$

Continuous Type A random variable x is of continuous type if its cumulative distribution function, $F(x) = P[X \leq x]$, is continuous everywhere.

$\lim_{x \to a} F(x) = F(a)$

$$F(x) = \begin{cases} 2 \text{ if } x \neq 0 \\ 0 \text{ if } x = 0 \end{cases}$$

$\lim_{x \to 0^-} F(x) = 2$

$\lim_{x \to 0^+} F(x) = 2$

$F(0) = 0$

Definition If x is a continuous random variable with a differentiable cumulative distribution function F, then f(x) = F'(x) is called the probability density function for the random variable X.

Properties:

i) $f(x) \geq 0$ because F is nondecreasing.

ii) $\displaystyle\int_{-\infty}^{\infty} f(x)dx = 1$
$F(x) = P(X \leq x)$
$F(+\infty) = P(X \leq \infty) = 1$
$F(-\infty) = P(x \leq -\infty) = 0$
$P(a < X \leq b) = P(X \leq b) - P(X \leq a) = F(b) - F(a)$
$P(-\infty < x < +\infty) = F(\infty) - F(-\infty) = 1$
$\int_{-\infty}^{\infty} F'(x)dx = \int_{-\infty}^{\infty} F(x)dx = 1$

(52) Let X be a random variable with the probability density function:

$$f(x) = \begin{cases} kx^2 \text{ if } 0 < x < 1 \\ 0 \text{ if } elsewhere \end{cases}$$

i) Find k

Since $1 = \int_{-\infty}^{\infty} f(x)dx = \int_0^1 kx^2dx = 1$

`$k \frac{x^3}{3} |_0^1 = k(1/3 - 0) = k/3 = 1 \rightarrow k=3$`

ii) Find $P(1/2 < x < 2/3)$
$= \int_{1/2}^{2/3} 3x^2dx = x^3|_{1/2}^{2/3} = (\frac{2}{3})$
Professor was moving too fast here, and therefore ii) is incomplete

iii) Find $E(x)$ and σ^2
$E(x-\mu)^2 = E(x^2) - \mu^2$
$E(x^2) = \int_0^1 x^2(3x^2)dx = \frac{3}{5}$
$\sigma^2 = \frac{3}{5} - \frac{9}{16} = \frac{48-45}{80} = \frac{3}{80}$

iv) Find cumulative distribution function.

$$F(x) = P(X \leq x) = \begin{cases} 0 \text{ if } x < 0 \\ \int_0^x 3t^2dt \text{ if } 0 < x < 1 \\ 1 \text{ if } x \geq 1 \end{cases}$$

$$F(x) = \begin{cases} 0 \text{ if } x < 0 \\ x^3 \text{ if } 0 \leq x \leq 1 \\ 1 \text{ if } x \geq 1 \end{cases}$$

$\int_0^x 3t^2dt = t^3|_0^x = x^3$

*Few notes and remarks for continuous random variables.

i) $P(X = a) = \int_a^a f(x)dx = 0$

$P(2 < X < 3) = P(2 \leq X < 3) = P(2 \leq X \leq 3)$

In continuous, the chance of getting the EXACTLY same result twice is 0. Someone will not die at EXACTLY 71, because a millisecond before or after is also possible. Therefore you can drop or add "or equal to" on the ranges.

ii) $P(a \leq X \leq b) = F(b) - F(a)$

iii) f(x) (probability density functions) can also take values larger than one, unlike continuous distribution functions.

(53) Uniform distribution example.

If x has a constant probability density functions on [a,b], we denote:

$X\ Unif[a, b]$ (x has a uniform distribution on a to b, meaning it is constant.)

$f(x) = \begin{cases} c & \text{if } a \leq x \leq b \\ 0 & \text{elsewhere} \end{cases}$

We can calculate c by noticing that:

$1 = \int_a^b cdx = cx|_a^b = cb - ca = 1 \rightarrow C = \frac{1}{b-a}$

$f(x) = \begin{cases} \frac{1}{b-a} & \text{if } a \leq x \leq b \\ 0 & \text{elsewhere} \end{cases}$

$E(x) = \int_a^b \frac{1}{b-a} xdex = \frac{x^2}{2(b-a)}|_a^b = \frac{a+b}{2}$

$E(x^2) = \int_a^b \frac{x^2}{b-a} dx = \frac{1}{b-a}(b^3/3 - a^3/3) = \frac{a^2+b^2+ab}{3}$

$\sigma^2 = E(x^2) - (\frac{a+b}{2})^2 = \frac{(b-a)^2}{12}$

(54) Pick a point from [a,b] at random. Call the point you picked X. With uniform density,

$P(X \leq x) = \frac{x-a}{b-a} = F(x)$ where $a \leq x \leq b$

$F(x) = \int_a^x \frac{dt}{b-a} \longrightarrow f(x) = \begin{cases} \frac{1}{b-a} & a \leq x \leq b \\ 0 & \text{elsewhere} \end{cases}$

$E(x) = \int_a^b xf(x)dx = \int_a^b \frac{x}{b-a} dx = \frac{x^2}{2(b-a)}|_a^b = \frac{b^2-a^2}{2(b-a)} = \frac{b+a}{2}$

$Var(X) = E(X^2) - (E(X))^2$

$E(X^2) = \int_a^b x^2 \frac{1}{b-a} dx = \frac{x^3}{3(b-a)}|_a^b = \frac{b^3-a^3}{3(b-a)} = \frac{a^2+b^2+ab}{3}$

$\sigma^2 = \frac{a^2+b^2+ab}{3} - (\frac{a+b}{2})^2 = \frac{4a^2+4b^2+4ab-3(a+b)^2}{12} = \frac{a^2+b^2-2ab}{12}$

(55) A student leaves home between 8 and 8:30am with uniform distribution. It takes between 30 and 40 minutes, uniformly, to arrive at school.

X = The departure time ~u[8,8.5]

$$f(x) = \begin{cases} \frac{1}{0.5} & \text{if } 8 \leq x \leq 8.5 \\ 0 & \text{elsewhere} \end{cases}$$

Y = the amount of time it takes to arrive at school.

$$g(y) = \begin{cases} \frac{1}{2/3-1/2} & \text{if } 1/2 < y < 2/3 \\ 0 & \text{elsewhere} \end{cases}$$

$$g(y) = \begin{cases} 6 & \text{if } 1/2 < y < 2/3 \\ 0 & \text{elsewhere} \end{cases}$$

i) **Find the mean and variance of the arrival time. X and Y are independent.**

$$\begin{aligned} E(X+Y) \quad = \quad E(X)+E(Y) &= 8.25 + \frac{1/2+2/3}{2} \\ &= 8.25 + \frac{7}{12} \\ &= 8.25 + 0.583 \\ &= 8.833 \\ &\approx 8.50am \end{aligned}$$

$Var(X+Y) = Var(X) + Var(Y)$

$Var(X) = \frac{(b-a)^2}{12} = \frac{0.5^2}{12} = \frac{1}{48}$

$Var(Y) = \frac{(2/3-1/2)^2}{12} = \frac{1}{12\times 36}$

$Var(X+Y) = \frac{1}{48} + \frac{1}{12\times 36}$

ii) **If a student leaves home between 8-8:30am in 5 days in a row, what is the probability that he leaves home after 8:15am at least 3 of those days?**

P(leaving home after 8:15am)

$q = 0.5$

$n = 5$

$\int_8 .25^8 .5 \frac{1}{0.5} dx = 0.5$

T = # days he leaves after 8:15am in 5 different days.

$P(T \geq 3) = \binom{5}{3} \frac{1}{2}^3 \frac{1}{2}^2 + \binom{5}{4} \frac{1}{2}^4 \frac{1/2}{+} \binom{5}{5} \frac{1}{2}^5$

$= \frac{10}{32} + \frac{5}{32} + + \frac{1}{32} = \frac{16}{32} = \frac{1}{2}$

Gaussian (Normal) Distribution

Flip 3 coins:

x	0	1	2	3
$P(X = x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

Flip 4 coins:

x	0	1	2	3	4
$P(X = x)$	$\frac{1}{16}$	$\frac{4}{16}$	$\frac{6}{16}$	$\frac{4}{16}$	$\frac{1}{16}$

Flip 5 coins:

x	0	1	2	3	4	5
$P(X = x)$	$\frac{1}{32}$	$\frac{5}{32}$	$\frac{10}{32}$	$\frac{10}{32}$	$\frac{5}{32}$	$\frac{1}{32}$

Notice the numbers from pascal's triangle.

Define: $f(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$ $z \in R$

- f is symmetrical around zero.
- $f(z) \to 0$ as $z \to \pm\infty$
- f(0) is the largest value of f.
- $\int_{-\infty}^{\infty} f(z) dz = 1$

So, f is a probability density function. This is called the standard normal distribution.

See pages 742-743.

$P(Z \leq 2)$

$P(-2 \leq Z \leq 1.4)$

$P(-1.645 \leq Z \leq 1.645$

Expected Value and Variance of Gaussian Distribution

$E(Z) = 0$

$E(Z^2) = 1$