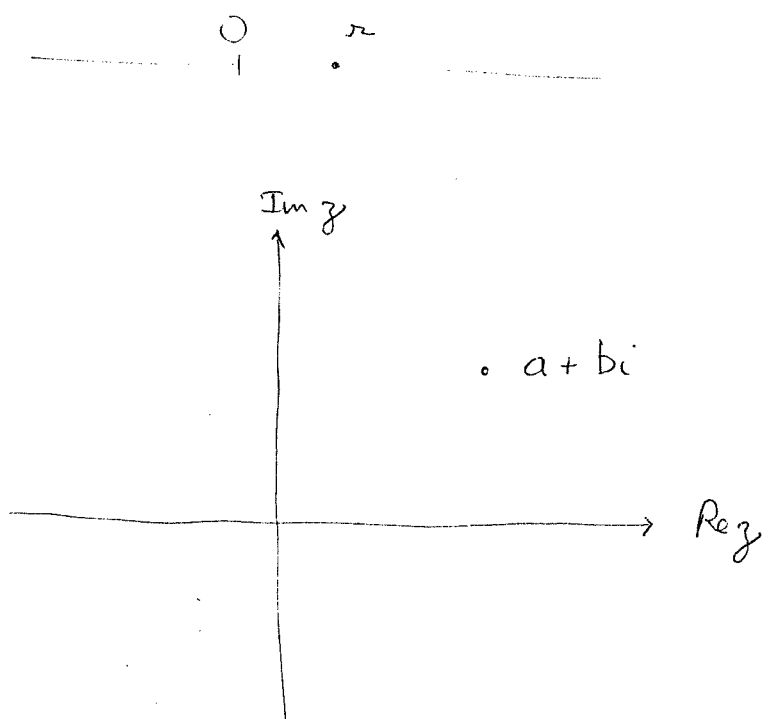


Complex plane  
Argand plane



Real numbers.  
can't solve  $x^2 + 1 = 0$ .

"Invent"  $i$  s.t.  
 $i^2 = -1$

"impossible #'s"

"i" - Euler 1777

$$i = \sqrt{-1}$$

Recall:

Def<sup>n</sup>  $\mathbb{C} = \{ a + bi \mid a, b \in \mathbb{R}, i^2 = -1 \}$  ← close bracket  
 ↑ set of all things like this such that conditions on "things like this" Set Notation

$$z = a + bi \in \mathbb{C}$$

↑                    ↑                    ↑  
 "belongs to"

$a =$  real part of  $z$      $b =$  imaginary part of  $z$      $\text{Im}(z)$  (both are real numbers!)

$\mathbb{R} \subset \mathbb{C}$  (subset of)  
 ( $a=0$ :  $\frac{z}{i} = bi$  pure imaginary)    ( $b=0$   $z=a$  is real)

Equality

$$a + bi = c + di \Leftrightarrow a = c \ \& \ b = d$$

Addition

$$(a + bi) + (c + di) = a + c + (b + d)i$$

Mult<sup>n</sup>

$$(a + bi)(c + di) = (ac - bd) + (ad + bc)i$$

Rank: They satisfy all the arithmetical properties of  $\mathbb{R}$  except there is no ordering in  $\mathbb{C}$ . i.e. " $z > 0$ " makes no sense.

e.g. solve  $z(3 + 4i) = 1$ , if possible.

Recall  $(a + bi)(a - bi) = a^2 - (bi)^2 = a^2 + b^2$  ( $\neq 0$  unless  $a = b = 0$ )

So<sup>f</sup> for  $z = a + bi$ , let  $\bar{z} = a - bi$  (complex conjugate of  $z$ )

Note:  $|z| > 0 \Leftrightarrow z \neq 0$ ,  $|z| = \sqrt{a^2 + b^2}$  (modulus of  $z$ , mod  $z$ )

Then, If  $z \neq 0$  ( $\Leftrightarrow$  at least one of  $a, b$  is not zero),  $|z| \neq 0$  so

$\rightarrow z \cdot \bar{z} = |z|^2$       so       $z \cdot \frac{\bar{z}}{|z|^2} = 1$

i.e.  $z \neq 0 \Rightarrow \frac{1}{z} = \frac{\bar{z}}{|z|^2}$

Hence if  $z(3+4i) = 1$ ,       $z = \frac{1}{3+4i} = \frac{3-4i}{3^2+4^2} = \frac{3}{25} - \frac{4i}{25}$

$\frac{L1}{L2}$  F04, F06

Other properties  
(easy to check)

$\overline{z+w} = \bar{z} + \bar{w}$   
 $\overline{zw} = \bar{z} \bar{w}$

$\overline{(\bar{z})} = z$

- $z \in \mathbb{R} \Leftrightarrow z = \bar{z}$
- $z$  pure im.  $\Leftrightarrow z = -\bar{z}$

$|z| = |\bar{z}|$

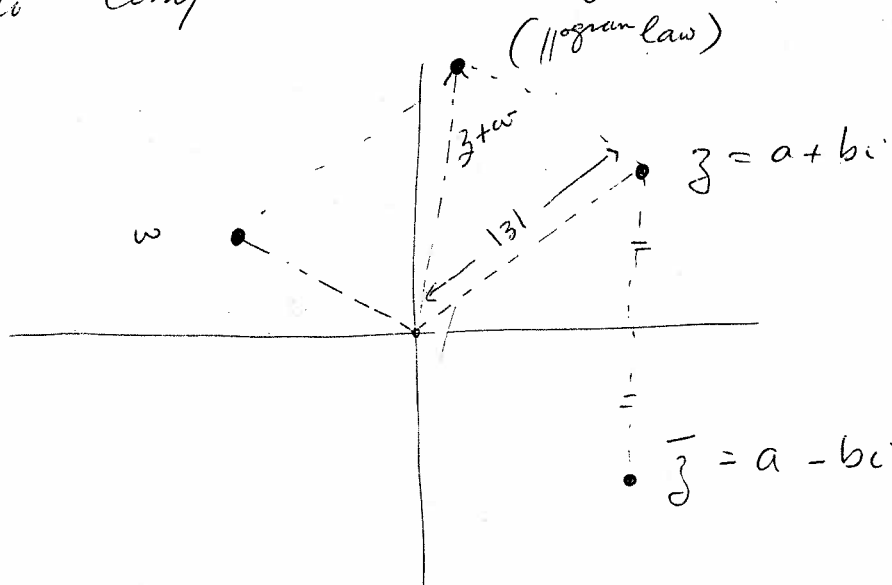
$|zw| = |z| |w|$

$|z| \geq 0$ ,  $|z| = 0 \Leftrightarrow z = 0$

$|z+w| \leq |z| + |w|$   
 $\Delta$  inequality

$\frac{L1}{L2}$

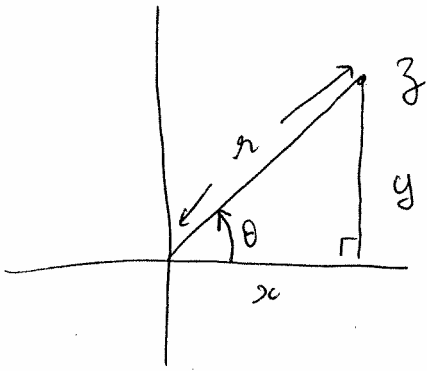
back to Complex Plane : remind ourselves of the geom. interpretations of  $z, \bar{z}, |z|$ .



12   Polar form " $z = re^{i\theta}$ " (polar coordinates,  $\mathbb{R}^3$ )

If  $z = x + iy$  ;  $x, y \in \mathbb{R}$  and

$r$  &  $\theta$  are as indicated,



$$\frac{x}{r} = \cos \theta$$

$$\frac{y}{r} = \sin \theta$$

$$r = |z| = \sqrt{x^2 + y^2}$$

$$\therefore z = x + iy = r \cos \theta + i r \sin \theta = r (\cos \theta + i \sin \theta)$$

- $\theta$  is not uniquely determined ;  $\theta = \arg(z)$ , argument of  $z$
- ( $\theta' = \theta + 2n\pi$ ,  $n \in \mathbb{Z}$  also works)
- if we assume in addition that  $-\pi < \theta \leq \pi$ ,

$$\theta = \text{Arg}(z) \quad \text{Principal argument of } z$$

11  

Euler 1748

$$e^z = 1 + z + \frac{z^2}{2} + \dots = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

$$e^{i\theta} = \cos \theta + i \sin \theta \quad (\text{compare power series...})$$

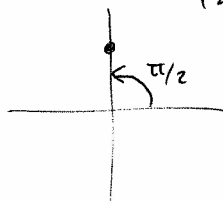
$$\therefore z = r e^{i\theta} \quad \leftarrow \quad \underline{\text{Polar form of } \text{ph} \# z}$$

$$r e^{i\theta} = r' e^{i\theta'} \Leftrightarrow r = r', \quad \theta - \theta' = 2n\pi, \quad n \in \mathbb{Z}$$

Polar form examples

e.g. (a)  $i$

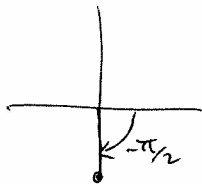
$$|i| = \sqrt{0^2 + 1^2} = 1$$



$$i = 1 \cdot e^{i\pi/2} = (\cos \pi/2 + i \sin \pi/2) \quad (24)$$

w04 ↓ Jan 6  
Jan 8

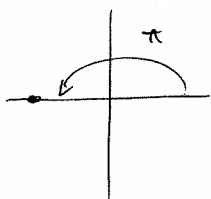
(b)  $-i$



$$-i = e^{-i\pi/2}$$

$$(-i = e^{i3\pi/2} \dots)$$

(c)  $-1$

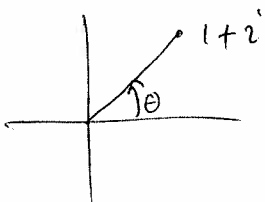


$$|-1| = 1$$

$$-1 = e^{i\pi}$$

$$(= e^{-i\pi}, e^{3i\pi}, \dots)$$

(d)



$$|1+i| = \sqrt{1^2 + 1^2} = \sqrt{2}$$

$$\cos \theta = \frac{1}{\sqrt{2}}$$

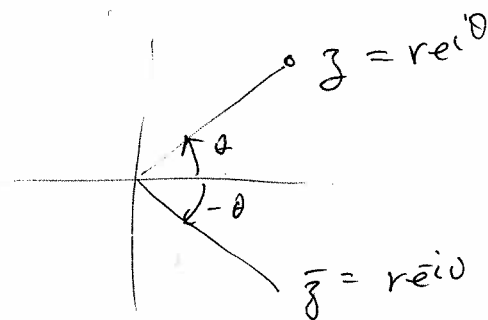
$$\sin \theta = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \theta = \pi/4$$

$$\therefore 1+i = \frac{\sqrt{2}}{2} e^{i\pi/4}$$

Rekh.  $\overline{(re^{i\theta})} = re^{-i\theta}$

$$|e^{i\theta}| = 1 \text{ for all } \theta.$$



$r \neq 0$   $\frac{y}{x} = \tan \theta$  not enough

mult<sup>n</sup> complex #'s : (easy in polar form)

(15)

$$z_1 = r_1 e^{i\theta_1}, \quad z_2 = r_2 e^{i\theta_2};$$

$$z_1 z_2 = r_1 e^{i\theta_1} r_2 e^{i\theta_2} = (r_1 r_2) e^{i\theta_1} e^{i\theta_2}$$

$$= (r_1 r_2) (\cos\theta_1 + i\sin\theta_1) (\cos\theta_2 + i\sin\theta_2) = (r_1 r_2) [(\cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2) + i(\sin\theta_1 \cos\theta_2 + \sin\theta_2 \cos\theta_1)]$$

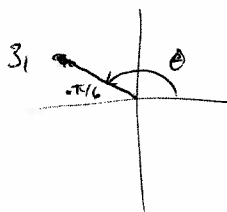
$$= r_1 r_2 [\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)] = r_1 r_2 e^{i(\theta_1 + \theta_2)} \quad !$$

Similarly, if  $z_2 \neq 0$ ,  $\frac{z_1}{z_2} = \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)}$

e.g. polar form of  $z = \frac{-\sqrt{3} + i}{1 + i} = \frac{z_1}{z_2};$

$-\sqrt{3} + i$ ;  $|-\sqrt{3} + i| = \sqrt{3+1} = 2$   $\left\{ \begin{array}{l} |1+i| = \sqrt{2}; \quad (1+i) = \sqrt{2} e^{i\pi/4} \\ \cos\theta = \frac{-\sqrt{3}}{2} \\ \sin\theta = \frac{1}{2} \end{array} \right\} \Rightarrow \theta = 5\pi/6$

$z_1 = 2 e^{i5\pi/6}$



$$\therefore z = \frac{2 e^{i5\pi/6}}{\sqrt{2} e^{i\pi/4}} = \sqrt{2} e^{i(5\pi/6 - \pi/4)} = \sqrt{2} e^{i7\pi/12}$$

Remark: Gauss (1797) If  $n \geq 1$ , Every polynomial  $p(z) = a_n z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0$  with complex coefficients can be completely factored using complex #'s, i.e.

$$p(z) = a_n (z - w_1)(z - w_2) \dots (z - w_n) \quad \text{with } a_n, w_i \in \mathbb{C}$$

$\swarrow \quad \nearrow \quad \dots \quad \searrow$   
 roots

# Vectors in $\mathbb{R}^n$

Vector (Latin: vehere: to convey, carrier) (R6)

We'll use intuition from low dimensions for higher dimensions (R7)

geometric insights

"geometric ideas"

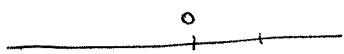
+ algebraic understandings

algebraic ✓

"Alg"

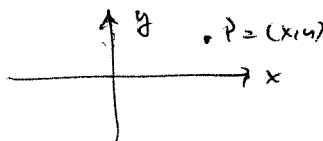
"Geom"

$\mathbb{R}$ -scalars



(17)

$\mathbb{R}^2 = \{(x,y) \mid x,y \in \mathbb{R}\}$  ;  $u = (x,y)$



Cartesian plane

$\mathbb{R}^3 = \{(x,y,z) \mid x,y,z \in \mathbb{R}\}$  ;  $u = (x,y,z)$



'3-space'

∴ 300 gears

Hamilton  $\mathbb{R}^4 = \{(x_1, x_2, x_3, x_4) \mid x_i \in \mathbb{R}\}$

space-time?

$n \in \mathbb{N}$   $\mathbb{R}^n = \{(x_1, \dots, x_n) \mid x_i \in \mathbb{R}\}$ ;  $u = (x_1, \dots, x_n)$

? "n-space" (4.1.2)

↑ ellipsis

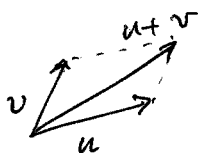
Manipulations of vectors in  $\mathbb{R}^n$ , Eq  $(x_1, \dots, x_n) = (x_1', \dots, x_n')$   $\Leftrightarrow$   $x_i = x_i'$   $i=1, \dots, n$ .

Add  $(x_1, \dots, x_n) + (x_1', \dots, x_n') = (x_1 + x_1', \dots, x_n + x_n')$

Zero vector  $(0, \dots, 0) \in \mathbb{R}^n$ , Negative  $-(x_1, \dots, x_n) = (-x_1, \dots, -x_n)$

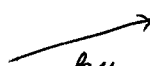
Mult<sup>n</sup> by scalar  $k \in \mathbb{R}$ ,  $k(x_1, \dots, x_n) = (kx_1, \dots, kx_n)$ .

n=2



$k (> 0)$

$k' (< 0)$



$k'u$

- ⓪ length  $ku = |k| \cdot \text{length } u$
- Ⓛ same dir<sup>n</sup>.

Def<sup>n</sup> If  $k_1, \dots, k_m \in \mathbb{R}$ , and  $u_1, \dots, u_m \in \mathbb{R}^n$  are vectors,  $k_1 u_1 + k_2 u_2 + \dots + k_m u_m$  is called a loc. of  $u_1, \dots, u_m$

No. B. All usual properties hold:  $\forall u, v, w \in \mathbb{R}^n, \forall k \in \mathbb{R}$

$(u+v)+w = u+(v+w)$

$u+v = v+u$

$k(k'u) = (kk')u$

$u+0 = u$

$k(u+0) = ku + k \cdot 0$

$1u = u$

$u+(-u) = 0$

$(k+k')u = ku + k'u$

### 3.2 + Dot (Inner Product) in $\mathbb{R}^2, \mathbb{R}^3, \mathbb{R}^n$ : (length + angles!) (R8)

$\bullet$   $\frac{n=3}{u=3}$   $u = (x_1, x_2, x_3)$   $v = (y_1, y_2, y_3)$   $\text{vecs. } \subset \mathbb{R}^3$

$u \cdot v = x_1 y_1 + x_2 y_2 + x_3 y_3$  dot product

$\|u\| = \sqrt{x_1^2 + x_2^2 + x_3^2}$

length, norm of  $u$

$\|u-v\| = \text{distance between } u \text{ \& } v$

$\frac{n}{u} = (x_1, \dots, x_n), v = (y_1, \dots, y_n)$  ;  $u \cdot v = x_1 y_1 + \dots + x_n y_n$

$\|u\| = \sqrt{x_1^2 + \dots + x_n^2}$

$\mathbb{R}^n$  with " $\cdot$ "

"Euclidean  $n$ -space"

N.B.:  $u \in \mathbb{R}^n$  is zero  $\Leftrightarrow \|u\| = 0$ .

L2

$u, v \neq 0$

N.B.  $n \leq 3$ , seen  $u \cdot v = 0 \Leftrightarrow u$  and  $v$  are perpendicular / orthogonal

$\frac{n}{u}; u, v$   $u \cdot v = 0$  say  $u, v$  are perpendicular.

e.g.  $(1, 0, 0, 1), (1, 0, 0, -1)$  are  $\perp$ , since  $(1, 0, 0, 1) \cdot (1, 0, 0, -1) = 0$

(1340)

other angles?

L2

Prob 26, Ex 3.2 (Cauchy-Schwarz) If  $u, v \in \mathbb{R}^n$ , then

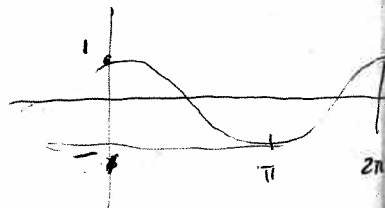
$|u \cdot v| \leq \|u\| \|v\|$ ;  $\Rightarrow \textcircled{1} \|u+v\| \leq \|u\| + \|v\|$

$\triangle$  inequality (like for angles #5).

$\Rightarrow$  Allows def<sup>n</sup> of angle between 2 non-zero vecs. If  $u, v \neq 0$ , angle  $\theta$  between  $u$  &  $v$  is  $\theta$  which satisfies

1)  $\cos \theta = \frac{u \cdot v}{\|u\| \|v\|}$  ( $-1 \leq 1$ )

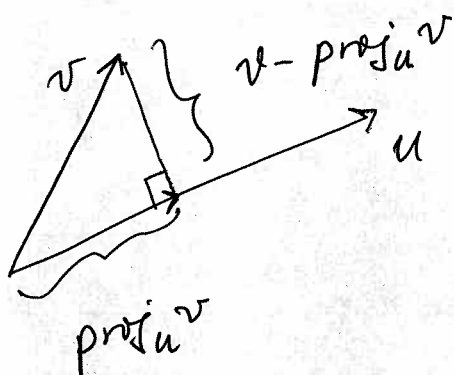
2)  $0 \leq \theta \leq \pi$  (so it's unique)  $\leq \pi$ .



# Orthogonal Projections of $v$ on $u$

3.2.3

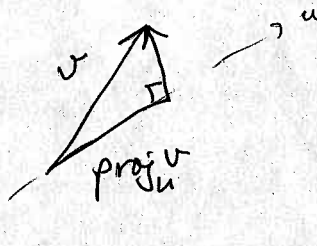
(R9)



$$\text{proj}_u v = \underbrace{\frac{v \cdot u}{\|u\|^2}}_{\text{scalar}} \underbrace{u}_{\text{vector}}$$

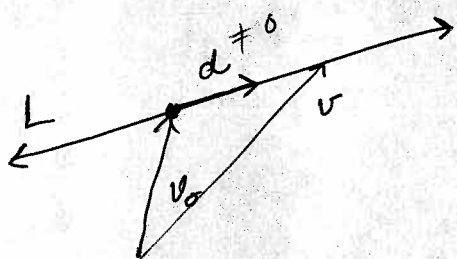
- Satisfies
- 1)  $\text{proj}_u v \parallel u$  ( $\Rightarrow$  formula above)
  - 2)  $v - \text{proj}_u v \perp u$

So  $v = \text{proj}_u v + (v - \text{proj}_u v)$   
 $\parallel v \quad \perp$



UMT System  
 transmission  
 a  
 b  
 c  
 d  
 e  
 f  
 g  
 h  
 i  
 j  
 k  
 l  
 m  
 n  
 o  
 p  
 q  
 r  
 s  
 t  
 u  
 v  
 w  
 x  
 y  
 z

Lines in  $\mathbb{R}^2, \mathbb{R}^3$  ( $\mathbb{R}^n$ )



$$v = v_0 + td$$

$$L = \{ v_0 + td \mid t \in \mathbb{R}, d \neq 0, d \in \mathbb{R}^n \text{ fixed} \}$$

↑  
dir<sup>n</sup> vector of line

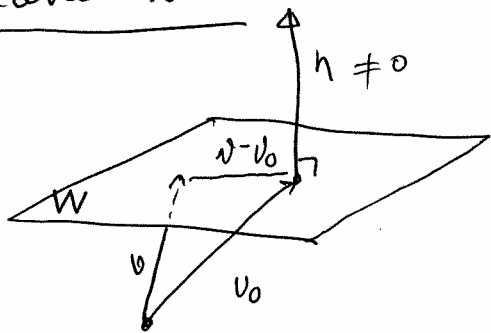
- vector form above

- write  $v = (x_1, x_2, x_3)$   $d = (d_1, d_2, d_3)$   
 $(m=3) v_0 = (a, b, c)$

$$\begin{aligned} x_1 &= a + td_1 \\ x_2 &= b + td_2 \\ x_3 &= c + td_3 \end{aligned}$$

Line through  $v_0$  with dir<sup>n</sup>  $d$

# Planes in $\mathbb{R}^3$



$$W = \{v \in \mathbb{R}^3 \mid (v - v_0) \cdot n = 0\}$$

Plane through  $v_0$  with normal  $n$ .

(non-zero multiples ok)

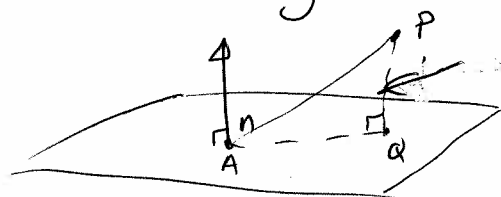
Cartesian form: eg  $n = (2, -1, 1)$ ,  $v_0 = (0, 0, 3)$

$$W = \{ (x, y, z) \in \mathbb{R}^3 \mid (x, y, z - 3) \cdot (2, -1, 1) = 0 \}$$

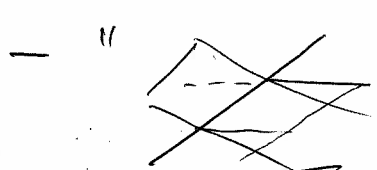
$$= \{ (x, y, z) \in \mathbb{R}^3 \mid 2x - y + z = 3 \}$$

$\uparrow$  coeff  $\Rightarrow$  normal.

- intersection of lines, lines & planes, planes & planes



shortest distance of  $P$  from plane is  
 e.g.  $\|P - Q\| = \| \text{proj}_n (P - A) \|$



"angle" between planes  $\equiv$  angle between normals.

## Cross Product in $\mathbb{R}^3$

3.3.4, 3.5 (~ area, volume)

$$u = (x_1, y_1, z_1), \quad v = (x_2, y_2, z_2)$$

$$u \times v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix} = (y_1 z_2 - y_2 z_1, -(x_1 z_2 - x_2 z_1), x_1 y_2 - x_2 y_1)$$

$$= \left( \begin{vmatrix} y_1 & z_1 \\ y_2 & z_2 \end{vmatrix}, - \begin{vmatrix} x_1 & z_1 \\ x_2 & z_2 \end{vmatrix}, \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} \right)$$

Thms 2, 3 5  
 176-177

$$(u \times v) \cdot u = 0 = (u \times v) \cdot v$$

$$u \times v = -v \times u$$

distrib law

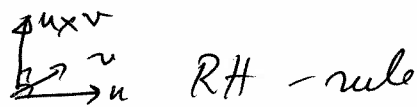
$$u \times (v + w) = u \times v + u \times w$$

Prop.  $u \times (v \times w) \neq (u \times v) \times w$  (not associative)

(RU)

eg  $\hat{i} \times (\hat{j} \times \hat{k}) = \hat{i} \times \hat{0} = \hat{0}$   
 $(\hat{i} \times \hat{j}) \times \hat{k} = \hat{j} \times \hat{k} = -\hat{i} \neq \hat{0}$

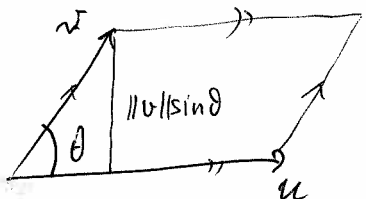
Thm 5  
178  
•  $\|u \times v\| = \|u\| \|v\| \sin \theta$



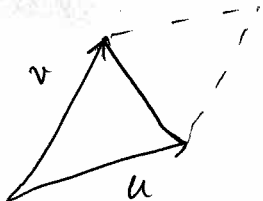
$\Rightarrow$  ①  $u, v$  "parallel"  $\Leftrightarrow u \times v = 0$

(later: (xx)  $u \times v = 0 \Rightarrow u$  is a multiple of  $v$  or  $v$  is a multiple of  $u$ .)

$\Rightarrow$  ②

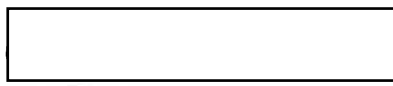


area  $\parallel$ ogram  $= \|u\| \|v\| \sin \theta$   
 $= \|u \times v\|$

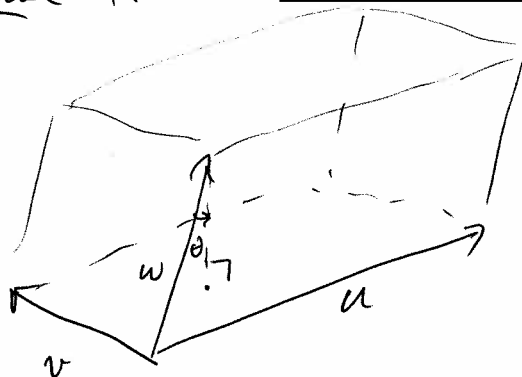


area  $\Delta = \frac{1}{2} \|u \times v\|$

Volume P.178



$u \times v \neq 0$   
 $w \neq 0$



vol = area base  $\cdot \perp$  hgt  
 $= \|u \times v\| \cdot \perp$  hgt

$\perp$  hgt  $= \|w\| |\cos \theta|$ ,  $\theta$  as shown

But  $|\cos \theta| = \left| \frac{(u \times v) \cdot w}{\|u \times v\| \|w\|} \right|$  so

Thm 6  
P.178  
(ignore "det")

vol =  $\frac{\|u \times v\| \cdot \|w\| \cdot |(u \times v) \cdot w|}{\|u \times v\| \|w\|} = |u \times v \cdot w|$

e.g. find vol <sup>u</sup> ~~1~~ piped det. by  $(1,1,1)$ ,  
 $(1,2,7)$  &  $(2,3,6)$ . (order isn't imp.)

(R12)

$$v \times w = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 7 \\ 2 & 3 & 6 \end{vmatrix} = (-9, -(-8), -1)$$

$$u \cdot (v \times w) = (1, 1, 1) \cdot (-9, 8, -1) = -2$$

$$\therefore \text{vol} = +2.$$

13/4

Remarks

$$\begin{aligned} (u \times v) \cdot w &= u \cdot (v \times w) \\ &= w \cdot u \times v \\ &= w \times u \cdot v \quad \text{etc} \end{aligned}$$

(Same cyclic order)

• 3 ~~vectors~~ are coplanar  
 (can be made to lie  
 in same plane)

$$\Leftrightarrow u \cdot v \times w = 0$$

later  
 (=)

There are scalars  $a, b, c$   
 not all zero st.

$$au + bv + cw = 0.$$

$$u \cdot v \times w = \begin{vmatrix} u \\ v \\ w \end{vmatrix}$$

e.g. above

3x3  
 determinant  
 (later)

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 7 \\ 2 & 3 & 6 \end{vmatrix}$$

Chapter 45  
(examples from 1.1)

Vector Spaces

①

$\mathbb{R}, \mathbb{R}^2, \mathbb{R}^3$  - geometric vectors

NON-geometric,  $\mathbb{R}^n$   $n \geq 4$  "vectors"  
NON- $\mathbb{R}^n$  vectors

"Spaces" of equations.

$$\begin{cases} E_1: x - y - z = 1 \\ E_2: 2x - y + z = 1 \\ E_3: -x + 2y + 4z = 4 \end{cases} \quad \left\{ \begin{array}{l} E_2 - 2E_1 = E_4: y + 3z = 3 \\ E_1 + E_3 = E_5: y + 3z = 3 \\ E_5: E_1 + E_3: x + 2z = 2 \end{array} \right.$$

we can write:  $E_1 + E_3 = E_5$   
 $3E_1 - E_2 + E_3 = 0$  ...

- Remarks:
- 1) can add eqns to get another eqn
  - 2) can multiply " by a scalar to get another eqn
  - 3) equations have negatives!

$$-E_1: -x - y + z = -1$$

$$E_1 + (-E_1) = ? \text{ "zero equation"}$$

4) "zero equation"  $0=0$  works!

5) Certain usual arithmetic holds:

$$\begin{aligned} E_1 + E_2 &= E_2 + E_1 \\ E_1 + (E_2 + E_3) &= (E_1 + E_2) + E_3 \\ k(E_1 + E_2) &= kE_1 + kE_2 \quad k \in \mathbb{R} \\ &\vdots \end{aligned}$$

There are exactly the (10) properties of <sup>②</sup>

$\mathbb{R}^n$  (Thm 1.1., P. 4) !

One could consider the space  $\mathcal{E}$  of all equations  
"obtainable from"  
generated by  $E_1, E_2$  &  $E_3$ . Note that

$\mathcal{E}$  behaves exactly like a space of vectors!  
 $\mathcal{E} = \{k_1 E_1 + k_2 E_2 + k_3 E_3 \mid k_i \in \mathbb{R}\}$  ! "linear combinations of equations"  
 $x = x_0$  ?

(Q: is there an eqn of the form  $y = y_0$  in  $\mathcal{E}$  ?  
is can we solve for  $y$  ?

Can we find  $y_0, a, b \in \mathbb{R}$  s.t.

is  
 $(y = y_0)$  is  $a E_1 + b E_2$  ?

exercise: no

e.g. Formation of HBr (Stoichiometry)

— see web notes P1, 2.

(+)

"vectors" don't have to be geometric, or even "n-tuples"

(3)

1.1 What do we really need here?

V-set of vectors of some sort; <sup>operations:</sup> "add<sup>n</sup>" of vectors, "mult<sup>n</sup>" of vectors by scalars

"Closure" { • the sum of 2 vectors should be a vector;  $u, v \in V \Rightarrow u+v \in V$   
• the scalar multiple of a vector should be a vector  $a \in \mathbb{R}, v \in V \Rightarrow a \cdot v \in V$

"Existence" { • there should be a zero vector  $0$ ;  $0+u = u$   
• every vector should have a negative given  $v \in V$ , there is  $-v \in V$  s.t.

$v + (-v) = 0$  (previously)  
 $-v = -1 \cdot v$ ?

- Arithmetic properties:
- $v+w = w+v$
  - $u+(v+w) = (u+v)+w$
  - $a(v+w) = av+aw$
  - $(a+b)v = av+bv$
  - $a(bv) = (ab)v$
  - $1 \cdot v = v$

Any set  $V$ , with 2 operations as above satisfying these

these conditions is called a VECTOR SPACE

(See P293, (5.1))

□ (4)  
e.g.  $\mathbb{R}^2, \mathbb{R}^3, \mathbb{C}$  with usual ops.

eg  $V = \{(x, 2x) \mid x \in \mathbb{R}\}$ , standard ops from  $\mathbb{R}^2$   
 $C \checkmark$   $E \checkmark$   $A = 0K$ , all works for  $\mathbb{R}^2$   $\checkmark$

eg  $U = \{(x, x+2) \mid x \in \mathbb{R}\}$ ; usual ops of  $\mathbb{R}^2$   
 Usual ops  $\Rightarrow$  usual zero =  $(0,0) \notin U$ .  $\therefore X$

Matrices

eg  $M_{22}(\mathbb{R}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid a, b, c, d \in \mathbb{R} \right\}$

Add<sup>n</sup>, mult<sup>n</sup> by scalars : componentwise  
 $C \checkmark$   $E \checkmark$   $A$  (like  $\mathbb{R}^4$ , stacked).

(  $M_{mn}(\mathbb{R}) = \{ A \mid A \text{ is an } m \times n \text{ matrix with real entries} \}$  )

eg Spaces of functions (#5, P.295)

$[a, b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}$

$F[a, b] = \{f \mid f : [a, b] \rightarrow \mathbb{R}\}$

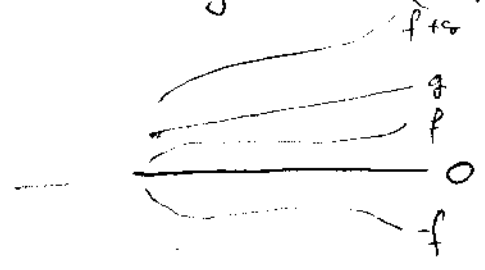
U.B.  $f = g \Leftrightarrow f(x) = g(x) \forall x \in [a, b]$

$f, g \in F[a, b]; k \in \mathbb{R}$   
 Add<sup>n</sup>  $(f+g)(x) = f(x) + g(x)$

$\therefore f+g \in F[a, b]$

multiply scalar  $(kf)(x) = k \cdot f(x)$

$\therefore kf \in F[a, b]$



$C \checkmark$   
 $E$   $0(x) = 0, \forall x \in [a, b]$   
 $(-f)(x) = -f(x)$   
 $A \checkmark$   
 $\checkmark$   
 $\checkmark$   
 $\checkmark$

o Can also consider

$F(\mathbb{R}) = \{f \mid f: \mathbb{R} \rightarrow \mathbb{R}\}$ , same ops ("ptwise")

$f(x) = \cos x$ ,  $g(x) = x + x^2$ ,  $f, g \in F(\mathbb{R})$ , all polynomial functions

$h(x) = \frac{1}{x}$ ,  $k(x) = \tan x$  }  $\notin F(\mathbb{R})$ ; not def'd everywhere.

S.1.2 Subspaces & Spanning Sets

Suppose  $V$  is a v.s. and  $W \subset V$  is a subset of  $V$

(eg  $\{(x, 2x) \mid x \in \mathbb{R}\}$ ). Give  $W$  same ops as  $V$ .

Is  $W$  a v.s.?

C - add? - mult by scalars?

E - zero? - yes  $-1 \cdot v = -v$  ... (true from axioms! Thm 2, P. 296)

A - true in  $V$ , so true in  $W$ .

Def 4 A subset  $W$  of a v.s.  $V$  is a subspace of  $V$  if  $W$  is

a v.s. when given the same ops as  $V$ .

e.g.  $\{(x, 2x) \mid x \in \mathbb{R}\}$  is a subspace of  $\mathbb{R}^2$  | e.g.  $\{0\}$  is a s.s. of  $V$

Thm 3 (Subspace Test) If  $V$  is a v.s. and  $W \subset V$ , then

P. 297  $W$  is a subspace of  $V$  if

- 1)  $0 \in W$
- 2)  $W$  is closed under add<sup>n</sup>
- 3) " " mult<sup>n</sup> by scalars.

SHORTCUT

$$\text{eg } T = \{ v \in \mathbb{R}^3 \mid v \cdot (1, -2, 1) = 0 \} \subset \mathbb{R}^3 \quad (6)$$

$$= \{ (x, y, z) \mid x - 2y + z = 0 \} \quad (\neq \mathbb{R}^3)$$

Subspace Test

1)  $0 \in T$  since  $(0, 0, 0) \cdot (1, -2, 1) = 0$   
 $(0, 0, 0)$

2) If  $u, v \in T$ , then

$$(u+v) \cdot (1, -2, 1) = u \cdot (1, -2, 1) + v \cdot (1, -2, 1)$$

$$= 0 + 0$$

$$\therefore u+v \in T \quad \therefore T \text{ closed under } +$$

3) If  $u \in T, k \in \mathbb{R}$ , then

$$(ku) \cdot (1, -2, 1) = k \cdot (u \cdot (1, -2, 1))$$

$$= k \cdot 0$$

$$= 0$$

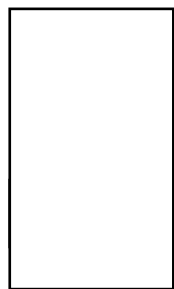
$$\therefore ku \in T \quad \therefore T \text{ closed under } \times \text{ by } \mathbb{R}$$

$\therefore T$  is a subspace of  $\mathbb{R}^3$

Remarks (1) Any plane through origin is a subspace of  $\mathbb{R}^3$  - there's nothing special about  $(1, -2, 1) = n$ .

(2) If a plane in  $\mathbb{R}^3$  doesn't contain 0, it's not a subspace of  $\mathbb{R}^3$

(3) Any line in  $\mathbb{R}^3$  (or  $\mathbb{R}^2$ ) thru 0 is a s.s. of  $\mathbb{R}^2$  (or  $\mathbb{R}^3$ )  
 (Same remark about lines not containing 0)



e.g.  $V = F[0, 1], \quad W = \{ f \in F[0, 1] \mid f(0) = 0 \}$

$0 : 0(0) = 0 \text{ so } 0 \in W$

$+ : f, g \in W \Rightarrow (f+g)(0) = f(0) + g(0) = 0$   
 $\therefore f+g \in W$

$k : f \in W, k \in \mathbb{R} \Rightarrow (kf)(0) = k \cdot f(0) = k \cdot 0 = 0 \therefore kf \in W$

$\therefore W$  is a subspace of  $F[0, 1]$

Transpose (p.7)  $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^t := \begin{bmatrix} a & c \\ b & d \end{bmatrix}$  ; can do it for  $m \times n$  matrices ⑦

e.g.  $S = \{ A \in M_{2 \times 2} \mid A^t = A \}$  Symmetric  $2 \times 2$  matrices

$$= \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_{\mathbb{R}} \mid \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \right\}$$
$$= \left\{ \begin{bmatrix} a & b \\ b & d \end{bmatrix} \mid a, b, d \in \mathbb{R} \right\}.$$

$0: \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}^t = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \therefore 0 \in S$

$+$ :  $\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix} = \begin{bmatrix} a+a' & b+b' \\ c+c' & d+d' \end{bmatrix} \in S \quad \therefore S$  closed under  $+$

$k$ :  $k \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ka & kb \\ kc & kd \end{bmatrix} \in S \quad \therefore S$  closed under mul. by scalar.

$\therefore S$  is a subspace of  $M_{2 \times 2}$

e.g. trace of square matrix

$$\text{trace} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = a + d ;$$

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{tr} A = a + d$$

$SL_2 = \{ A \in M_{2 \times 2} \mid \text{tr} A = 0 \}$  is a subspace of  $M_{2 \times 2}$

$$= \left\{ \begin{bmatrix} a & b \\ c & -a \end{bmatrix} \mid a, b, c \in \mathbb{R} \right\} \dots$$

(can do it for  $n \times n$ 's)

see #.

# Span & linear combinations

Infinite  $\rightarrow$  finite

(P. 299)

⑧

Recall

$$T = \{ (x, y, z) \mid x - 2y + z = 0 \} \quad \text{ss of } \mathbb{R}^3$$

$$S = \left\{ \begin{bmatrix} a & b \\ b & d \end{bmatrix} \mid a, b, d \in \mathbb{R} \right\} \quad \text{ss of } M_{22}$$

Rewrite T:

$$T = \{ (x, y, z) \mid x = 2y - z \}$$

$$= \{ (2y - z, y, z) \mid y, z \in \mathbb{R} \} \quad \text{no conditions left}$$

$$= \{ (2y, y, 0) + (-z, 0, z) \mid y, z \in \mathbb{R} \}$$

$$= \{ \underbrace{y}_{\text{linear comb of } (2, 1, 0)} + \underbrace{z}_{\text{linear comb of } (-1, 0, 1)} \mid y, z \in \mathbb{R} \}$$

linear comb of  $(2, 1, 0)$  &  $(-1, 0, 1)$

$$= \text{span} \{ (2, 1, 0), (-1, 0, 1) \}$$

$$S = \left\{ a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + d \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \mid a, b, d \in \mathbb{R} \right\}$$

$$= \text{span} \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

Def<sup>n</sup> (P. 299) ① If  $v_1, \dots, v_m$  are vectors in a v.s.  $V$ ,

and  $a_1, \dots, a_m$  are scalars, the vector

$$a_1 v_1 + \dots + a_m v_m \quad \text{is called a}$$

linear combination of  $v_1, \dots, v_m$ .

②  $\{ a_1 v_1 + \dots + a_m v_m \mid a_1, \dots, a_m \in \mathbb{R} \}$  is called

the span of  $v_1, \dots, v_m$ ; we write

$$\text{span} \{ v_1, \dots, v_m \} = \{ a_1 v_1 + \dots + a_m v_m \mid a_1, \dots, a_m \in \mathbb{R} \}$$

$\{ v_1, \dots, v_m \}$  is called a spanning set for  $\text{span} \{ v_1, \dots, v_m \}$

OR

"

spans

"

③ A vector space (or subspace)  $W$  is spanned ⑨  
 by  $v_1, \dots, v_m \in W$  if  $W = \text{span}\{v_1, \dots, v_m\}$  ;  
 " $\{v_1, \dots, v_m\}$  spans  $W$ "

e.g.  $T = \{y(2, 1, 0) + z(-1, 0, 1) \mid y, z \in \mathbb{R}\}$

$= \text{span}\{ \overset{v_1}{(2, 1, 0)}, \overset{v_2}{(-1, 0, 1)} \}$  ;  $\{v_1, v_2\}$  spans  $T$

$\{v_1, v_2\}$  is a spanning set for  $T$

$S = \text{span}\left\{ \underset{M_1}{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}, \underset{M_2}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}, \underset{M_3}{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}} \right\}$

$\{M_1, M_2, M_3\}$  spans  $S$ ;  $S$  is spanned by  $M_1, M_2$  &  $M_3$ .

e.g. - Let  $L = \text{span}\{(0, 1, 0)\} = \{y(0, 1, 0) \mid y \in \mathbb{R}\}$   
 $= \{(0, y, 0) \mid y \in \mathbb{R}\}$  ( $y$ -axis)

is a s.s. of  $\mathbb{R}^3$ ! (like others...)

Thm 4 P. 299 Let  $V$  be a v.s. If  $\{v_1, \dots, v_m\} \subset V$ , then

(1)  $U = \text{span}\{v_1, \dots, v_m\}$  is always a subspace of  $V$

(2) If  $W$  is any s.s. of  $V$  s.t.  $W \supseteq \{v_1, \dots, v_m\}$  then

$$W \supseteq U$$

i.e.  $U = \text{span}\{v_1, \dots, v_m\}$  is the smallest s.s. of  $V$  containing  $v_1, \dots, v_m$ .

Pf. (1) (check s.s. test.) I  $0 = 0v_1 + 0v_2 + \dots + 0v_m \in U$

II If  $u = a_1v_1 + \dots + a_mv_m$  &  $v = b_1v_1 + \dots + b_mv_m$ , then

$$u+v = (a_1+b_1)v_1 + \dots + (a_m+b_m)v_m \in U$$

III If  $u = a_1v_1 + \dots + a_mv_m$  &  $k \in \mathbb{R}$ ,

$$ku = (ka_1)v_1 + \dots + (ka_m)v_m \in U.$$

(2) - exercise.

e.g.  $\{(x, y, x-y) \mid x, y \in \mathbb{R}\} = \text{span}\{(1, 0, 1), (0, 1, -1)\}$  (b)  
 $\therefore$  is a s.s. of  $\mathbb{R}^3$

e.g.  $V = F(\mathbb{R})$ ,  $H = \text{span}\{f, g\}$  is a s.s. of  $F(\mathbb{R})!$

$f(x) = \cos x$

$g(x) = \sin x$

$= \{af + bg \mid a, b \in \mathbb{R}\}$

(ex. show,  $h(x) = \sin(x+1)$ ,  
 $= (\sin x) \cos 1 + (\cos x) \sin 1$ )

then  $h \in H!$

hint: trig identity for  $\sin(a+b)$

$= a \sin x + b \cos x$

( $a = \cos 1$ ,  $b = \sin 1$ )

note: if  $k(x) = 1$ ,  $\forall x \in \mathbb{R}$ , then  $k \notin H$ :

Suppose  $k(x) = a \cos x + b \sin x$ ,  $\forall x \in \mathbb{R}$   
 $k = af + bg$ , fixed  $a, b \in \mathbb{R}$

Choose convenient values of  $x$

$x = 0$  :  $k(0) = 1 = a \cos 0 + b \sin 0$   
 $\Rightarrow 1 = a$

$x = \frac{\pi}{2}$  :  $k(\frac{\pi}{2}) = 1 = a \cos \frac{\pi}{2} + b \sin \frac{\pi}{2}$   
 $\Rightarrow 1 = b$

$x = \pi$  :  $k(\pi) = 1 = a \cos \pi + b \sin \pi$   
 $\Rightarrow 1 = -a$

!

$\therefore k \notin \text{span}\{f, g\} = H$ .

e.g.

e.g. Diagonal matrices (2x2)

$$D_2 = \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \mid a, b \in \mathbb{R} \right\} \text{ is a SS of } M_{22}$$

$$\begin{aligned} \text{because } D_2 &= \left\{ a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \mid a, b \in \mathbb{R} \right\} \\ &= \text{span} \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\} \end{aligned}$$

$$- D_3 = \left\{ \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} \mid a, b, c \in \mathbb{R} \right\}$$

$$= \text{span} \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\} \text{ is a}$$

S.S. of  $M_{33}$

e.g. Subspaces of  $\mathbb{R}$

$\{0\}$  ✓

$\mathbb{R}$  ✓

any others?

Suppose  $U$  is a SS of  $\mathbb{R}$ , and that  $U \neq \{0\}$ ; i.e. there is  $u \neq 0, u \in U$ . Claim:  $U = \mathbb{R}$ . Let  $r \in \mathbb{R}$  be any real number.

e.g. Subspaces of  $\mathbb{R}^2$   $\{(0,0)\}$ ,  $\mathbb{R}^2$ , lines thru  $(0,0)$ . (14)

Theorem: these are the only ss. of  $\mathbb{R}^2$ .

Suppose  $W$  is a ss. of  $\mathbb{R}^2$  and  $W \neq \{(0,0)\}$  i.e.  $W$  contains a non-zero vector  $w_0$  ( $\therefore$  as many!), so  $\text{span}\{w_0\} \subset W$

Case 1  $W = \text{span}\{w_0\}$ . In this case  $W$  is the line thru  $0$  with dir<sup>n</sup>  $w_0$ .

Case 2 There is a vector in  $W$  which is not in  $\text{span}\{w_0\}$  i.e.

there is  $0 \neq w_1 \in W$  s.t.  $w_1$  and  $w_0$  are not parallel.

Then, <sup>as  $W$  is a ss.</sup>  $\text{span}\{w_0, w_1\} \subset W$ ; claim:  $\text{span}\{w_0, w_1\} = \mathbb{R}^2 = W$ .

Since  $\text{span}\{w_0, w_1\} \subset W \subset \mathbb{R}^2$ , it's enough to show that  $\mathbb{R}^2 \subset \text{span}\{w_0, w_1\}$  i.e. any  $v \in \mathbb{R}^2$  is a l.c. of  $w_0$  &  $w_1$

The fact that  $w_0, w_1$  aren't  $\parallel$  means that

area  $\left( \begin{array}{c} \vec{w}_1 \\ \vec{w}_0 \end{array} \right) \neq 0$  i.e. if  $w_0 = (x_0, y_0)$  then  $w_1 = (x_1, y_1)$

$$x_0 y_1 - x_1 y_0 \neq 0.$$

Then, if  $(x, y) \in \mathbb{R}^2$ , one can, <sup>easily</sup> check that

$$(x, y) = \left( \frac{x y_1 - y x_1}{x_0 y_1 - x_1 y_0} \right) w_0 + \left( \frac{x_0 y - y_0 x}{x_0 y_1 - x_1 y_0} \right) w_1$$

check (x comp:  $\frac{(x y_1 - y x_1) x_0 + (x_0 y - y_0 x) x_1}{x_0 y_1 - x_1 y_0} = x \frac{(x_0 y_1 - x_1 y_0)}{x_0 y_1 - x_1 y_0} = x$ )

e.g. Subspaces of  $\mathbb{R}^3$

$\{(0,0,0)\}, \mathbb{R}^3$ , lines thru 0, & planes thru 0. (15)

Theorem (later) These are the only subspaces of  $\mathbb{R}^3$ .

for now: if  $V = \text{span}\{v_0\}$ ,  $v_0 \neq 0$  in  $\mathbb{R}^3$ ,  $V$  is a line

if  $V = \text{span}\{v_0, v_1\}$  and neither  $v_0, v_1$  aren't parallel,

then: claim  $V = \{v \in \mathbb{R}^3 \mid v \cdot (v_0 \times v_1) = 0\}$  is the plane through 0 with normal  $v_0 \times v_1$ !

(i)  $V \subseteq$  plane described above: let  $v = av_0 + bv_1$ . then

$$v \cdot (v_0 \times v_1) = (av_0 + bv_1) \cdot (v_0 \times v_1) = a(v_0 \cdot v_0 \times v_1) + b(v_1 \cdot v_0 \times v_1) = a \cdot 0 + b \cdot 0 = 0$$

$\therefore v \in$  plane.

(ii) plane above  $\subseteq V$  i.e.  $v \cdot (v_0 \times v_1) = 0 \implies v \in \text{span}\{v_0, v_1\}$

Let  $n = v_0 \times v_1$ . Then  $n \neq 0$  because  $v_0, v_1$  aren't parallel.

Recall  $u_1 \times (u_2 \times u_3) = (u_1 \cdot u_3)u_2 - (u_1 \cdot u_2)u_3$  \* (1.65(f))

Suppose  $v \cdot n = 0$ .

$$\text{Then } n \times (v \times n) = \|n\|^2 v - (n \cdot v)n = \|n\|^2 v$$

Now expand

$$\text{span}\{(1,1,2), (-1,1,0)\}$$

$$= \{v \mid v \cdot (-1, -1, 2) = 0\}$$

$$= \{v \mid (v_1, v_2, v_3) \cdot (-1, -1, 2) = 0\}$$

$$= \text{span}\{(1,1,2), (-1,1,0)\}$$

$$n \times (v \times n)$$

using get

$$v \times n = av_0 + bv_1$$

$$= n \times (av_0 + bv_1)$$

$$= -a v_0 \times n - b v_1 \times n$$

$$= a'v_0 + b'v_1$$

expand again

See web page for a'

# 4.7 Linear Dependence & Independence

16

generalize geometric notions

2 vectors <sup>are /</sup> aren't parallel, lie / in a line collinear

3 vectors <sup>lie / don't lie</sup> in a plane  
<sup>are / are not</sup> coplanar

need to express these ideas algebraically

- necessitates talking about all vectors, including  $0$ , include degenerate case

Suppose 2 vectors  $u, v$  are collinear

This is equivalent to: either  $u = kv$  for some  $k \in \mathbb{R}$  (unless  $v = 0$ )  
or  $v = lu$  for some  $l \in \mathbb{R}$  ( $l \neq 0$ )

These two statements can be summarized as follows:

There <sup>are</sup> scalars  $a, b \in \mathbb{R}$ , not both zero s.t.  $au + bv = 0$   
( $a \neq 0 \Rightarrow u = (-\frac{b}{a})v$ ; or  $b \neq 0 \Rightarrow v = (-\frac{a}{b})u$ )

Similarly, "3 vectors  $u, v, w$  lie in a plane" is equivalent

to there are scalars  $a, b, c \in \mathbb{R}$ , not all zero, s.t.  
 $au + bv + cw = 0$ .

CONTRAPOSITIVE is also important 2 vectors  $u, v$  do not lie in a line  $\Leftrightarrow$

2 non zero vectors  $u, v$  are not parallel  $\Leftrightarrow$

$$au + bv = 0 \Rightarrow a = b = 0 \quad (\Leftarrow \text{clearly true, but has no info})$$

3 vectors  $u, v, w$  do not lie in a plane  $\Leftrightarrow$  for scalars  $a, b, c$ ,

$$au + bv + cw = 0 \Rightarrow a = b = c = 0$$

"

If  $V$  is a vector space,  $v_1, \dots, v_m$  are vectors in  $V$

(17)

Defn P.126  $\{v_1, \dots, v_m\}$  is linearly dependent  
 $v_1, \dots, v_m$  are

if there are scalars  $a_1, \dots, a_m$ , not all zero, s.t.

$$a_1 v_1 + \dots + a_m v_m = 0.$$

$\{v_1, \dots, v_m\}$  is linearly independent  
 $v_1, \dots, v_m$  are

if for scalars  $a_1, \dots, a_m$ ,  $a_1 v_1 + \dots + a_m v_m = 0 \Rightarrow a_1 = a_2 = \dots = a_m = 0$   
 implies

$$\text{L.I.} = \neg(\text{L.D.})$$

e.g.  $\{(1,0), (0,1)\}$  is L.I. since  $a_1(1,0) + a_2(0,1) = (0,0)$   
 $\Rightarrow (a_1, a_2) = (0,0) \Rightarrow a_1 = a_2 = 0$

$\{(1,-1), (1,1)\}$  is L.I. since  $a_1(1,-1) + a_2(1,1) = (0,0)$   
 $\Rightarrow \begin{cases} a_1 + a_2 = 0 \\ -a_1 + a_2 = 0 \end{cases} \Rightarrow a_1 = a_2 = 0$

$\{(1,0), (0,1), (1,1)\}$  is L.D. since  $1 \cdot (1,0) + 1 \cdot (0,1) - 1 \cdot (1,1) = (0,0)$ !  
rmk:  $(1,0) \in \text{span}\{(0,1), (1,1)\}$  (1.9)

$\left\{ \begin{matrix} [10] \\ [00] \end{matrix}, \begin{matrix} [01] \\ [00] \end{matrix}, \begin{matrix} [00] \\ [10] \end{matrix}, \begin{matrix} [00] \\ [01] \end{matrix} \right\}$  is L.I. since

$$a_1 M_1 + a_2 M_2 + a_3 M_3 + a_4 M_4 = 0 \Rightarrow \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow a_1 = \dots = a_4 = 0.$$

$\left\{ \begin{matrix} [10] \\ [00] \end{matrix}, \begin{matrix} [00] \\ [01] \end{matrix}, \begin{matrix} [-10] \\ [01] \end{matrix} \right\}$  is L.D. since  $A_1 - A_2 + A_3 = 0$   
rmk:  $A_1 \in \text{span}\{A_2, A_3\}$

e.g.  $\{1, \sin x, \cos x\}$  is l.o.c. in  $F[0, 2\pi]$  :

Suppose  $a1 + b \sin x + c \cos x = 0 \quad \forall x \in [0, 2\pi]$  \*

$$\left. \begin{array}{l} \text{at } x=0 \rightarrow a + c = 0 \\ x = \pi/2 \rightarrow a + b = 0 \\ x = \pi \rightarrow a - c = 0 \end{array} \right\} \Rightarrow a=b=c=0.$$

e.g.  $\{1, \sin^2 x, \cos^2 x\}$  is l.o.d. in  $F[0, 2\pi]$

Since  $1 - \sin^2 x - \cos^2 x = 0, \forall x \in [0, 2\pi]$

i.e.  $f - g - h = 0$  as functions.

rnks:  $\cos^2 x \in \text{span}\{1, \sin^2 x\}$

$\left\{ \begin{array}{l} \sin^2 x \\ x-\pi \\ \dots \end{array} \right\}$

$\Rightarrow \text{span}\{1, \sin^2 x, \cos^2 x\} = \text{span}\{1, \sin^2 x\}$

e.g.  $\{(1, 2, 1)\}$  is l.o.c. since  $a(1, 2, 1) = (0, 0, 0) \Rightarrow a=0$

FACT If  $v \in V$  is  $\{0\}$  is l.o.c.  $\Leftrightarrow v \neq 0$ .

e.g.  $\{(0, 0, 0, 0), (1, 2, 3, 4)\}$  is l.o.d. since  $1(0, 0, 0, 0) + 0(1, 2, 3, 4) = 0$

FACT Any set containing the zero vector is l.o.d.

FACT  $\{u, v\}$  is l.o.d.  $\Leftrightarrow u = kv$  or  $v = lu$  (in  $\mathbb{R}^3 \Leftrightarrow u, v$  are collinear) (in  $\mathbb{R}^2$ : for  $z=0, u, v$ )

FACT  $\{u, v, w\}$  is l.o.d.  $\Leftrightarrow u \in \text{span}\{v, w\}$  or  $v \in \text{span}\{u, w\}$  or  $w \in \text{span}\{u, v\}$  \*

If  $u, v, w$  live in  $\mathbb{R}^3$ , then

(19)

$\{u, v, w\}$  l.d.  $\Leftrightarrow$   $u, v, w$  are coplanar.  
(or collinear).

In general vector spaces can only say  $\neq$

19. Lemma (4.10) A set  $\{v_1, \dots, v_n\} \subset V$  is l.d.

$\Leftrightarrow$  at least one of the vectors is a l.c. of the

others. Pf: " $\Rightarrow$ " If  $a_1 v_1 + \dots + a_n v_n = 0$ , and  $a_n \neq 0$ , then  $v_n = -\frac{a_1}{a_n} v_1 - \frac{a_2}{a_n} v_2 - \dots - \frac{a_{n-1}}{a_n} v_{n-1}$

Notes

$\{v_1, \dots, v_n\}$  l.d.  $\Rightarrow$  (say  $v_n \in \text{span}\{v_1, \dots, v_{n-1}\}$ ) then  
 $v_n = b_1 v_1 + \dots + b_{n-1} v_{n-1}$

$$\text{span}\{v_1, \dots, v_n\} = \text{span}\{v_1, \dots, v_{n-1}\}.$$

Since  $a_1 v_1 + \dots + a_n v_n = (a_1 + b_1) v_1 + \dots + (a_{n-1} + b_{n-1}) v_{n-1}$

SoB. This says a linearly dependent spanning set may be reduced  
in size i.e. Suppose  $W = \text{span}\{v_1, \dots, v_n\}$

If  $\{v_1, \dots, v_n\}$  is l.d., (suppose  $v_n \in \text{span}\{v_1, \dots, v_{n-1}\}$ )

then  $W = \text{span}\{v_1, \dots, v_{n-1}\}$

$$\text{e.g. } W = \text{span}\left\{ \begin{matrix} v_1 \\ (1, 0, 0) \end{matrix}, \begin{matrix} v_2 \\ (0, 1, 0) \end{matrix}, \begin{matrix} v_3 \\ (1, 1, 0) \end{matrix} \right\}$$

But  $v_3 = v_1 + v_2$ , so  $W = \text{span}\{v_1, v_2\}$  ( $x-y$  plane)

e.g. Since  $1 = \cos^2 x + \sin^2 x$  in  $F(\mathbb{C}, 2\pi)$ ,

$$\text{span}\{1, \sin^2 x, \cos^2 x\} = \text{span}\{\cos^2 x, \sin^2 x\}$$

$$= \text{span}\{1, \sin^2 x\}$$

We can decrease the size of a l.o.s. spanning set  
 How can we increase the size of a l.o.s. set? (20)

FACT If  $\{v_1, \dots, v_n\}$  is l.o.s. in  $V \in V$ , then

$$\{v_1, v_1, \dots, v_n\} \text{ is l.o.s. } \Leftrightarrow v \notin \text{span}\{v_1, \dots, v_n\}$$

$\Rightarrow$  "  $\Leftarrow$  " <sup>Suppose</sup>  $av + a_1v_1 + \dots + a_nv_n = 0$ , if  $a \neq 0$ ,  $v \in \text{span}$ . So  $a = 0$ . Then  $a_1 = \dots = a_n = 0$ ; so  $a = a_1 = \dots = a_n = 0$ .

i.e. we can increase the size of a l.o.s. set if we can find a vector not in its span

e.g.  $\left\{ \begin{matrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \end{matrix} \right\}$  is l.o.s.;  $M_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \notin \text{span}\{M_1, M_2\}$ ,  
 $\left( \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ 0 \\ b \end{bmatrix} \right)$  has no sol<sup>n</sup>.

So  $\{M_1, M_2, M_3\}$  is l.o.s.

e.g.  $\left\{ \begin{matrix} v_1 \\ v_2 \end{matrix} \right\} = \{(2, 1, 0), (-1, 0, 1)\}$  is l.o.s. (neither is a mult<sup>le</sup> of other) <sup>OK for 2</sup>

Can we find  $v_3 \in \mathbb{R}^3$  st.  $v_3 \notin \text{span}\{v_1, v_2\}$ ? We know

$$\begin{aligned} \text{span}\{v_1, v_2\} &= \{v \in \mathbb{R}^3 \mid v_1 \times v_2 \cdot v = 0\} & \begin{vmatrix} v_i & v_j & v_k \\ 2 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix} &= (1, -2, 1) \\ &= \{(x, y, z) \mid x - 2y + z = 0\} = S \end{aligned}$$

Choose  $v = (1, 0, 0) \notin S$  e.s.  $\therefore$

$\{(2, 1, 0), (-1, 0, 1), (1, 0, 0)\}$  is l.o.s.

(many choices)

e.s.  $\{(2, 1, 0), (-1, 0, 1), (1, -2, 1)\}$  is l.o.s.

(21)

e.g.  $\{(1, 0), (0, 1)\}$  is l.i. in  $\mathbb{R}^2$ . Can we

find  $v_3 \in \mathbb{R}^2$  s.t.  $\{v_1, v_2, v_3\}$  is l.i.?

NO: since we know  $\text{span}\{v_1, v_2\} = \mathbb{R}^2$ !

$\therefore \forall v \in \mathbb{R}^2$ ,  $\{v_1, v_2, v\}$  is l.d.

Same for  $\{(1, -1), (1, 1)\}$  :  $\{w_1, w_2, w\}$  is l.d.  $\forall w \in \mathbb{R}^2$ .

Indeed, if  $u_1, u_2$  are any 2 non zero, non parallel v. in  $\mathbb{R}^2$ ,  $\{u_1, u_2, u\}$  is l.d. for any choice of  $u \in \mathbb{R}^2$  s.t.  $\text{span}\{u_1, u_2\} = \mathbb{R}^2$ .

Claim : Any 3 vectors in  $\mathbb{R}^2$  are l.d.!

pf Suppose  $\{v_1, v_2, v_3\}$  is l.i. in  $\mathbb{R}^2$ . Then,  $\{v_1, v_2\}$  is l.i. (any subset of a l.i. set is l.i.) So, as before  $\{v_1, v_2, v_3\}$  is l.d. for any  $v \in \mathbb{R}^2$ .

FACT: 4 or more vectors in  $\mathbb{R}^3$  are l.d.

Thm 4.13 Lemma If a v.s.  $V$  can be spanned by  $n$  vectors, then any l.i. subset has

at most  $n$  vectors. ( $\Leftrightarrow$  if  $V$  has a subset of  $m$  l.i., any spanning set has at least  $m$  vectors)

Size of any spanning set  $\geq$  size of any l.i.o. subset of  $V$  (22)

(pf: later...  $\mathbb{R}^3$  is spanned by  $P_1, P_2, P_3$   
so any 4 or more vectors are l.i.o.)

eg. •  $\mathbb{R}^4$  is spanned by 4 vectors - so any 5 or more vectors in  $\mathbb{R}^4$  are l.i.o.

•  $M_{22}$  is spanned by 4 vectors, so any 5 or more matrices in  $M_{22}$  are l.i.o.

•  $D = \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \in M_{22} \mid a, b \in \mathbb{R} \right\}$  is spanned by 2 matrices, so any 3 or more matrices in

$D$  are l.i.o.

•  $U = \{ (x, y, z) \mid x+z=0 \} = \text{span}\{ (1, 0, -1), (0, 1, 0) \}$   
is spanned by 2 vectors, so any 3 or more vectors in  $U$  are l.i.o. (while  $\exists 3$  l.i.o.

in  $\mathbb{R}^3$ , ... eg  $\{ (1, 0, -1), (0, 1, 0), (1, 0, 1) \}$  is l.i.o. in  $\mathbb{R}^3$   
not in  $U$ .

# Basis & Dimension

Def<sup>n</sup> A set  $\{v_1, \dots, v_n\} \subset V$  is a basis of  $V$  if

(1)  $\{v_1, \dots, v_n\}$  is l.i.

(2) " spans  $V$

(largest l.i. set, smallest spanning set)

eg  $\{(1,0), (0,1)\}$  is a basis of  $\mathbb{R}^2$

$\{(1,1), (1,-1)\}$  " "

Do all bases have same # of elts?

Thm 2 (P. 304 P. 195) If  $\{v_1, \dots, v_n\}$  and  $\{w_1, \dots, w_m\}$  are bases of  $V$ , then  $n = m$ .

Pf. Since  $\{v_1, \dots, v_n\}$  spans  $V$  and  $\{w_1, \dots, w_m\}$  is l.i.  
 $n \geq m$

"  $\{w_1, \dots, w_m\}$  spans  $V$  and  $\{v_1, \dots, v_n\}$  " "  
 $m \geq n$ .

Def<sup>n</sup> P. 304 P. 195 If  $V$  has a finite basis  $\{v_1, \dots, v_n\}$ , the dimension of  $V$  is  $n$  (" $\dim V = n$ ")

$V$  is s.t.b finite dimensional.

$$\begin{aligned} \text{e.g. } SL_2 &= \left\{ A \in M_{2 \times 2} \mid \text{tr } A = 0 \right\} \quad \text{tr} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = a+d \quad (24) \\ &= \left\{ \begin{bmatrix} a & b \\ c & -a \end{bmatrix} \mid a, b, c \in \mathbb{R} \right\} \quad (\text{seen before}) \\ &= \text{span} \left\{ \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}}_{M_1}, \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{M_2}, \underbrace{\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}}_{M_3} \right\}; \end{aligned}$$

So  $\{M_1, M_2, M_3\}$  spans  $SL_2$ ; moreover,

$$aM_1 + bM_2 + cM_3 = 0 \Rightarrow \begin{bmatrix} a & b \\ c & -a \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow a = b = c = 0$$

$\therefore \{M_1, M_2, M_3\}$  is l.i.

$\therefore \{M_1, M_2, M_3\}$  is a basis of  $SL_2$

$$\therefore \dim SL_2 = 3$$

$$\text{e.g. } W = \text{span} \{1, \sin x, \cos x\} \quad (\text{s.s. of } F[0, 2\pi])$$

$\{1, \sin x, \cos x\}$  spans  $W$  (by defn of  $W$ )

$\{1, \sin x, \cos x\}$  is l.i. (shown last time)

$\therefore \{1, \sin x, \cos x\}$  is a basis of  $W$ .

$$\therefore \dim W = 3$$

$V = \mathbb{R}^n$ ;  
e.g.  $e_i = (0, 0, \dots, 0, 1, 0, \dots, 0)$   
i<sup>th</sup> position

$\{e_1, \dots, e_n\}$  is a basis of  $\mathbb{R}^n$

eg  $D = \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \mid a, b \in \mathbb{R} \right\}$

(25)

$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$  is a basis of  $D$  l.i. ✓  
span ✓

$\therefore \dim D = 2$

eg  $U = \left\{ (x, y, z) \mid x + z = 0 \right\}$

$= \text{span} \left\{ \overset{v_1}{(1, 0, -1)}, \overset{v_2}{(0, 1, 0)} \right\}$ ;  $\left\{ (1, 0, -1), (0, 1, 0) \right\}$  is  
also l.i.  $\therefore \{v_1, v_2\}$  is a basis of  $U$ .

$\dim U = 2$

eg  $\dim M_{22} = 4$  since  $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \dots \right\}$  is a basis

$\dim M_{mn} = mn$  since  $\left\{ E_{ij} \in M_{mn} \mid E_{ij} \text{ has } 1 \text{ in } i^{\text{th}} \text{ row \& } j^{\text{th}} \text{ col; zero elsewhere} \right\}$   
is a basis

e.g.  $F(\mathbb{R}) = \{f: \mathbb{R} \rightarrow \mathbb{R}\}$  does not have finite

dimension since,  $\{1, x, x^2, \dots, x^n\}$  is l.i. for

any  $n \in \mathbb{N}$ : Suppose  $p(x) = a_0 + a_1x + \dots + a_nx^n = 0$  for all  $x$

$\in \mathbb{R}$ . Then  $p(x)$ , a polynomial of degree (at most  $n$ ) has  
more than  $n$  roots! Impossible  $\therefore a_0 = a_1 = \dots = a_n = 0$ .

(Under the name)

$\dots \rightarrow 0$  to  $\dots \rightarrow 0 = 0$

Remark

RECALL

Size of any spanning set  $\geq \dim V \geq$  size of an l.i. set for  $V$

26

How can we find bases for  $V$ ?

Thms 3.4 Pgs 306-307 (197-198) Suppose  $\dim V = n < \infty$ . Then

(1) Any l.i. set  $\{v_1, \dots, v_n\} \subset V$  is a basis (i.e. spans too!)

(2) Any spanning set  $\{w_1, \dots, w_n\} \subset V$  " (i.e. is l.i. too)

(3) If  $\{v_1, \dots, v_m\}$  is l.i., there are  $v_{m+1}, \dots, v_n$  s.t.  $\{v_1, \dots, v_m, v_{m+1}, \dots, v_n\}$  is a basis (every l.i. set can be extended to a basis)

(4) If  $S = \{w_1, \dots, w_n\}$  spans  $V$ , there is a subset of  $S$  that is a basis of  $V$ .

f) 1) If  $\{v_1, \dots, v_n\}$  didn't span  $V$ ,  $\exists v \in V$  s.t.  $\{v, v_1, \dots, v_n\}$  is l.i., contradicting \*

2) If  $\{w_1, \dots, w_n\}$  is l.i., then (say)  $w_1 \in \text{span}\{w_2, \dots, w_n\}$ ; so

$V = \text{span}\{w_1, \dots, w_n\} = \text{span}\{w_2, \dots, w_n\}$  is spanned by  $(n-1)$  vectors, contradicting \*.

3) ... later;

4) ...

2/2/04

e.g. (for 1)  $U = \{(x, y, z) \mid x + z = 0\}$  ; know  $\dim U = 2$ .

Since  $(1, 1, -1)$  &  $(0, 1, 0)$  both belong to  $U$  and are l.i.,

$\{(1, 1, -1), (0, 1, 0)\}$  is another basis for  $U$ .

e.g. for (3.) Note:  $U \subset \mathbb{R}^3$ ,  $\{(1, 1, -1), (0, 1, 0)\}$  is l.i.. let's extend this (basis of  $U$ ) to a basis of  $\mathbb{R}^3$ : need  $v \notin U$  e.g.  $v = (1, 0, 0)$ .

Then, we know  $\{v, v_1, v_2\}$  is l.i.. Since  $\dim \mathbb{R}^3 = 3$ ,  $\{v, v_1, v_2\}$  is a basis of  $\mathbb{R}^3$ .

8/10/22

What if  $\dim W = n$  and  $U \subseteq W$  is a subspace?

Thm 5  
(P. 307)

If  $\dim W = n$ , and  $U$  is a s.s. of  $W$  then  
1)  $\dim U \leq \dim W$

2)  $\dim U = \dim W \iff U = W$ .

Pf. 1) take a basis of  $U$ ,  $\{u_1, \dots, u_m\}$ , extend to a basis of  $W$ . So  $\dim U \leq \dim W$  (since  $U \subseteq W$ )

2) If  $\{u_1, \dots, u_n\}$  is a basis of  $U$ ; since  $\dim W = n$  also, by 4.14-4.16 (1), it's a basis of  $W$ . But  $U = \text{span}\{u_1, \dots, u_n\} = W$ , so  $U = W$ .

e.g.  $U = \{(x, y, z) \mid x+z=0\}$   $\dim U = 2$ ,  $\{(1, 1, -1), (0, 1, 0)\}$  basis; can extend to a basis of  $\mathbb{R}^3$ : need  $v \notin U$  e.g.  $v = (1, 0, 0) \dots$

Corollary If  $\{u, v, w\}$  is non-coplanar, then  $\{u, v, w\}$  is a basis of  $\mathbb{R}^3$ .  
( $\{u, v, w\}$  non-coplanar  $\iff \{u, v, w\}$  l.i.  $\iff u \times v \cdot w \neq 0$ )

e.g. Any 4-dim'l subspace of  $M_{22}$  is  $M_{22}$

e.g. Any 2-dim'l s.s. of  $U = \{(x, y, z) \mid x+z=0\} \subseteq U$ .

What are bases good for? (5.4.3)

Suppose  $B = \{v_1, \dots, v_n\}$  is a basis of  $V$ ; Then for every  $v \in V$

there are unique scalars  $x_1, \dots, x_n \in \mathbb{R}$  s.t.

$$v = x_1 v_1 + x_2 v_2 + \dots + x_n v_n$$

$$(v = y_1 v_1 + \dots + y_n v_n$$

$$\implies 0 = (x_1 - y_1)v_1 + \dots + (x_n - y_n)v_n$$

$$\implies x_i = y_i$$

- exist:  $B$  spans  $V$   
- unique because  $B$  l.i.

$$\text{Fix } \mathcal{B} = \{v_1, \dots, v_n\}$$

(idea)  $\checkmark$

$\mathbb{R}^n$

(28)

(29)

$$v = x_1 v_1 + \dots + x_n v_n \iff (x_1, \dots, x_n)$$

$M_{2 \times 2}$

$\mathbb{R}^4$

$\mathcal{B} = \text{Standard basis}$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\iff (a, b, c, d)$$

$SL_2$

$\mathbb{R}^3$

$$= \left\{ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\}$$

$$\begin{bmatrix} a & b \\ c & -a \end{bmatrix}$$

$$\implies (a, b, c)$$

All vector spaces of dimension  $n$   
(no matter how weird they look)  
are exactly like  $\mathbb{R}^n$ !

1.1 - some examples

## 1.2 Linear Systems

( $6 \times 10^6$  eqs  
 $5 \times 10^6$  variables)

air flow  $v, t, p$   $\rightarrow$   $\ast$   
adj.  $h, p, \dots$   
1000 nodes  
1000 nodes  
(30)

eg.  $\ast$   $\left\{ \begin{array}{l} X_1 + X_2 + X_4 = 1 \\ X_3 - X_4 = -1 \end{array} \right.$   $\left. \begin{array}{l} 2 \text{ eqs} \\ 4 \text{ unknowns} \end{array} \right\}$

Annotations:  $X_1, X_2, X_3, X_4$  are unknowns; coefficients are  $1, 1, 0, 1$  and  $0, 1, 0, -1$ ; RHS (constant term) are  $1$  and  $-1$ .

$(1, 0, 0, 0)$  is not a sol<sup>n</sup> of  $\ast$  (doesn't satisfy 2<sup>nd</sup> eqn)

$(2, 0, -2, -1)$  is a sol<sup>n</sup> of  $\ast$

• General sol<sup>n</sup> to a system is the set of all sol<sup>n</sup>s:

Claim:  $S = \{ (1-s-t, s, t-1, t) \mid s, t \in \mathbb{R} \}$  is the general sol<sup>n</sup>

to  $\ast$  & check (I)  $(1-s-t, s, t-1, t)$  is a sol<sup>n</sup> of  $\ast$  for every  $s, t \in \mathbb{R}$

$$X_1 + X_2 + X_4 = (1-s-t) + s + t = 1 \quad \checkmark$$

$$X_3 - X_4 = (t-1) - t = -1 \quad \checkmark$$

(II) Every sol<sup>n</sup> of  $\ast$  is in  $S$ : Use first eqn to

write  $X_1 = 1 - X_2 - X_4$   $\therefore (X_1, X_2, X_3, X_4)$  is a sol<sup>n</sup> to  $\ast$

2<sup>nd</sup>  $X_3 = -1 + X_4$   $\Leftrightarrow (X_1, X_2, X_3, X_4) = (1 - X_2 - X_4, X_2, -1 + X_4, X_4)$

$\ast$  Change names of  $X_2, X_4$  to  $s, t$  to obtain  $S$  above.

10/05  $s, t \leftarrow$  called the parameters in gen'l sol<sup>n</sup>.

Note:  $\ast$  has only many sol<sup>n</sup>s

eg.  $X_1 + 2X_2 + 3X_3 = 4$

$$X_2 + X_3 = 5$$

$$X_3 = 1$$

$3 \times 3$  system. (3 plans)

has a unique sol<sup>n</sup>  $(-7, 4, 1)$   
(only one)

e.g.  $x + y = 1$

$2x + 2y = 1$

(2x2 system)

(// lines)

has no solutions at all

Def<sup>n</sup> (2.11) • A system that has no solutions is inconsistent.

A system that has at least 1 sol<sup>n</sup> is consistent.

e.g.  $x_1 + x_2 + x_4 = 0$

$x_3 - x_4 = 0$

2x4

is called a homogeneous system (RHS consists of 0's)

FACT

• Homogeneous systems are always consistent since

$0 = (0, 0, 0, 0)$  is always a sol<sup>n</sup>.

12/2/04

these examples exhibit all types of behaviour!

$a_1x_1 + a_2x_2 + \dots + a_nx_n = b$   
unknowns, variables

Thm 4, p19 Any linear system with real (or complex) coefficients has

- (Field = R) either (i) a unique sol<sup>n</sup>  
(ii) no sol<sup>n</sup> or  
(iii)  $\infty$  many sol<sup>n</sup>

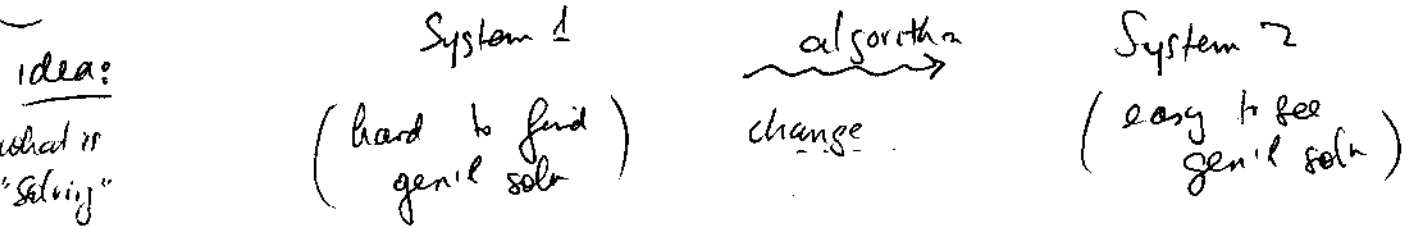
(NoB. not true for non-linear systems.  $x^2 - 64$  has exactly 2 sol<sup>n</sup>)

Remark: • a degenerate <sup>linear</sup> eqn is of the form

$0x_1 + 0x_2 + \dots + 0x_n = b$

no sol<sup>n</sup> if  $b \neq 0$  or  $\infty$  sol<sup>n</sup> if  $b = 0$

(1.2.2) Equivalent linear systems, Gaussian Elimination



such that N.B. gen'l soln sys 1 = gen'l soln sys 2  
 i.e. sys 1 & sys 2 have the same set of solutions

What do we need ① A special "easy" form for sys 2 that's easy to solve: so called echelon or row-RE form. <sup>(with)</sup> reduced row echelon

② The algorithm that will take System 1 to sys 2 - in reduced row echelon form.

③ Determine general soln from RRE form.  
 • To be efficient, we'll drop the variable names, addition signs, "equals" signs

eg.  $x_1 + 2x_2 + 3x_3 + 4x_4 = 5$   
 $x_1 + x_2 + x_3 = 4$

will be written

$$\left[ \begin{array}{cccc|c} 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 2 & 3 & 4 \end{array} \right] \equiv [A \ b]$$

Coefficient matrix  $\underbrace{\hspace{10em}}$  RHS  
 (2x4)

$A = \begin{bmatrix} \dots \\ \dots \end{bmatrix}$   $b = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$

augmented matrix of the system (P. 12)

examples where gen'l soln is easy to find:

(34)

$$\begin{bmatrix} 1 & 0 & 0 & | & a \\ 0 & 1 & 0 & | & b \\ 0 & 0 & 1 & | & c \end{bmatrix}$$

$$(x_1, x_2, x_3) = (a, b, c) \quad (\text{unique soln})$$

$$\begin{bmatrix} 1 & a & 0 & b & | & e \\ 0 & 0 & 1 & d & | & f \\ 0 & 0 & 0 & 0 & | & g \end{bmatrix}$$

2 cases I  $g \neq 0$  : inconsistent  
(last eqn is  $0 = g \neq 0$ )

$$\text{II } g = 0$$

Gen'l soln is

(set  $x_3 = s$   
 $x_4 = t$ )

$$\begin{cases} x_1 = e - ad - bt \\ x_2 = b \\ x_3 = f - dt \\ x_4 = t \end{cases} ; s, t \in \mathbb{R}$$

$$\begin{bmatrix} 1 & a & 0 & 0 & | & b \\ 0 & 0 & 1 & 0 & | & c \\ 0 & 0 & 0 & 1 & | & d \end{bmatrix}$$

gen'l soln:

$$\begin{cases} x_1 = b - ad \\ x_2 = c \\ x_3 = c \\ x_4 = d \end{cases} ; c \in \mathbb{R}$$

↓ L12 FOT

With a small difference

Def<sup>n</sup> (P.15) A matrix is in echelon form (row-echelon) if (RE)

1. All zero rows are at the bottom
2. The first non-zero entry in each row is a 1 "leading 1" (pivot)
3. Each leading 1 is to the right of the leading 1 in rows above.

if, in addition

↳ Each leading 1 is the only non-zero entry in its column,

26/2/02 J413

RC

e.g. RRE :

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

(35)

RE  
(not RC)

$$\begin{bmatrix} 1 & 6 & 2 & -1 \\ 0 & 1 & 4 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

not RE

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 2 & 3 \end{bmatrix}$$

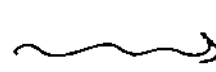
$$\begin{bmatrix} 1 & 2 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Remark: It's easier to write down the gen'l soln to a system if the coefficient matrix  $A$  of  $[A|b]$  is in RE form rather than just RE form. (Back-substitute already done.)

J413  
19/2/04

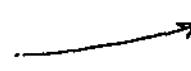
how to go from

$$[A|b]$$



$$[A|b]$$

with



in RE form?

P.15 Gaussian Algorithm

e.g.

$$\begin{bmatrix} 0 & 1 & 2 & 3 & | & 4 \\ 1 & 2 & 3 & 4 & | & 5 \\ 2 & 3 & 4 & 5 & | & 6 \end{bmatrix}$$

Step 1

If the matrix is zero, stop.

Step 2

Locate left most non-zero column.

(column 1)

Step 3 Interchange the top row with another, if necessary, to bring a non-zero <sup>entry</sup> to the top

$$R_1 \leftrightarrow R_2 \quad \begin{bmatrix} 1 & 2 & 3 & 4 & | & 5 \\ 0 & 1 & 2 & 3 & | & 4 \end{bmatrix}$$

Step 3 Divide the first row by whatever we need to get a leading one in the 1st row.

(nothing to do here)

(36)

Step 4 Add suitable multiples of row 1 to rows below it to make all entries below leading 1 zero:

$$-2R_1 + R_3 \rightarrow R_3 \quad \left[ \begin{array}{cccc|c} 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & -3 & -4 \end{array} \right]$$

Step 5 This completes ops for row 1 with the 1st row.

Ignore it & proceed to step 1.

(Step 1) (Step 2)

$$(Step 3) \quad R_2 + R_3 \rightarrow R_3 \quad \left[ \begin{array}{cccc|c} 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

(Step 1) ~~Step 2~~

--- stop.

Coeff matrix is now in RE form

(A3.5)

Step 6 RE  $\rightarrow$  RC. Begin with the right-most leading one, and add suitable multiples of the row it's in to rows above so that this leading one is the only non-zero entry in its column.

$$-R_2 + R_1 \rightarrow R_1 \quad \left[ \begin{array}{cccc|c} 1 & 0 & -1 & -2 & -3 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

Coeff matrix is now in RC form.

Topic: solving the  
General Sol<sup>n</sup> for RRE form.

(37)

1. Decide if system is inconsistent or not (rows of zeros in A)  
 If cons:  
 2. Assign parameters to "non-leading" variables  
 3. Solve for "leading variables" in terms of parameters

$$\left[ \begin{array}{cccc|c} \textcircled{1} & 0 & -1 & -2 & -3 \\ 0 & \textcircled{1} & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

$$\begin{aligned} x_1 &= -3 + 0 + 2t \\ x_2 &= 4 - 2s - 3t \\ x_3 &= s \\ x_4 &= t \end{aligned} \quad ; s, t \in \mathbb{R}.$$

gen'l soln.

So  $[A|b] \xrightarrow{\text{Gauss}} [\tilde{A}|\tilde{b}]$ . Are the general solns the same?

Def<sup>n</sup> 2 <sup>linear</sup> systems are equivalent if they have same gen'l soln.

"Gauss" is a sequence of "elementary row operation" (P.13)

- I. Interchange 2 rows  $R_i \leftrightarrow R_j$
- II. Multiply one row by a non-zero scalar  $aR_i \rightarrow bR_i \quad a \neq 0$
- III. Add a multiple of one row to another  $bR_i + R_j \rightarrow R_j$

Thm 1 (P.13) If an elem. row op<sup>n</sup> is performed on augmented matrix of a linear system, the resulting linear system is equivalent to the original one.

$$\left( \sqrt{2x+1} = x-7 \quad \dots \quad 0 = (x-4)(x-12) \right)$$

not equivalent

Def<sup>n</sup> 2 matrices A, B are row-equivalent ( $A \sim B$ ) if B can be obtained from A by a finite sequence of elem row ops.

$$\text{ex. } \left[ \begin{array}{ccc} 1 & 4 & 1 \\ & & \end{array} \right] \sim \left[ \begin{array}{ccc} 1 & 4 & 1 \\ 0 & -5 & -2 \end{array} \right] \sim \left[ \begin{array}{ccc} 1 & 4 & 1 \\ 0 & 1 & 2/5 \end{array} \right] \sim \left[ \begin{array}{ccc} 1 & 0 & -3/5 \\ 0 & 1 & 2/5 \end{array} \right]$$

The Gaussian algorithm always stops

Thm 2. P.16) Every matrix is row-equivalent to a (unique) matrix in RRE form. (RE forms aren't unique)

$$\text{eg } \begin{bmatrix} 1 & 4 & 1 \\ 2 & 3 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2/5 \\ 0 & 1 & 2/5 \end{bmatrix} \leftarrow \text{RRE}$$

No. of leading ones : this means that the # of leading 1's in any RE form = # leading ones in RC form.

1.2.4

Def<sup>n</sup> : (P.18) The rank of a matrix  $A$ ,  $\text{rank } A$ , is the # of leading ones ('pivots') in any RE form of  $A$

e.g.  $\text{rank} \begin{bmatrix} 1 & 4 & 1 \\ 2 & 3 & 0 \end{bmatrix} = \text{rank} \begin{bmatrix} 1 & 0 & -2/5 \\ 0 & 1 & 2/5 \end{bmatrix} = \text{rank} \begin{bmatrix} 1 & 4 & 1 \\ 0 & 1 & 2/5 \end{bmatrix} = 2$

rk doesn't change, becomes easier to see

$\text{rank} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} = 3$ , etc.

note:  $\text{rank } A \leq \# \text{ rows of } A$  (leading ones are in different rows.)  
 $\leq \# \text{ cols of } A$  ( " " " cols )

Connection of rank  $A$ , rank  $[A|b]$  to general solution of  $[A|b]$ .

$$[A|b] \rightsquigarrow \left[ \begin{array}{ccc|c} * & & & * \\ 0 & \dots & 0 & 0 \\ 0 & & & 0 \end{array} \right]$$

$$\sim \left[ \begin{array}{ccc|c} * & & & * \\ 0 & \dots & 0 & 0 \\ 0 & \dots & 0 & 0 \end{array} \right]$$

$$\text{rank } A \leq \text{rank } [A|b] \leq \text{rank } A + 1$$

## 3 examples

$$\textcircled{1} \left[ \begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & -1 & 1 & -1 & 1 \end{array} \right] \sim \left[ \begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & -1 & 1 \end{array} \right] \sim \left[ \begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array} \right] \quad [A|b]$$

$\text{rank} A = 2 \quad \text{rank} [A|b] = 3 \Rightarrow$  System is inconsistent

$$\textcircled{2} [A|b] = \left[ \begin{array}{cccc|c} 0 & 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 6 \end{array} \right] \sim \dots \sim \left[ \begin{array}{cccc|c} 1 & 0 & -1 & -2 & -3 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

$\text{rank} A = 2 = \text{rank} [A|b] \Rightarrow$  System is consistent

$\# \text{ params in gen'l sol'n} = \# \text{ vars} - \text{rank} A$   
 $= \# \text{ cols} A - \text{rank} A$

$$\textcircled{3} [A|b] = \left[ \begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 1 & 3 & 3 & 1 \\ 1 & 2 & 4 & 2 \\ 2 & 5 & 7 & 3 \end{array} \right] \sim \left[ \begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 1 & 1 & 3 \end{array} \right] \sim \dots \sim \left[ \begin{array}{ccc|c} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$\text{rank} A = 3 = \text{rank} [A|b] \Rightarrow$  system is consistent

$\# \text{ params in gen'l sol'n} = \# \text{ cols} A - \text{rank} A = 0 \quad ! \quad \underline{\text{Unique sol'n}}$

Summary A system  $[A|b]$ 

1) is consistent  $\Leftrightarrow \text{rank} A = \text{rank} [A|b]$

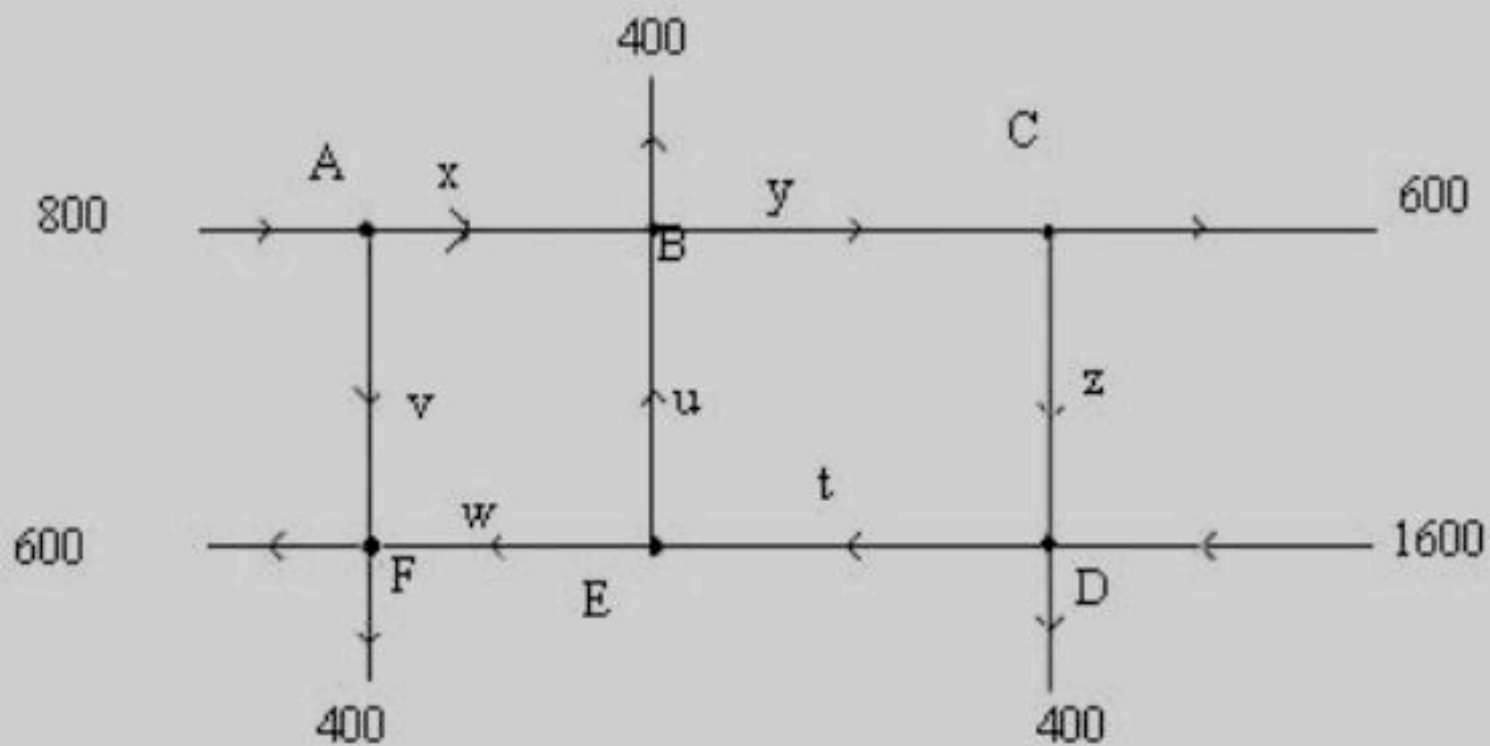
2) has only many solns  $\Leftrightarrow \text{rank} A = \text{rank} [A|b]$  (inconsistent)  
 $\& \text{rank} A < \# \text{ cols} A$

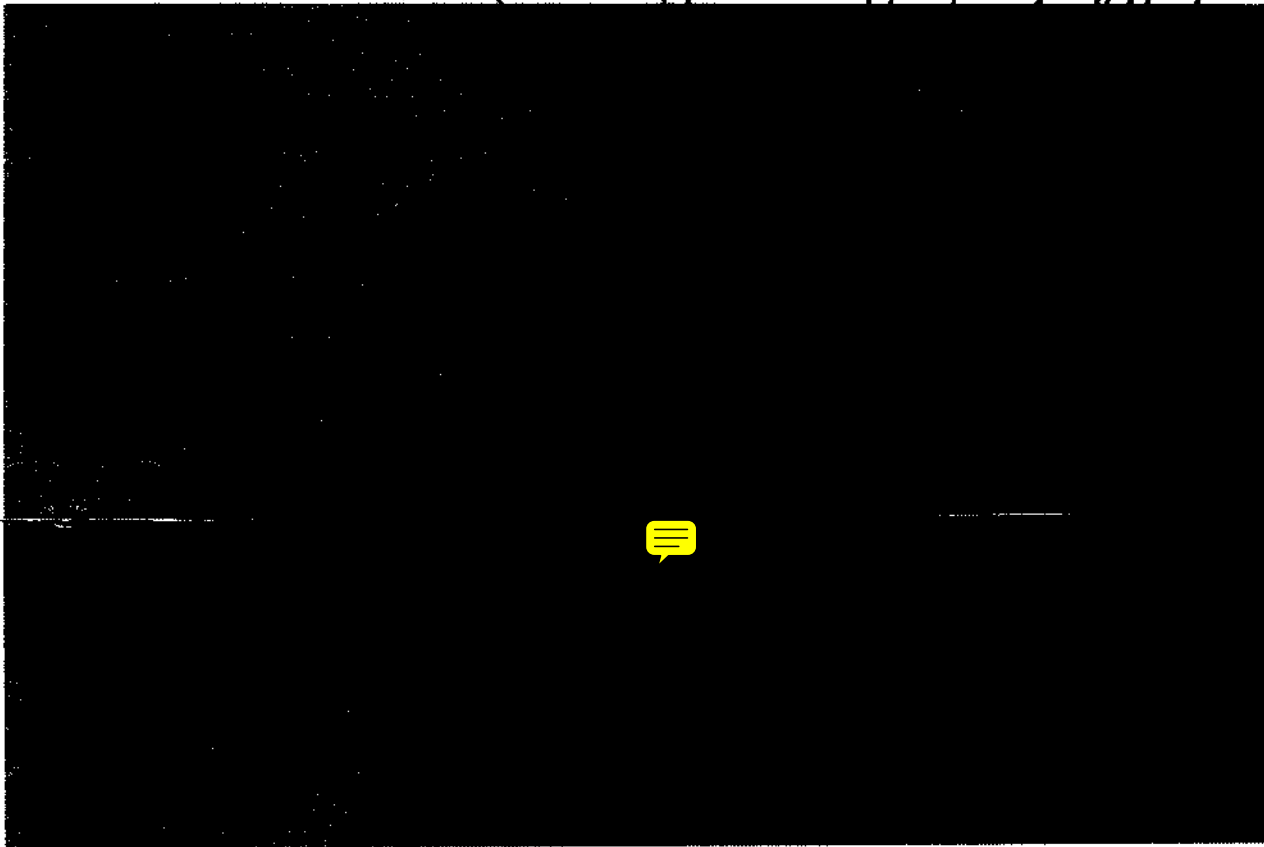
3) has a unique sol'n  $\Leftrightarrow \text{rank} A = \text{rank} [A|b]$  (consistent)  
 $= \# \text{ cols} A.$

N.B.

1 Reinterpret  $[A|b]$  as matrix

$$\text{rank } A = \text{rank } [A|b]$$





By Kirchhoff's first Law, the flow into an intersection is equal to the flow out. This gives the following system

$$\begin{aligned}
 800 &= x + v & (A) \\
 u + x &= y + 400 & (B) \\
 y &= z + 600 & (C) \\
 1600 + z &= t + 400 & (D) \\
 t &= u + w & (E) \\
 v + w &= 1000 & (F)
 \end{aligned}$$

The augmented matrix of the above system is

$$\left[ \begin{array}{ccccccc|c}
 1 & 0 & 0 & 0 & 1 & 0 & 0 & 800 \\
 1 & -1 & 0 & 1 & 0 & 0 & 0 & 400 \\
 0 & 1 & -1 & 0 & 0 & 0 & 0 & 600 \\
 0 & 0 & 1 & 0 & 0 & 0 & -1 & -1200 \\
 0 & 0 & 0 & 1 & 0 & 1 & -1 & 0 \\
 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1000
 \end{array} \right]$$

$$\begin{bmatrix}
 1 & 0 & 0 & 0 & 1 & 0 & 0 & 800 \\
 0 & 1 & 0 & -1 & 1 & 0 & 0 & 400 \\
 0 & 0 & 1 & -1 & 1 & 0 & 0 & -200 \\
 0 & 0 & 0 & 1 & -1 & 0 & -1 & -1000 \\
 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1000 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
 \end{bmatrix}$$

Therefore, the solution can be written as

$$\begin{aligned}
 x &= w - 200 \\
 y &= t - 600 \\
 z &= t - 1200 \\
 u &= t - w \\
 v &= 1000 - w
 \end{aligned}
 \quad w, t \in \mathbb{R}$$

One way streets:  $w \geq 200, t \geq 1200, (t \geq w), w \leq 1000$

For instance, if  $w=300$  and  $t=1300$  (in vehicles per hour), then  $x=100, y=700, z=100, u=1000, v=700$ .

$$\left. \begin{aligned}
 1000 &\geq w \geq 200 \\
 t &\geq 1200
 \end{aligned} \right\} w, t \in \mathbb{Z}$$

Suppose now that the streets from A to B and from B to C must be closed (for construction for instance), that is  $x=0$  and  $y=0$ . How might the traffic be rerouted?

To answer this question, set  $x=y=0$  in the above solution, we get  $w=200, t=600, z=-600, u=400$  and  $v=800$ . Of course, the negative value for  $z$  is not normal. In order to avoid negative flow, we must reverse directions on the streets connecting C and D; this change makes  $z=600$  instead of  $z=-600$ .

$$\begin{aligned}
 x &= 0 \\
 y &= 0 \\
 x=y=0 \\
 w &= 200 \\
 t &= 600 \\
 z &= t - 1200 = -600! \\
 u &= +400 \\
 v &= 800
 \end{aligned}$$

need to change direction of at least one street

12/2/02

# 1.4 Matrix Multiplication

(40)

- Matrices - useful for
- probability (transition matrices)
  - economics (Leontief)
  - geometry
  - quantum theory
  - solving linear systems
  - vector spaces (keeping track of l.c.)
  - computing your final grade.
- Matrices - array of #'s
- collection of rows
  - collection of cols
  - "gen"

Recall  $m \times n$  matrix  $A$ :  $A$  has  $m$  rows ( $n$  entries)  $n$  cols ( $m$  entries)

notation  $A = [a_{ij}]$ ,  $a_{ij}$  =  $(i,j)$  entry  $\begin{bmatrix} 1 & 2 & 7 \\ 4 & 5 & 6 \end{bmatrix} = [a_{ij}]$ ;  $a_{11} = 1$ , etc.

Def<sup>n</sup> (p.27) If  $A$  is an  $m \times n$  matrix and  $B$  is an  $n \times p$  matrix, then  $AB$  is the  $m \times p$  matrix whose

$(i,j)$  entry is the dot product of the  $i^{\text{th}}$  row of  $A$  and the  $j^{\text{th}}$  col. of  $B$ .

i.e. if  $A = [a_{ij}]$  &  $B = [b_{ij}]$  then

$$AB = [c_{ij}] \text{ where } c_{ij} = \sum_{k=1}^n a_{ik} b_{kj} = [a_{i1} \dots a_{in}] \cdot \begin{bmatrix} b_{1j} \\ \vdots \\ b_{nj} \end{bmatrix}$$

Remark  $AB$  makes sense iff  $\# \text{ cols } A = \# \text{ rows } B$ .

eg.  $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x + 2y + 3z \\ 4x + 5y + 6z \\ 7x + 8y + 9z \end{bmatrix} = x \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix} + y \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix} + z \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix}$

$$\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ \dots & \dots & \dots \end{bmatrix} = [a + 2d + 3g \quad b + 2e + 3h \quad c + 2f + 3j] = 1 \cdot [a \ b \ c] + 2 [d \ e \ f] + 3 [g \ h \ j]$$

(l.c. of cols of B)

$$\begin{matrix}
 \begin{bmatrix} 1 & 2 \\ 2 & 6 \\ 1 & -1 \end{bmatrix} & \begin{bmatrix} 4 & 1 & 4 & 3 \\ 0 & -1 & 3 & 1 \end{bmatrix} & = & \begin{bmatrix} 4 & -1 & 10 & 5 \\ 8 & -4 & 26 & 12 \\ 4 & 2 & 1 & 2 \end{bmatrix} \\
 A & B & & AB \\
 3 \times 2 & 2 \times 4 & & 3 \times 4.
 \end{matrix}$$

BA doesn't make sense here, "isn't defined".

1-s.  $[1 \ 2 \ 3] \begin{bmatrix} 4 \\ 5 \\ -6 \end{bmatrix} = [-4] = -4 = (1, 2, 3) \cdot (4, 5, 6)$

↑  
convention  
(row by col)

dot product.

i.e. if vectors are written as rows,  $u \cdot v = u v^t$

↑ dot product

↑ matrix product.

↓  $u \cdot v = u^t v$

col

eg.  $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 5 \\ -6 \end{pmatrix} = [1 \ 2 \ 3] \begin{bmatrix} 4 \\ 5 \\ -6 \end{bmatrix} = -4$

dot

↓

w · x

wt

↑

matrix product

w · x = w<sup>t</sup> · x

(col)

eg  $\begin{bmatrix} 4 \\ 5 \\ -6 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 4 & 8 & 12 \\ 5 & 10 & 15 \\ -6 & -12 & -18 \end{bmatrix} \neq \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \\ -6 \end{bmatrix} = -4$

3x1    1x3

A    B

also defined

AB

BA

① Matrix mult<sup>n</sup> is not commutative

(P28) i.e.  $AB$  is not always the same as  $BA$

(even if they have the same size)

e.g.

$$\begin{matrix} & A & & B \\ \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} & = & \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \end{matrix}$$

$$\begin{matrix} & B & & A \\ \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} & = & \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \end{matrix}$$

$AB \neq BA$  (always)

(turns out to be very useful)

$$\textcircled{2} \quad \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$C \quad D \quad = \quad 0$

$$CD = 0 \quad \not\Rightarrow \quad C = 0 \quad \text{or} \quad D = 0!$$

Recall if  $A = [a_{ij}]$  is  $m \times n$

then  $A^t = [b_{ij}]$  where  $b_{ij} = a_{ji}$

1.1.4 Transpose

Already discussed  
 $(A+B)^t = A^t + B^t$  |  $(A^t)^t = A$



"Square matrices : can multiply as numbers  
of fixed size

(44)

e.g. P35  $A = \begin{bmatrix} 1 & 2 \\ 3 & -4 \end{bmatrix}$      $A^2 = \begin{bmatrix} 7 & -6 \\ -9 & 22 \end{bmatrix}$  ,     $A^3 = \begin{bmatrix} -11 & 38 \\ 57 & -106 \end{bmatrix}$

interesting fact (2x2 for now)

$\text{Tr } A = -3$

$\det A = -10$

$A^2 - \text{tr}(A)A + \det A \cdot I_2$

$= \begin{bmatrix} 7 & -6 \\ -9 & 22 \end{bmatrix} + 3 \begin{bmatrix} 1 & 2 \\ 3 & -4 \end{bmatrix} - 10 \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} !$

Inverses: defn

exercise: show true for any 2x2.

l.f.f

~~l.f.f~~

Block mult'n

sometimes (carefully)  
- Gen. treat parts of matrices  
as if they were numbers!

e.g.  $A = \begin{bmatrix} 1 & 0 & 0 & | & 0 & 0 \\ 0 & 1 & 0 & | & 0 & 0 \\ 0 & 0 & 1 & | & 0 & 0 \\ \hline 0 & 0 & 0 & | & 0 & 1 \\ 0 & 0 & 0 & | & 0 & 0 \end{bmatrix}$

$\begin{bmatrix} B & 0 \\ 0 & C \end{bmatrix}$

Then  $A^2 = \begin{bmatrix} B & 0 \\ 0 & C \end{bmatrix} \begin{bmatrix} B & 0 \\ 0 & C \end{bmatrix} = \begin{bmatrix} B^2 & 0 \\ 0 & C^2 \end{bmatrix}$  works!

$A^{10^9} = \begin{bmatrix} B^{10^9} & 0 \\ 0 & C^{10^9} \end{bmatrix}$

$B^{10^{100}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3$

$A^2$  25x5 = 125 mult.  
1 entry 5 mult.

1 terahertz

$= 10^{12}$   
 $\frac{10^{100} \times 125}{10^{12}}$  secs

$= 10^{88} \times 1.25$

$\approx \frac{10^{90} \times 1.25}{\pi \times 10^8}$  yrs

$\approx 4 \times 10^{81}$  yrs

$A^{10^{10}} = [I_3, 0]$

1.4.3 Matrix eqn of a linear system, etc

e.g.  $x_1 + 2x_2 + 3x_3 = 2$   
 $x_2 + x_3 = -1$   
 $x_1 + \dots + x_3 = 4$

Sys

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}$$

A  $\quad$   $\quad$   $\quad$  =  $\quad$  b  
matrix eqn of Sys

$[A \quad | \quad b]$

 $AX = b$

$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$        $A \text{ } m \times n,$   
 $x \text{ } n \times 1$        $b \text{ } m \times 1.$

$$x_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}$$

Saw before: use block mult w/o knowing.

$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} = [c_1 \ c_2 \ c_3];$   
 $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$c_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$        $c_2 = \dots$   
 $c_i = i^{\text{th}} \text{ col of } A$

$AX = [c_1 \ c_2 \ c_3] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1 c_1 + x_2 c_2 + x_3 c_3$   
 $= x_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$   
 $\in \text{span} \{c_1, c_2, c_3\}$

In general

$A \text{ } m \times n;$        $A = [c_1 \ \dots \ c_n]$        $c_i = i^{\text{th}} \text{ col. of } A, c_i \in \mathbb{R}^m$   
 $x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$

$AX = [c_1 \ \dots \ c_n] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x_1 c_1 + \dots + x_n c_n$



# 1.4 Homogeneous Systems

(47)

Suppose  $A$  is  $m \times n$

$Ax = 0$  clearly has  $x = 0$  as sol<sup>n</sup>

Def<sup>n</sup>  $\ker A = \{x \in \mathbb{R}^n \mid Ax = 0\}$  is the kernel of  $A$  (null space)

P. 181  
(4.1) Thm For any  $m \times n$  matrix  $A$ ,  $\ker A$  is a subspace of  $\mathbb{R}^n$ .

Pf. Let  $Ax = 0$  be the matrix eqn of the system. For

$\ker A = \{x \in \mathbb{R}^n \mid Ax = 0\}$  satisfies ①  $0 \in \ker A$  since  $A0 = 0$

② If  $u, v \in \ker A$  and  $k \in \mathbb{R}$

$$\begin{aligned} A(u + kv) &= Au + kAv \\ &= 0 + k \cdot 0 = 0 \end{aligned}$$

Hence  $u + kv \in \ker A$  Thus  $\ker A$  is subspace of  $\mathbb{R}^n$ .

e.g.  $A0 = 0$ 's  $\left[ \begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{array} \right] \sim \left[ \begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \sim \left[ \begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$

Gen<sup>l</sup> soln is  $x = -\Delta$   
 $y = -\Delta$  ;  $\Delta \in \mathbb{R}$ ;  $\ker A = \{v \in \mathbb{R}^3 \mid Av = 0\}$   
 $z = \Delta$

is  $\{(-\Delta, \Delta, \Delta) \mid \Delta \in \mathbb{R}\}$

$= \text{span}\{(-1, 1, 1)\} ! *$

(line through origin with dir<sup>n</sup>  $(-1, 1, 1)$ ).

Basis of  $\ker A = \{(-1, 1, 1)\}$

(spans by  $*$

∴ is since  $(-1, 1, 1) \neq 0$ )

$\dim \ker A = 1$

9/3/04

e.g.  $\left[ \begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 2 & 2 & 3 & 0 \end{array} \right] = [A \ b]$  rank  $A = 2 < 3 = \# \text{variable}$  48  
 $\therefore$  will have only many sols

Theorem 3.2 (P.22) A homogeneous system  $Ax=0$  with fewer eqns than unknowns always has only many sols; the number of parameters in the gen'l soln =  $\# \text{variables} - \text{rank } A$ .

Pf.  $A$  is  $m \times n$  so  $m < n$  is the assumption; then  
 $\text{rank } A \leq m$  (&  $\text{rank } A \leq n$ ) so  $\text{rank } A \leq m < n \therefore n - \text{rank } A > 0$   
 $\therefore \# \text{params} = n - \text{rank } A > 0. \therefore$  at least 1 param in gen'l soln.

Remark This is the heart of the fundamental fact

size of a spanning set  $\geq$  size of a l.i.s set  
 (4.4.3, P.200)

Recall  $Ax=0 \Leftrightarrow$  cols of  $A$  are l.i.s  $A = [v_1 \dots v_n]$   $v_i \in \mathbb{R}^m$   
has unique soln

If  $m < n$   $Ax=0$  never has a unique soln so cols of  $A$  are always l.d.  
 i.e. if  $m < n$ ,  $n$  vectors in  $\mathbb{R}^m$  are always l.d.

So  $\ker A = \{x \mid Ax=0\}$  is a subspace of  $\mathbb{R}^n$ ; what is its dimension?

e.g.  $\left[ \begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 2 & 2 & 3 & 0 \end{array} \right] \sim \left[ \begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 \end{array} \right]$   $\begin{matrix} x_1 = -\Delta - 2t \\ x_2 = \Delta \\ x_3 = -t \\ x_4 = t \end{matrix}$   $\Delta, t \in \mathbb{R}$

$\ker A = \{(-\Delta - 2t, \Delta, -t, t) \mid \Delta, t \in \mathbb{R}\} = \text{span} \left\{ \overset{v_1}{(-1, 1, 0, 0)}, \overset{v_2}{(-2, 0, -1, 1)} \right\}$

So  $\{v_1, v_2\}$  spans  $\ker A$ . Moreover,  $s v_1 + t v_2 = 0 \Leftrightarrow \Delta = t = 0$   
 $\therefore \{v_1, v_2\}$  is a basis of  $\ker A \therefore \dim \ker A = 2$

= # parameters!  
 $\neq$   
 N.B.

①  $A = [c_1 \dots c_n]$   $x \in \mathbb{R}^n$   $x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$

②  $(Ax=0 \Rightarrow x=0) \Leftrightarrow c_1, \dots, c_n$  are l.i.  
 $\Updownarrow \Rightarrow$   
 $\text{rank } A = n$

②  $Ax=b$  is consistent  $\Leftrightarrow b \in \text{span}\{c_1, \dots, c_n\}$

③  $\text{ker } A = \{ x \in \mathbb{R}^n \mid Ax=0 \} \subset \mathbb{R}^n$   
ss of  $\mathbb{R}^n$

$\begin{bmatrix} 1 & 1 & 1 & 3 & | & 0 \\ 2 & 2 & 3 & 7 & | & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 & 2 & | & 0 \\ 0 & 0 & 1 & 1 & | & 0 \end{bmatrix}$   
 $x_1 = -\Delta - 2t$   
 $x_2 = \Delta$   
 $x_3 = -t$   
 $x_4 = t$

$= \{ (-\Delta - 2t, \Delta, -t, t) \mid \Delta, t \in \mathbb{R} \}$

$\text{ker } A = \text{span} \{ \underset{v_1}{(-1, 1, 0, 0)}, \underset{v_2}{(-2, 0, -1, 1)} \}$  "basic" solutions

$s v_1 + t v_2 = 0 \Leftrightarrow \Delta = t = 0$

$\{v_1, v_2\}$  basis for  $\text{ker } A$ ;  $\dim \text{ker } A = \# \text{ param} = 2$

Sol<sup>n</sup> obtained this way are sometimes called the basic sol<sup>n</sup>

Theorem 2.6 (P.25) (P.209) 49  
If a homogeneous linear system  $AX=0$  is a basis of the set of all solutions, which has dimension  $n - \text{rank } A$ . (If  $A$  is  $m \times n$ .)

Rank ← See opposite (cf of Lemma 4.4.3 P.200)  
(16/11/09, 19/3/02)

1.5 Matrix Inverses (for square matrices only)

P.38  
There's a profit <sup>in</sup> comparing  $AX=b$   
( $A$   $n \times n$ ,  $x \in \mathbb{R}^n$ ,  $b \in \mathbb{R}^n$ ) to  $ax=c$ ;  $a, x, c \in \mathbb{R}$

If  $a \neq 0$ ,  $x = \frac{c}{a}$  (is the unique sol<sup>n</sup>). Can we

"divide by  $A$ ?" Careful:  $AB=C$

$$B = \begin{pmatrix} c \\ \vdots \\ c \end{pmatrix} = AC \text{ or } CA$$

Recall  $I_n = I = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix} = \text{diagonal matrix with } n \text{ '1's'}$

Def<sup>n</sup> R.39) If  $A$  is an  $n \times n$  matrix and  $B$  is an  $n \times n$  matrix s.t.  $AB = BA = I$ , then  $B$  is called an inverse of  $A$ , and we write  $B = \underline{A^{-1}}$ .  $A$  is then called invertible.

(P.41, 60)

Rank: if  $AB=I$ , then  $BA=I$  too. See well

e.g.  $I_n = \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix}$  is invertible,  $I_n^{-1} = I_n$ .

e.g.  $\begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ , so  $A, B$  are both invertible  
 $A = B^{-1}, B = A^{-1}$

e.g.  $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$  is not invertible since  $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$   
 $A \quad x_0 = 0$   
 $(x_0 \neq 0)$

(if  $A^{-1}$  existed,  $x_0 = A^{-1}(Ax_0) = A^{-1}(0) = 0$ ).

Fact

(P.40)  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$  is invertible  $\Leftrightarrow ad - bc \neq 0$ ;

If  $ad - bc \neq 0$ ,  $A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$  check.

(checking formula works is easy; ex. show  $ad - bc = 0 \Rightarrow A$  is not invertible)  
 $A \begin{bmatrix} d \\ -c \end{bmatrix}$

(1.5.4)

Some facts (I) (Thm 3, P43) If  $a \neq 0$  is a scalar,  $k \in \mathbb{R}$ ,  $A$  &  $B$  are invertible

- then so is
- 1)  $A^{-1}$  and  $(A^{-1})^{-1} = A$
- 2)  $A^k$  and  $(A^k)^{-1} = (A^{-1})^k$
- 3)  $A^t$  and  $(A^t)^{-1} = (A^{-1})^t$
- 4)  $(aA)$  and  $(aA)^{-1} = a^{-1}A^{-1}$
- 5)  $AB$  and  $(AB)^{-1} = B^{-1}A^{-1}$

Thm 2.1.4 If  $A$  is an invertible  $n \times n$  matrix, then

for any  $b \in \mathbb{R}^n$ , the linear system  $Ax = b$

1) is always consistent

2) has the unique sol<sup>n</sup>  $x = A^{-1}b$

pf. 1) Check:  $A \cdot (A^{-1}b) = (AA^{-1})b = I \cdot b = b$  so  $x = A^{-1}b$  is a sol<sup>n</sup>  
 2) Suppose  $Ax' = b$ , then  $A^{-1}(Ax') = A^{-1}b \Rightarrow (A^{-1}A)x' = A^{-1}b \Rightarrow x' = b^f$ , so  
 $A^{-1}b$  is the only sol<sup>n</sup>.

Corollaries If  $A$  is invertible,  $A = [c_1 \dots c_n]$

1) The columns of  $A$  span  $\mathbb{R}^n$

(since  $Ax = b$  is always consistent)

2) The columns of  $A$  are l.o.i.

(since  $Ax = 0$  has the unique sol<sup>n</sup>  $0 = A^{-1}b$ )

1) & 2)  $\Rightarrow$  3) The cols of  $A$  are a basis of  $\mathbb{R}^n$

L17/1.5.3

Finding Inverses (when they exist)

Note  $A^{-1}$  exists  $\Rightarrow$  rank  $A = n$

$\Rightarrow$  RRE form of  $A$  is  $I_n$

?

YES!

L17 ↓ For

e.g. let  $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 5 \\ 2 & 2 & 5 \end{bmatrix}$  ;

(52)

we solve  $AC = I_3$  for  $C$ . Write  $C = [c_1 \ c_2 \ c_3]$

$I_3 = [e_1 \ e_2 \ e_3]$   $e_i, c_i$   $i$ th cols of  $I_3, C$  resp.

Then  $AC = A[c_1 \ c_2 \ c_3] = [Ac_1 \ Ac_2 \ Ac_3] = I_3 = [e_1 \ e_2 \ e_3]$

is equivalent to  $Ac_1 = e_1$   
 $Ac_2 = e_2$   
 $Ac_3 = e_3$

Solve  $[A | e_1]$   $\textcircled{1} \sim [\tilde{A} | \tilde{e}_1] \stackrel{?}{\sim} [I_3 | c_1]$  Note: the row  
 $[A | e_2]$   $\textcircled{2} \sim [\tilde{A} | \tilde{e}_2] \sim [I_3 | c_2]$  operations used in  
 $[A | e_3]$   $\textcircled{3} \sim [\tilde{A} | \tilde{e}_3] \sim [I_3 | c_3]$  each are completely  
determined by  $A$ ,

so are the same in each of  $\textcircled{1}, \textcircled{2}, \textcircled{3}$  : Do them all at once!

Use "superaugmented" matrix  $[A | e_1 \ e_2 \ e_3] \stackrel{!}{=} [A | I_3] \stackrel{?}{=} [I_3 |$

$$\left[ \begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 1 & 2 & 5 & 0 & 1 & 0 \\ 2 & 2 & 5 & 0 & 0 & 1 \end{array} \right] \sim \left[ \begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 3 & -1 & 1 & 0 \\ 0 & 0 & 1 & -2 & 0 & 1 \end{array} \right] \sim \left[ \begin{array}{ccc|ccc} 1 & 0 & -1 & 5 & 0 & -2 \\ 0 & 1 & 0 & 5 & 1 & -3 \\ 0 & 0 & 1 & -2 & 0 & 1 \end{array} \right] \sim \left[ \begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & 5 & 1 & -3 \\ 0 & 0 & 1 & -2 & 0 & 1 \end{array} \right]$$

unique  
soln!  $\uparrow \uparrow \uparrow$   
 $c_1 \ c_2 \ c_3$

$$\therefore A^{-1} = \begin{bmatrix} 0 & -1 & 1 \\ 5 & 1 & -3 \\ -2 & 0 & 1 \end{bmatrix}$$

Thm; <sup>is an</sup> If  $A$  <sup>is an</sup>  $n \times n$  matrix and  $\text{rank } A = n$ , then  $A$  is invertible, and  $A^{-1}$  can be  
computed via the algorithm above i.e.  $[A | I_n] \sim [I_n | A^{-1}]$

$A$  is not invertible

Remark ①: The algorithm decides if  $A^{-1}$  exists,  
 since it tells you rank  $A$ :

$$\text{eg } [A|I] = \left[ \begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 4 & 5 & 6 & 0 & 1 & 0 \\ 1 & -1 & -3 & 0 & 0 & 1 \end{array} \right] \sim \left[ \begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -3 & -6 & -4 & 1 & 0 \\ 0 & -3 & -6 & -1 & 0 & 1 \end{array} \right]$$

$$\sim \left[ \begin{array}{ccc|ccc} 1 & 2 & 3 & * & * & * \\ 0 & 1 & 2 & * & * & * \\ 0 & 0 & 0 & * & * & * \end{array} \right] ; \text{rank } A < 3 \text{ so}$$

$A$  is not invertible.

② If  $AC = I$  ( $A, C$   $n \times n$  matrices)  
 then  $CA = I$  too!

Pf: Consider  $Cx = 0$ . Then  $ACx = A \cdot 0 = 0$ . But  
 $Ax = Ix = x$ , so  $Cx = 0 \Rightarrow x = 0$ . Hence rank  $C = n$ ,  
 so by our algorithm, we can find  $B$  s.t.

$$CB = I.$$

Then  $ACB = A \cdot I$

$$\underbrace{I} \cdot B = A$$

$$B = A$$

i.e.  $CA = I$

$A, C$   $n \times n$  matrices

Remark: If  $AC = I$ , then  $CA = I$  too:

(54)

Pf: If  $AC = I$  ~~so if~~ <sup>then if</sup>  $Cx = 0$ ,  $ACx = Ix = x = 0$  so

$Cx = 0 \Rightarrow x = 0$ ; so  $\text{rank } C = n$ . Hence

can solve  $CB = I^*$  for  $B$ . Then

$$ACB = AI$$

$$I \cdot B = A$$

$$\text{i.e. } B = A$$

so \* says  $CA = I$ .

---

e.g.  $\{(1, 1, 2), (1, 2, 5), (2, 5, 5)\}$  basis of  $\mathbb{R}^3$

since matrix with these rows is invertible

$\{(1, 2, 3), (4, 5, 6), (1, -1, -3)\}$  is not

$\{(1, 4, 1, \dots), (2, 5, -1), (3, 6, -3)\}$  is not. ...

$\{(1, 1, 2), (1, 2, 2), (2, 5, 5)\}$  is ...

Summary (Thm 5, P 46 + Thm 2 P 192: 4.2.2)

The following are equivalent for an  $n \times n$  matrix  $A$

- |  |                                     |   |
|--|-------------------------------------|---|
| 1) $A$ is invertible                                 | $\Downarrow \checkmark$             | } $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4 \Rightarrow 5$<br>$\Downarrow$<br>$1 \Leftrightarrow 6$<br>$2 \Leftrightarrow 2^a$<br>$6, 2^a \Leftrightarrow 2^b$<br>$6, 5^a \Leftrightarrow 5^b$<br>$2^a, 3^a \Leftrightarrow 7$<br>$2^a, 5^a \Leftrightarrow 8$ |
| 2) $AX=0 \Rightarrow X=0$                            | $\Downarrow \checkmark$             |   |
| 3) $\text{rank } A = n$                              | $\Downarrow \dots$                  |   |
| 4) $A \sim I_n$                                      | $\Downarrow \checkmark$ (algorithm) |   |
| 5) $AX=b$ is consistent for all $b \in \mathbb{R}^n$ |                                     |   |

6)  $A^{-1}$  is invertible      3)<sup>b</sup>  $\text{rank } A^t = n$

2<sup>a</sup>) the columns of  $A$  are l.i.      2<sup>b</sup>) the rows of  $A$  are l.i.

5<sup>a</sup>) the columns of  $A$  span  $\mathbb{R}^n$       5<sup>b</sup>) the rows of  $A$  span  $\mathbb{R}^n$

7) the columns of  $A$  are a basis for  $\mathbb{R}^n$

8) the rows of  $A$  are a basis for  $\mathbb{R}^n$

Rank  $\text{Rank } A = \text{Rank } A^t!$

e.g.  $\{(1, 1, 2), (1, 2, 5), (2, 5, 5)\}$  is a basis of  $\mathbb{R}^3$  (56)

since  $\text{rank} \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 5 \\ 2 & 5 & 5 \end{bmatrix} = 3$

$\{(1, 2, 3), (4, 5, 6), (1, -1, 3)\}$  is not since

$$\text{rank} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 1 & -1 & -3 \end{bmatrix} < 3$$

$\{(1, 4, 1), (2, 5, -1), (3, 6, -3)\}$  is not

$\{(1, 1, 2), (1, 2, 2), (2, 5, 5)\}$  is ✓

Finding bases for subspaces of  $\mathbb{R}^n$

- 3 problems
- (I) Given a spanning set  $\{w_1, \dots, w_m\}$  of  $W$ , find any basis of  $W$
  - (II) Given a spanning set  $\{w_1, \dots, w_m\}$  of  $W$ , find a basis of  $W$  which is a subset of  $\{w_1, \dots, w_m\}$
  - (III) Given a basis  $B$  of  $W$ , extend  $B$  to a basis of  $\mathbb{R}^n$ .

We'll use matrices, and elem. row ops extensively.

Given an  $m \times n$  matrix  $A$ , there are 2 (new) subspaces associated to it

Write  $A = \begin{bmatrix} r_1 \\ \vdots \\ r_m \end{bmatrix}$   $r_i \in \mathbb{R}^n$ ,  $i^{\text{th}}$  row of  $A$

$A = [c_1 \dots c_n]$   $c_j \in \mathbb{R}^m$   $c_j$   $j^{\text{th}}$  col of  $A$

Defn  $\text{Row}(A) = \text{row space of } A = \text{span}\{r_1, \dots, r_m\} \subset \mathbb{R}^n$

$\text{Col}(A) = \text{col space of } A = \text{span}\{c_1, \dots, c_n\} \subset \mathbb{R}^m$   
 $= \{Ax \mid x \in \mathbb{R}^n\}$

Consider problem (I): Given  $\{w_1, \dots, w_m\}$ , a spanning set of  $W \subset \mathbb{R}^n$ , find any basis of  $W$

Row space algorithm: Set  $A = \begin{bmatrix} w_1 \\ \vdots \\ w_m \end{bmatrix}$ , (5.8)

(Lemma 11.209)

So  $\text{row}(A) = W$ ; consider effect of an elem row op<sup>n</sup> on  $A$ :

$A \xrightarrow[\text{row op}]{\text{elem}} A'$ ; rows of  $A'$  are l.c. of rows of  $A$

$$\therefore \text{row}(A') \subseteq \text{row}(A)$$

BUT: row op<sup>n</sup> are reversible,

$A' \xrightarrow[\text{row op}]{\text{elem}} A$

$\therefore$  rows of  $A$  are l.c. of rows of  $A'$

$$\therefore \text{row}(A) \subseteq \text{row}(A')$$

$$\begin{bmatrix} 1 & 3 & 3 & 4 \\ 1 & -1 & -1 & 0 \\ 2 & 1 & 1 & 3 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 3 & 3 & 4 \\ 0 & -4 & -4 & -4 \\ 2 & 1 & 1 & 3 \end{bmatrix}$$

$$\therefore \text{row}(A') = \text{row}(A) = W$$

So take  $A$  to RE or RRE form: get "simplest" spanning set for  $W$  (will see too);  $W = \text{span} \{ (1, 3, 3, 4), (1, -1, -1, 0), (2, 1, 1, 3) \}$ ; (RE)

$$\text{e.g. set } A = \begin{bmatrix} 1 & 3 & 3 & 4 \\ 1 & -1 & -1 & 0 \\ 2 & 1 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 3 & 4 \\ 0 & -4 & -4 & -4 \\ 0 & -5 & -5 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 3 & 4 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

row(A) = W       $A'$        $A''$       row(A'') = W       $\tilde{A} = \begin{bmatrix} \tilde{r}_1 \\ \tilde{r}_2 \\ \tilde{r}_3 \end{bmatrix}$

$$N = \text{row}(A) = \text{row}(\tilde{A}) = \text{span} \{ \tilde{r}_1, \tilde{r}_2 \} = \text{span} \{ \tilde{r}_1, \tilde{r}_2 \} = \text{span} \{ (1, 0, 0, 1), (0, 1, 1, 1) \}$$

$$a \tilde{r}_1 + b \tilde{r}_2 = 0 \Rightarrow \begin{pmatrix} a & 0 & 0 & a \\ 0 & b & b & b \end{pmatrix} + \dots \Rightarrow a - b = 0$$

Lemma 1(1) P203

Thm 1 P204

Row operations do not change

the row space of a matrix. Moreover

1) The non-zero rows of a matrix in (RE) RRE form a basis for row(A)

2) rank A = dim row(A)

e.g. Find a basis for  $W = \text{span}\{(1, 3, 7), (1, -1, 5), (2, 1, 1)\}$

$$\text{Set } A = \begin{bmatrix} 1 & 3 & 7 \\ 1 & -1 & 5 \\ 2 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 7 \\ 0 & -4 & 12 \\ 0 & -5 & 15 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 7 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 16 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix}$$

ed stop here.

$CW = \text{row}(A)$

$$\Sigma \{(1, 3, 7), (0, 1, -3)\} \text{ OR } \{(1, 0, 16), (0, 1, -3)\}$$

is a basis for W.

Remark:  $\text{col}(A) \neq \text{col}(A')$  :  $(1, 1, 2) \in \text{col}(A)$ , but  $(1, 1, 2) \notin \text{col}(A') = x-y \text{ plane}$

Problem II Given a spanning set  $\{u_1, \dots, u_m\}$  of a subspace of  $\mathbb{R}^n$ ,

find a basis which is a subset of the given spanning set.

e.g.  $W = \text{span}\{ \overset{u_1}{(1, 1, 1, 0)}, \overset{u_2}{(1, 1, 2, 1)}, \overset{u_3}{(1, 1, -1, 1)}, \overset{u_4}{(1, 1, 2, 2)} \}$   $u_i = w_i^t$

"Casting-out algorithm". Set  $B = [u_1^t \ u_2^t \ u_3^t \ u_4^t]$   
 $u_j = j^{\text{th}} \text{ col of } A$

$\text{col}(B) = W!$  First; reduce B to RE form

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & -2 & 1 \end{bmatrix} = B'$$

idea: recall  $[A \ b]$

consistent  $\Leftrightarrow$

(60)

$$\text{rank } A = \text{rank } [A \ b]$$

$\Leftrightarrow$

$b$  is a l.c. of cols of  $A$

We consider  $\text{rank } [u_1]$ ,  $\text{rank } [u_1 \ u_2]$ ,  $\text{rank } [u_1 \ u_2 \ u_3]$ , etc.

- when rank increases, new col is not dependent on previous (add to basis)

- when rank doesn't increase - new col is dependent on previous cols (don't add to basis)

(rank increases whenever a new leading one occurs!)

$$[u_1] = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}; \quad \text{rank } [u_1] = 1; \quad \text{add}$$

$$[u_1 \ u_2] = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 2 \\ 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \quad (\text{first 2 cols of } B') \quad \text{rank } [u_1, u_2] = 2 > \text{rank } [u_1]$$

$\therefore \{u_1, u_2\}$  l.i.

$$[u_1 \ u_2 \ u_3] = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 2 & -1 \\ 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad (\text{1st 3 cols of } B') \quad \text{rank } [u_1, u_2, u_3] > \text{rank } [u_1, u_2]$$

$\therefore \{u_1, u_2, u_3\}$  l.i.

$$B = [u_1 \ u_2 \ u_3 \ u_4] \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (B') \quad \text{rank } [u_1, u_2, u_3, u_4] = \text{rank } [u_1, u_2, u_3]$$

$\therefore u_4 \in \text{span } \{u_1, u_2, u_3\}$

$\therefore$  basis of  $\text{col}(B) = W$  is  $\{u_1, u_2, u_3\}$

Note: these are the cols of  $B$  where the leading ones occur

~~Last class: (I) find any basis of  $W = \text{span}\{w_1, \dots, w_m\} \subseteq \mathbb{R}^n$  (61)~~

- ~~Row space algorithm: set  $A = \begin{bmatrix} w_1 \\ \vdots \\ w_m \end{bmatrix}$   $w_i = i^{\text{th}}$  row of  $A$~~ 
  - ~~$W = \text{row}(A)$~~
  - ~~$A \sim A'$  (REF form of  $A$ ); basis of  $W = \text{basis of row}(A) = \text{row}(A) = \text{non-zero rows of } A'$~~
  - ~~$\dim W = \text{rank } A = \dim \text{row}(A)$~~

~~(II) find a basis of  $W = \text{span}\{w_1, \dots, w_m\} \subseteq \mathbb{R}^n$  that is a subset of  $\{w_1, \dots, w_m\}$ .~~

~~Column out algorithm set  $B = [w_1 \dots w_m]$  ( $w_j = j^{\text{th}}$  col of  $B$ )~~  
 ~~$\text{col}(B) = W$~~

~~$B' = \text{REF form of } B$ ; If leading ones occur in columns  $j_1, \dots, j_r$  of  $B'$ , then cols  $j_1, \dots, j_r$  of  $B$  are a basis of  $W$ .~~

~~$\dim W = \dim \text{col}(B) = \text{rank } B$  !~~

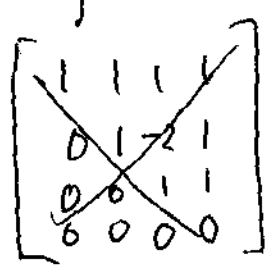
prob. (4.45) eg

$W = \text{span}\{u_1, u_2, u_3, u_4, u_5\}$       $B = [u_1 \ u_2 \ u_3 \ u_4 \ u_5]$

$$B = \begin{bmatrix} 1 & 2 & 1 & 2 & 3 \\ 2 & 4 & 3 & 7 & 7 \\ 1 & 2 & 2 & 5 & 5 \\ 3 & 6 & 6 & 15 & 14 \\ u_1 & u_2 & u_3 & u_4 & u_5 \end{bmatrix} \sim \dots \sim \begin{bmatrix} 1 & 2 & 1 & 2 & 3 \\ 0 & 0 & -1 & 3 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = B'$$

Building Basis

- $\text{rank}[u_1] = 1$       $\{u_1\}$  l.i.      $\{u_1\}$
- $\text{rank}[u_1, u_2] = 2$       $\{u_1, u_2\}$  l.d.      $\{u_2\}$
- $\text{rank}[u_1, u_2, u_3] = 2$       $\{u_1, u_3\}$  l.d.      $\{u_1, u_3\}$
- $\text{rank}[u_1, u_2, u_3, u_4] = 2$       $\{u_1, u_3, u_4\}$  l.d.      $\{u_1, u_3\}$
- $\text{rank } B = 3$       $\{u_1, u_3, u_5\}$  l.i.      $\{u_1, u_3, u_5\}$



$\therefore \{u_1, u_3, u_5\}$  is a basis of  $W (= \text{col}(B))$  which is a subset of

Thm 2.1.1 (4) 1) If  $B'$ 's any RE form of a matrix  $B$  and  
 P.205 the leading ones occur in columns  $j_1, \dots, j_r$  of  $B'$ , then  
 the columns  $j_1, \dots, j_r$  of  $B$  are a basis of  $\text{col}(B)$

$\therefore$  2)  $\text{rank } B = \dim \text{col}(B) (= \dim \text{row}(B))!$

Corollary: AMAZING FACT: for any matrix  $A$ ,  $\dim \text{row}(A) = \text{rank } A = \dim \text{col}(A)!$

Problem III Extending Bases (usually of a subspace to one for  $\mathbb{R}^p$ ).  
 P.04 L.19

e.g. Given  $W = \text{span} \{ (0, 0, 1, 0), (0, 1, 1, 0) \}$ .  $\dim W = 2$   
 $\{ (0, 0, 1, 0), (0, 1, 1, 0) \}$  Basis.

Extend the basis  $\{ \overset{v_1}{(0, 0, 1, 0)}, \overset{v_2}{(0, 1, 1, 0)} \}$  to a basis of  $\mathbb{R}^4$ .

Write  $A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ v_3 \\ v_4 \end{pmatrix}$ ; want to choose  $v_3, v_4$   
 so that  $\text{row}(A) = \mathbb{R}^4$   
 $\therefore \text{rank } A = 4$

$\sim \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ v_3 \\ v_4 \end{bmatrix}$  need leading ones in cols 1 & 4  
 $\therefore$  set  $v_3 = (1, 0, 0, 0)$   
 $v_4 = (0, 0, 0, 1)$

$\sim \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} = A'$ ;  $\text{rank } A' = 4$

$\{ v_1, v_2, v_3, v_4 \}$  is a basis of  $\mathbb{R}^4$

# Summary on subspaces associated to an $m \times n$ matrix $A$

(63)

- 1)  $\text{row}(A)$  is a s.s. of  $\mathbb{R}^n$ ;  $\dim \text{row}(A) = \text{rank } A$
2.  $\text{col}(A)$  s.s. of  $\mathbb{R}^m$ ;  $\text{col}(A) = \{Ax \mid x \in \mathbb{R}^n\}$ ;  $\dim \text{col}(A) = \text{rank } A$
3.  $\ker(A)$  is a s.s. of  $\mathbb{R}^n$ ;  $\ker A = \{x \in \mathbb{R}^n \mid Ax = 0\}$   
 $\dim \ker A = n - \text{rank } A$

or  $\dim \ker A + \text{rank } A = n$

Thm 6  
p. 209

$$\left. \begin{array}{l} \dim \ker A + \dim \text{col}(A) = n \\ \dim \ker A + \dim \text{im}(A) = n \end{array} \right\}$$

\*  $\dim \{x \mid Ax = 0\} + \dim \{Ax \mid x \in \mathbb{R}^n\} = n$

"conservation of dimension"

$A \text{ } m \times n$

$$\begin{array}{ccc} x & \mapsto & Ax \in \mathbb{R}^m \\ \in \mathbb{R}^n & \mapsto & \mathbb{R}^m \end{array}$$

Remark  $\ker A = \{x \mid Ax = 0\}$ ;  $A = \begin{bmatrix} r_1 \\ \vdots \\ r_m \end{bmatrix}$

$$= \left\{ \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n \mid \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_m \end{bmatrix} x = 0 \right\}$$

$$= \left\{ x \in \mathbb{R}^n \mid \begin{bmatrix} r_1 \cdot x \\ r_2 \cdot x \\ \vdots \\ r_m \cdot x \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \right\}$$

$$= \{x \in \mathbb{R}^n \mid x \text{ is orthogonal to all the rows of } A\}$$

Is.  $\dim \ker A + \dim \text{row}(A) = n$

e.g.  $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ -1 & -2 & 1 \end{bmatrix}$

$\text{row}(A) \subseteq \mathbb{R}^3$   
 $\text{col}(A) \subseteq \mathbb{R}^3$   
 $\text{ker } A \subseteq \mathbb{R}^3$

(1) (624)  
 not same!

basis of  $\sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$

$x = -0$   
 $y = 1$   
 $z = 1$

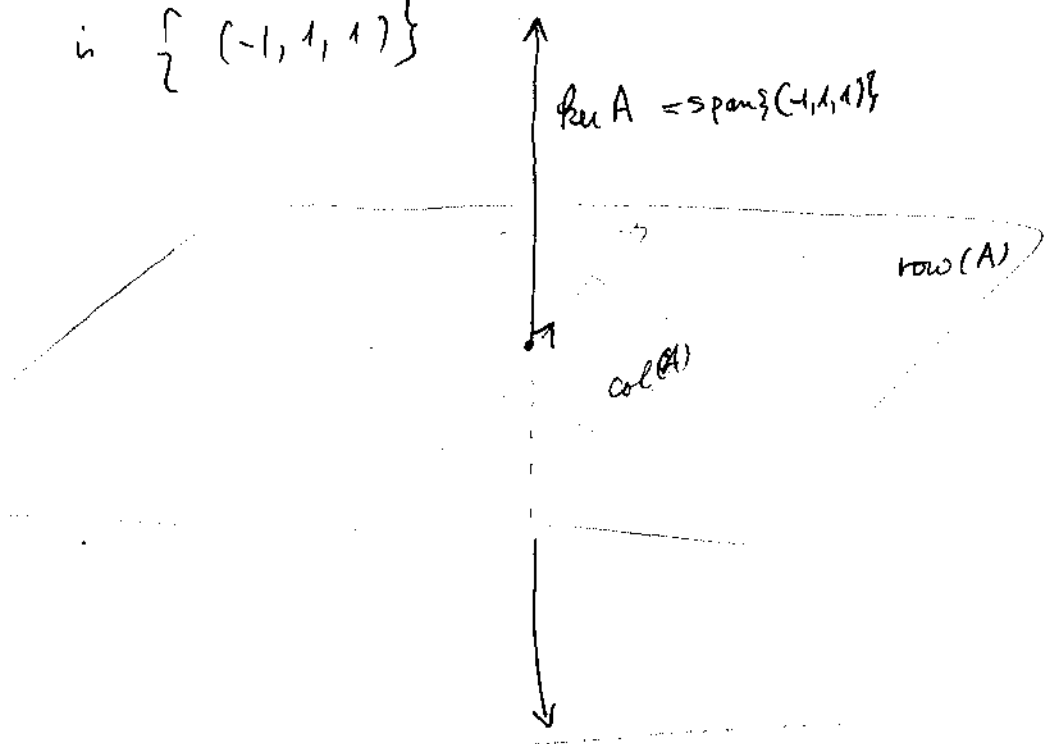
$\sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$

$\begin{bmatrix} x & y & z \\ 1 & 1 & -1 \\ 1 & 0 & -2 \end{bmatrix} \begin{matrix} \text{col } A \\ \hat{e}_1 \\ \hat{e}_2 \\ \hat{e}_3 \end{matrix} \quad (-2, +1, 1)$

$\text{row}(A) \text{ is } \{ (1, 1, 0), (0, 1, -1) \}$

$\text{col}(A) \text{ is } \{ (1, 1, -1), (1, 0, -2) \}$

$\text{ker } A \text{ is } \{ (-1, 1, 1) \}$



~~Extend basis of  $\text{col } A$  to basis of  $\mathbb{R}^3$ : note  $\{e_1, e_2, e_3\}$   
 $\sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$  which has rank 3  $\therefore \{e_1, e_2, e_3\}$  is a basis of  $\mathbb{R}^3$   
 $\text{ker } A$  to basis of  $\mathbb{R}^3$ :  $[-1 \ 1 \ 1] = [v]$~~

## 4.5 Orthogonality 4.5.1 - done

## 4.5.2 Orthogonal sets and projections

(65)

- Def<sup>n</sup>  $\{u_1, \dots, u_k\} \subset \mathbb{R}^n$  is orthogonal if

$$\textcircled{1} \quad u_i \cdot u_j = 0 \quad i \neq j$$

$$\textcircled{2} \quad u_i \neq 0, \quad \forall i \quad (\Leftrightarrow u_i \cdot u_i \neq 0)$$

in addition

(if  $\|u_i\| = \sqrt{u_i \cdot u_i} = 1$ , also called orthonormal)

Ex. (Thm 4) If  $\{u_1, \dots, u_k\}$  is orthogonal,  $\|u_1 + u_2 + \dots + u_k\|^2 = \|u_1\|^2 + \dots + \|u_k\|^2$

For Thm 5 p. 215

If  $\{u_1, \dots, u_k\}$  is orthogonal, it is l.o.i.

Pf. Suppose  $c_1 u_1 + \dots + c_k u_k = 0$  for some scalars  $c_1, \dots, c_k$ .

Take dot product of both sides with  $u_i$  to obtain

$$c_i (u_i \cdot u_i) = 0 \quad \Leftrightarrow \quad c_i = 0, \quad \forall i = 1, \dots, k.$$

Con  $\{u_1, \dots, u_n\} \subset \mathbb{R}^n$  is orthogonal, it is a basis of  $\mathbb{R}^n$ !

Thm 6 (p. 216) If  $W$  is a subspace of  $\mathbb{R}^n$  and  $\{u_1, \dots, u_k\}$  is an orthogonal basis of  $W$ , for any  $w \in W$  we have

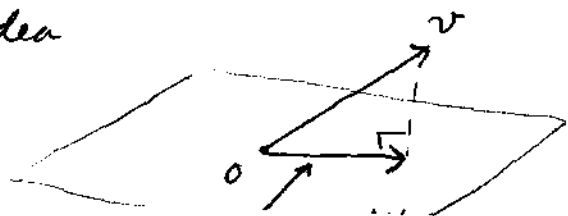
$$w = \underbrace{\left(\frac{w \cdot u_1}{\|u_1\|^2}\right)}_{\text{Fourier coefficient}} u_1 + \underbrace{\left(\frac{w \cdot u_2}{\|u_2\|^2}\right)}_{\text{Fourier coefficient}} u_2 + \dots + \underbrace{\left(\frac{w \cdot u_k}{\|u_k\|^2}\right)}_{\text{Fourier coefficient}} u_k$$

Pf. We know  $u = c_1 u_1 + \dots + c_k u_k$  for some scalars.

Take dot product on both sides with  $u_i$  to obtain

$$u \cdot u_i = c_i (u_i \cdot u_i) \quad \text{so} \quad c_i = \frac{u \cdot u_i}{\|u_i\|^2} \quad \square$$

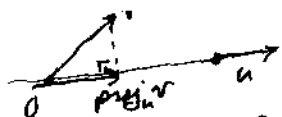
Idea



$W$  - "interesting" subspace

Given  $v \in \mathbb{R}^n$ , find vector in  $W$  closest to  $v$ : the best

eg  $W = \text{span}\{u\}$



$$\text{proj}_W v = \left( \frac{v \cdot u}{\|u\|^2} \right) u.$$

(66)

Def<sup>n</sup> 4.6.2 [223] If  $\{w_1, \dots, w_k\}$  is an orthogonal

basis of a subspace  $W$ , and  $v \in \mathbb{R}^n$ , then

$$\text{proj}_W v = \left( \frac{v \cdot w_1}{\|w_1\|^2} \right) w_1 + \dots + \left( \frac{v \cdot w_k}{\|w_k\|^2} \right) w_k$$

is the (orthogonal) projection of  $v$  on  $W$

("proj(v, W)")  
in book

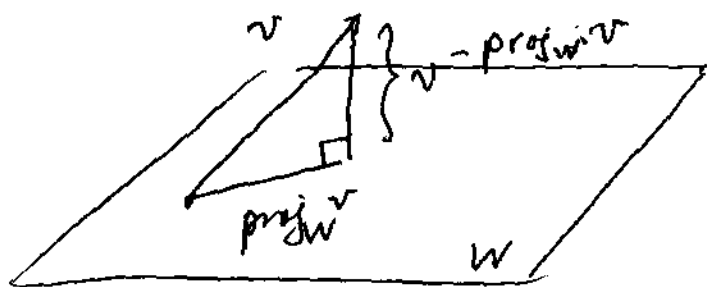
eg.  $W = \{ (x, y, z) \in \mathbb{R}^3 \mid x - z = 0 \}$

$\{ \underset{w_1}{(1, 0, 1)}, \underset{w_2}{(0, 1, 0)} \}$  is an orthogonal basis of  $W$ ;

$$\therefore \text{proj}_W (x, y, z) = \left( \frac{(x, y, z) \cdot w_1}{\|w_1\|^2} \right) w_1 + \left( \frac{(x, y, z) \cdot w_2}{\|w_2\|^2} \right) w_2$$

$$= \left( \frac{x+z}{2} \right) (1, 0, 1) + y (0, 1, 0)$$

$$= \left( \frac{x+z}{2}, y, \frac{x+z}{2} \right)$$



Approximation Thm: 1)  $v - \text{proj}_W v$  is  $\perp$  to every vector in  $W$

2)  $\text{proj}_W v$  is the best approximation to  $v$  by vectors in

$W$ : i.e.  $\forall w \in W$

$$\|v - \text{proj}_W v\| < \|v - w\|$$

Pf: (1) calc.  
(2)  $v - w = (v - \text{proj}_W v) + (\text{proj}_W v - w)$ ;

e.g.  $W = \{ (x, y, z) \mid x - z = 0 \}$

The best approximation to  $(1, 0, 0)$  in  $W$  is

$$\text{proj}_W(1, 0, 0) = (\frac{1}{2}, 0, \frac{1}{2})$$

— How can we find orthogonal bases for a subspace  $W$ ?

### 4.5.3 Gram-Schmidt Orthogonalization algorithm

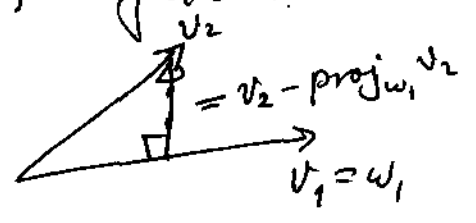
(p. 216) idea: convert any basis of  $W$  into an orthogonal one.

Let  $\{v_1, \dots, v_n\}$  be a basis for a subspace  $W$  of  $\mathbb{R}^n$

We produce an orthogonal basis  $\{w_1, \dots, w_n\}$  as follows:

Set  $\bullet w_1 = v_1$

$\bullet w_2 = v_2 - \text{proj}_{w_1} v_2$   
 $= v_2 - \left( \frac{v_2 \cdot w_1}{\|w_1\|^2} \right) w_1$



$\bullet w_3 = v_3 - \text{proj}_{w_1} v_3 - \text{proj}_{w_2} v_3$   
 $= v_3 - \left( \frac{v_3 \cdot w_1}{\|w_1\|^2} \right) w_1 - \left( \frac{v_3 \cdot w_2}{\|w_2\|^2} \right) w_2$

l.g.

(68)

$$W = \{ (x, y, z, w) \in \mathbb{R}^4 \mid x + y + z + w = 0 \}$$

$$\begin{matrix} x & y & z & w \\ [1 & 1 & 1 & 1 \mid 0] \end{matrix}$$

① Find a basis:

$$\begin{aligned} x_1 &= -z - w \\ x_2 &= z \\ x_3 &= w \\ x_4 &= t \end{aligned}$$

$$\left\{ \overset{v_1}{(-1, 1, 0, 0)}, \overset{v_2}{(-1, 0, 1, 0)}, \overset{v_3}{(-1, 0, 0, 1)} \right\}$$

G-S  $w_1 = (-1, 1, 0, 0)$

$$\begin{aligned} w_2 &= v_2 - \frac{v_2 \cdot w_1}{\|w_1\|^2} w_1 = (-1, 0, 1, 0) - \frac{1}{2} (-1, 1, 0, 0) \\ &= \left(-\frac{1}{2}, \frac{1}{2}, 1, 0\right) \quad \|w_2\|^2 = \frac{3}{2} \end{aligned}$$

$$\begin{aligned} w_3 &= v_3 - \left( \frac{v_3 \cdot w_1}{\|w_1\|^2} \right) w_1 - \left( \frac{v_3 \cdot w_2}{\|w_2\|^2} \right) w_2 \\ &= (-1, 0, 0, 1) - \frac{1}{2} (-1, 1, 0, 0) - \frac{\frac{1}{2}}{\frac{3}{2}} \left(-\frac{1}{2}, \frac{1}{2}, 1, 0\right) \\ &= \left(-\frac{1}{3}, \frac{1}{3}, -\frac{1}{3}, 1\right) \end{aligned}$$

$$\therefore \left\{ \overset{w_1}{(-1, 1, 0, 0)}, \overset{w_2}{\left(-\frac{1}{2}, \frac{1}{2}, 1, 0\right)}, \overset{w_3}{\left(-\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}, 1\right)} \right\}$$

is an orthonormal basis of  $W$ .

ex. Find best approx to  $u = (1, 0, 0, 0)$  in  $W$ .

6/11/03 <sup>lect 18</sup>  
Chp 2 Determinants

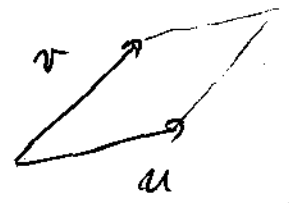
a number associated to a matrix A

(69, 70)  
 7

Geometrically - are about area, volume, hyper-volume

Algebraically - tell us when A is invertible.

Recall:



Area =  $\|u \times v\|$

$u = (a, b, 0)$

$u \times v = (0, 0, ad - bc)$

$v = (c, d, 0)$

$\therefore \text{area} = |ad - bc|$

Defn The determinant of  $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$  is  $ad - bc$ .

Written

$\begin{vmatrix} a & b \\ c & d \end{vmatrix}$

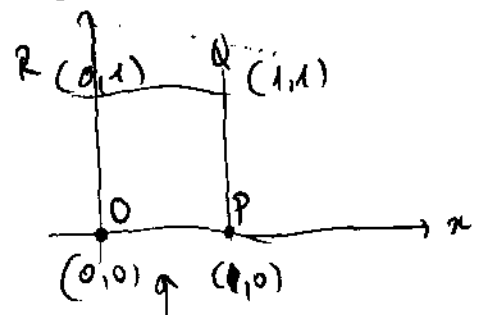
,  $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

; (note:  $\det A = \det A^t$  !)

Defn Determinants are related to area in another way:

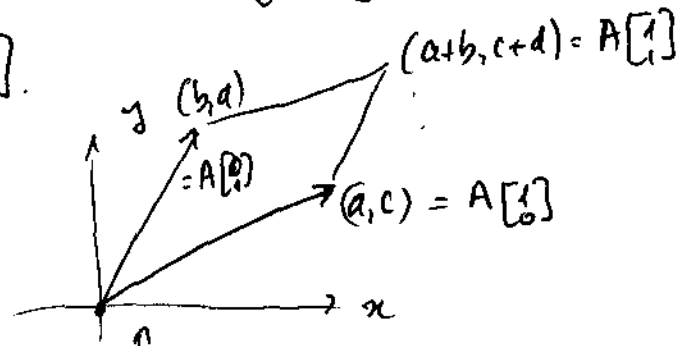
Suppose we transform the <sup>unit</sup> square by multiplying each point in the plane by a matrix  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$T_A$  i.e. send  $(x, y)$  to  $A \begin{bmatrix} x \\ y \end{bmatrix}$ .



area = 1

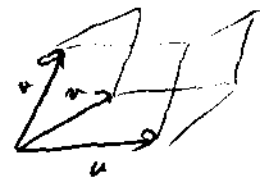
$T_A$   
 Linear transformation



area =  $|\det \begin{bmatrix} a & b \\ c & d \end{bmatrix}|$  !

exercise 1) Show that  $\text{area } T_A \left( \begin{matrix} R \\ Q \\ P \\ O \end{matrix} \right) = |\det A| \text{ area } \begin{matrix} R \\ Q \\ P \\ O \end{matrix}$

12/11/02 23/11/00  
 In  $\mathbb{R}^3$  the vol. of parallelepiped



(7.2)  
72

is  $|u \cdot v \times w|$ . (see 8.13 P 288)

If  $u = (a_1, b_1, c_1)$ ,  $v = (a_2, b_2, c_2)$ ,  $w = (a_3, b_3, c_3)$

$$\begin{aligned} \text{then } u \cdot v \times w &= (a_1, b_1, c_1) \cdot \underbrace{\left( \begin{matrix} |b_2 c_2| & -|a_2 c_2| & |a_2 b_2| \\ |b_3 c_3| & -|a_3 c_3| & |a_3 b_3| \end{matrix} \right)}_{v \times w!} \\ &= a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}. \end{aligned}$$

(3.3)

Def<sup>n</sup> The determinant of

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{matrix} a_1, b_1, c_1 \\ a_2, b_2, c_2 \\ a_3, b_3, c_3 \end{matrix}$$



(see P. 278-279)

3x3 det

(Computed, defined in terms of 2x2!).

(1.5.8)

26/3/02 =  $[a_{ij}]$

Def<sup>n</sup> Suppose  $A$  is an  $n \times n$  matrix and we know how

(Thm 8.8)

to compute  $(n-1) \times (n-1)$  determinants. Then

$$\det A = a_{11} C_{11}(A) + a_{12} C_{12}(A) + \dots + a_{1n} C_{1n}(A)$$

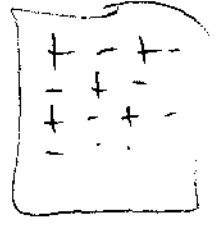
where

$$C_{ij}(A) = (-1)^{i+j} \det \begin{matrix} A_{ij} \text{ mtr} \end{matrix}$$

$((n-1) \times (n-1)$  matrix obtained by deleting  $i$ th row &  $j$ th column of  $A$ )

"(i,j) cofactor"

- pattern  $\rightarrow$



Properties of determinants - what operations don't change, then which do: Goal: how to evaluate them quickly

(73)

pbm: count # of mults

$2 \times 2$  - 2 mults of 2 numbers  
 $3 \times 3$  -  $6 = 3 \cdot 2$  " 3 "  
 $4 \times 4$  -  $4 \times 3 \times 2$  " 4 "  
 $25 \times 25$  -  $25!$  " 25

Now  $25! \approx 1.6 \times 10^{25}$ ; so for a  $25 \times 25$  matrix, there are  $\sim 4 \times 10^{26}$  ops  
 On a 1 THz machine @ 1 mult / clock cycle

$$\frac{4 \times 10^{26}}{10^{12} \times 7 \times 10^7} \text{ years} > 10^7 \text{ years.}$$

- main strategies

• expand along any row or column - get same answer (pick row, col with many zeros)

• adding multiple of one row (resp. column) to another row (resp. col) doesn't change det!

• interchanging rows (resp. cols) changes sign  
 • mult row by  $k$  changes det by  $k$ .

effect of row ops

• upper (lower) triangular matrices:

$$\det \begin{pmatrix} a_{11} & * & * \\ 0 & a_{22} & * \\ 0 & 0 & a_{33} \end{pmatrix} \text{ determinants are easy to compute}$$

$$= a_{11} \cdot a_{22} \cdot a_{33} !$$

e.g.  $|A| = \begin{vmatrix} 1 & 0 & 1 \\ 2 & 0 & 2 \\ 1 & -1 & 3 \end{vmatrix} \stackrel{\text{def}}{=} 1 C_{11}(A) + 0 C_{12} + 1 C_{13}(A)$

$$= 1 \begin{vmatrix} 0 & 2 \\ -1 & 3 \end{vmatrix} + 0 \begin{vmatrix} 1 & 2 \\ 1 & 2 \end{vmatrix} + 1 \begin{vmatrix} 1 & 0 \\ 2 & 0 \end{vmatrix}$$

22 For  
 • another row

74

$$\begin{vmatrix} 1 & 0 & 1 \\ 2 & 6 & 2 \\ 1 & -1 & 3 \end{vmatrix} \quad \text{row 3} \\ = 1 \cdot \begin{vmatrix} 0 & 1 \\ 6 & 2 \end{vmatrix} - (-1) \begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix} + 3 \begin{vmatrix} 1 & 0 \\ 2 & 6 \end{vmatrix} \\ = -6 + 18 = 12!$$

• a column

$$\begin{vmatrix} 1 & 0 & 1 \\ 2 & 6 & 2 \\ 1 & -1 & 3 \end{vmatrix} \quad \text{col 2} \\ = 0 \cdot \begin{vmatrix} * & * \\ * & * \end{vmatrix} + 6 \cdot \begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix} - (-1) \begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix} \\ = 12! \quad \text{ex.: col 1: } \begin{vmatrix} 6 & 2 \\ -1 & 3 \end{vmatrix} - 2 \begin{vmatrix} 0 & 1 \\ -1 & 3 \end{vmatrix} + 1 \begin{vmatrix} 0 & 1 \\ 6 & 2 \end{vmatrix} = 20 - 6 + 12$$

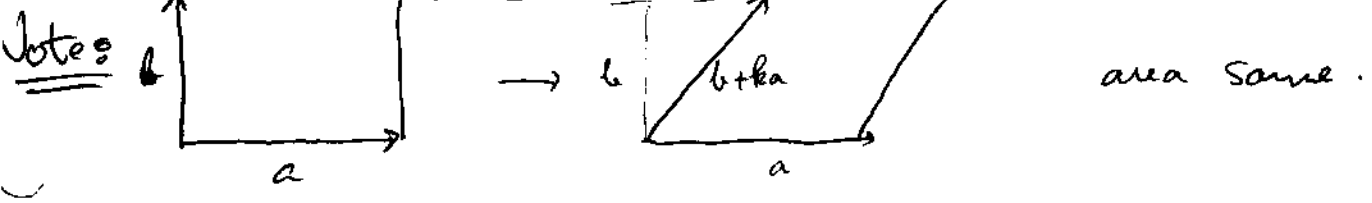
N.B. This means  $\det A = \det A^t$  !

• Upper Triangular

$$\begin{vmatrix} 7 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{vmatrix} \quad \text{col 1!} \\ = 7 \cdot \begin{vmatrix} 4 & 5 \\ 0 & 6 \end{vmatrix} + 0 + 0 \\ = 7 \cdot 4 \cdot 6 = \text{product of diag. elts.}$$

Thm 1 P. 75 (1)  $\det A$  can be expanded along any row or column. (2) If  $A$  is upper (or lower) triangular,  $\det A$  is the product of the diagonal entries.

Corollary: If  $A$  has a row or col. of zeros,  $\det A = 0$ .



# Effect of row operations on $\det A$ :

(76)

$$\bullet \det A = \begin{vmatrix} 1 & 0 & 1 \\ 2 & 6 & 2 \\ 1 & -1 & 3 \end{vmatrix} \xrightarrow{-2R_1 + R_2 \rightarrow R_2} \begin{vmatrix} 1 & 0 & 1 \\ 0 & 6 & 0 \\ 1 & -1 & 3 \end{vmatrix} \xrightarrow{-R_1 + R_3 \rightarrow R_3} \begin{vmatrix} 1 & 0 & 1 \\ 0 & 6 & 0 \\ 0 & -1 & 2 \end{vmatrix} = \begin{vmatrix} 6 & 0 \\ -1 & 2 \end{vmatrix} = 12 - 0 = 12$$

$$\bullet \begin{vmatrix} 1 & 0 & 1 \\ 0 & 6 & 0 \\ 0 & -1 & 2 \end{vmatrix} \xrightarrow{-R_1 + R_3 \rightarrow R_3} \begin{vmatrix} 1 & 0 & 1 \\ 0 & 6 & 0 \\ 0 & -1 & 2 \end{vmatrix} \xrightarrow{\text{col}} \begin{vmatrix} 6 & 0 \\ -1 & 2 \end{vmatrix} = 12!$$

$$\bullet \begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & -1 & 2 \end{vmatrix} \xrightarrow{\left(\frac{1}{6}\right)R_2 \rightarrow R_2} \begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & -1 & 2 \end{vmatrix} \xrightarrow{\text{col}} \begin{vmatrix} 1 & 0 \\ 1 & 2 \end{vmatrix} = 2 = 12 \cdot \left(\frac{1}{6}\right)$$

$$\bullet R_1 \leftrightarrow R_3 \begin{vmatrix} 0 & -1 & 2 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix} \xrightarrow{\text{col}} \begin{vmatrix} -1 & 2 \\ 1 & 0 \end{vmatrix} = -2 \quad (\text{change of sign})$$

Thm 2 P77 (a) adding a multiple of a row to a different row does not change  $\det A$

(b) multiply a single row of  $A$  by  $k$  changes  $\det A$  by the factor  $k$

(c) interchanging 2 rows of  $A$  changes the sign of  $\det A$ .

Remark: same statements hold for "column" replacing "row".

Cor: (1) if  $A$  has a row which is a multiple of another,  $\det A = 0$  eg  $\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 2 & 2 & 2 \end{vmatrix} = 0$

(2)  $\det(kA) = k^n \det A$  if  $A$  is  $n \times n$ ,  $k \in \mathbb{R}$ .

Remark  $A \sim A'$  (REF form: upper triangular!)

If we use  $j$  row interchanges, and multiply various rows

by  $k_1, \dots, k_r$ , then  $\det A' = (-1)^j k_1 \dots k_r \det A$  ( $k_i \neq 0$ !)

# Strategy for computing $\det A$

(76)

- ① Use row/col ops to (almost) clean or close rows or col
- ② expand along close col/row
- ③ return to ① if necessary

e.g.  $\begin{vmatrix} 1 & -2 & 3 & -1 \\ 1 & 1 & -2 & 0 \\ 2 & 0 & 4 & -5 \\ 1 & 4 & 4 & -6 \end{vmatrix} \xrightarrow{\substack{2C_1+C_2 \rightarrow C_2 \\ \text{clean row 2}}} \begin{vmatrix} 1 & -3 & 5 & -1 \\ 1 & 0 & 0 & 0 \\ 2 & -2 & 8 & -5 \\ 1 & 3 & 6 & -6 \end{vmatrix} \xrightarrow{\text{row 2}} \begin{vmatrix} -3 & 5 & -1 \\ -2 & 8 & -5 \\ 3 & 6 & -6 \end{vmatrix}$

$\xrightarrow{C_1+C_2 \rightarrow C_1} \begin{vmatrix} 0 & 0 & -1 \\ 13 & -17 & -5 \\ 21 & -24 & -6 \end{vmatrix} = (-1) \begin{vmatrix} 13 & -17 \\ 21 & -24 \end{vmatrix} = 3 \cdot \begin{vmatrix} 13 & -17 \\ 7 & -8 \end{vmatrix} = 3 \begin{vmatrix} -1 & -1 \\ 7 & -8 \end{vmatrix} = 3 \cdot (15) = 45$

$-2R_2 + R_1 \rightarrow R_1$

e.g. (comp. later)  $\lambda$  unknown  $\begin{vmatrix} 3-\lambda & 2 & 4 \\ 2 & -\lambda & 2 \\ 4 & 2 & 3-\lambda \end{vmatrix} \xrightarrow{-R_1+R_3 \rightarrow R_3} \begin{vmatrix} 3-\lambda & 2 & 4 \\ 2 & -\lambda & 2 \\ 1+\lambda & 0 & -1-\lambda \end{vmatrix} \xrightarrow{C_1+C_2 \rightarrow C_1} \begin{vmatrix} 3-\lambda & 2 & 7-\lambda \\ 2 & -\lambda & 4 \\ 1+\lambda & 0 & 0 \end{vmatrix}$

$= (1+\lambda) \begin{vmatrix} 2 & 7-\lambda \\ -\lambda & 4 \end{vmatrix} = (1+\lambda) \{ 8 + \lambda(7-\lambda) \} = (1+\lambda) (-\lambda^2 + 7\lambda + 8)$   
 $= (1+\lambda) (8-\lambda)(1+\lambda)$   
 $= (1+\lambda)^2 (8-\lambda)$

Link  $\begin{vmatrix} a & b & c & 0 & 0 \\ 0 & a & b & c & 0 \\ 0 & 0 & a & b & c \\ 0 & 0 & 0 & a & b \\ 0 & 0 & 0 & 0 & a \end{vmatrix} \begin{vmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}; \det = -1.$

②  $\begin{vmatrix} 0 & 15 \\ 3 & -10 \\ 2 & 0 & 1 \end{vmatrix} = \begin{vmatrix} -10 & 15 \\ 3 & -10 \\ 0 & 0 & 1 \end{vmatrix} = 7; \quad -2R_2 + 3R_3 \rightarrow R_2 \begin{vmatrix} 0 & 15 \\ 0 & 23 \\ 2 & 0 & 1 \end{vmatrix} = 2 \cdot (-7) = -14$

# Determinants and Matrix Inverses

(77)

Thm 2: (P. 82) An  $n \times n$  matrix  $A$  is invertible  $\Leftrightarrow \det A \neq 0$ . (add to list...)

PF " $\Rightarrow$ ": Assume  $A$  invertible; then  $A \sim I_n$ , i.e. RC form of  $A$  is  $I_n$

$\therefore$  from above  $\det A = (-1)^{\sum k_i} \frac{1}{k_1} \dots \frac{1}{k_n} \cdot \det(I_n) \neq 0$ .

" $\Leftarrow$ ": If  $\det A \neq 0$ ,  $\det A = (-1)^{\sum k_i} \frac{1}{k_1} \dots \frac{1}{k_r} \det(\tilde{A})$  ( $\tilde{A}$  RC form of  $A$ )  
 implies  $\det \tilde{A} \neq 0$ ; since  $\tilde{A}$  is upper triangular,  $\Rightarrow \tilde{A}$  has no row of zeros,  
 so  $\text{rank } A = n$ ;  $\therefore A$  is invertible

(27) Thm 1  $\det(AB) = \det A \det B$  \* (See web!)  
 (P. 82)

Different pf from book: If  $A$  is not invertible, then neither is  $AB$  (remember?)

so both sides of eqn are zero.

Suppose  $A$  is invertible. Consider  $[A|AB]$ ; now  $A \sim I_n$  ... ?

$\sim [I_n | M]$ ; claim:  $M = B$ ! We know row reduction doesn't change  $\ker$ ; so  $\ker [A|AB] = \ker [I|M]$ ; Note:  $\{e_1, \dots, e_n\}$  is a basis of  $\mathbb{R}^n$

$[A|AB] \begin{bmatrix} B e_i \\ -e_i \end{bmatrix} = A B e_i - A e_i = 0; i=1, \dots, n.$

Hence,  $0 = [I|M] \begin{bmatrix} B e_i \\ -e_i \end{bmatrix} = B e_i - M e_i = 0; i=1, \dots, n \therefore M = B.$

$[A|AB] \sim [I_n | B]$ ;  $\det A = (-1)^{\sum k_i} \frac{1}{k_1} \dots \frac{1}{k_n} \det I_n$ ; same row ops take

# Chpt. 3 Evls, Freqs, Diagonalization (square matrices) $\mathbb{R}$

• Many processes modelled by  $x \mapsto Ax \mapsto A^2x \mapsto \dots \mapsto A^N x$  ?  
input      output  $N \gg 1$

• if  $A = \begin{bmatrix} a_{11} & & 0 \\ & \ddots & \\ 0 & & a_{nn} \end{bmatrix}$ ,  $A^N = \begin{bmatrix} a_{11}^N & & 0 \\ & \ddots & \\ 0 & & a_{nn}^N \end{bmatrix}$  easy  $\lambda_{ii}$  largest det's behavior.

• not all matrices are diagonal: what are the special properties of diagonal matrices? Goal: "diagonalize" matrices where possible

eg  $D = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$   $D \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$   $D \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 4 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$   
 $D e_1 = e_1$   $D e_2 = 4 e_2$   $\{e_1, e_2\}$  basis of  $\mathbb{R}^2$

eg  $A = \begin{bmatrix} 3 & -1 \\ -2 & 2 \end{bmatrix}$   $A \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 1 \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix}$   $A \begin{bmatrix} -1 \\ 1 \end{bmatrix} = 4 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$   
 $A v_1 = v_1$   $A v_2 = 4 v_2$

$\downarrow$  w/  $P = [v_1 \ v_2]$   
 $A [v_1 \ v_2] = [A v_1 \ A v_2] = [v_1 \ 4 v_2] = [v_1 \ v_2] \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} = D$

note.  $P = [v_1 \ v_2] = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}$  inv (s.  $\{v_1, v_2\}$  basis of  $\mathbb{R}^2$ ) so

$$AP = PD \quad \text{or} \quad D = P^{-1}AP$$

$$A = PDP^{-1}$$

$$A^N = \underbrace{PDP^{-1}} \underbrace{PDP^{-1}} \dots \underbrace{PDP^{-1}} = PD^N P^{-1} \text{ easy!}$$

Def 4 P.90 An  $n \times n$  matrix is diagonalizable ("diag'ble") if there is an invertible matrix  $P$  and a diagonal matrix  $D$  s.t.

$$P^{-1}AP = D$$

eg.  $D, A$  above are diag'ble (every diagonal matrix is diag'ble)  $P \neq I$

How can we (try to) diagonalize any matrix  $A$ ?

(79)

Need special vectors  $v_1, v_2$  above:

Def<sup>n</sup> 1.90 • A non-zero vector  $v$  is an eigen vector of a matrix  $A$  if  $Av = \lambda v$  for some scalar  $\lambda$ , called an eigenvalue of  $A$  (conseq. to evcs (eig  $v$  corresponds to eval  $\lambda$ )).

eg.  $A = \begin{bmatrix} 3 & -1 \\ -2 & 2 \end{bmatrix}$ ,  $\lambda$  is an eval  $v_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$  or evcs corres. to  $\lambda = 4$   
 $\lambda = 1$  " "  $v_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$  " "  $\lambda = 1$

eg  $D = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}$   $\lambda_1, \lambda_2$  are evals  $\downarrow$   $e_1, e_2$  are conseq. evcs. Same evals  
differs evcs

• UB in our example above, cols of  $P = [v_1 \ v_2]$  were evcs of  $A$

Thm 3 (P. 94)  $A$  is diag<sup>l</sup>ble  $\Leftrightarrow$  there is a basis  $\{v_1, \dots, v_n\}$  of  $\mathbb{R}^n$  consisting of evcs of  $A$ . Moreover, if it is, if  $P = [v_1 \ \dots \ v_n]$  ( $v_j = j^{\text{th}}$  col)

then  $P^{-1}AP = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$  is diagonal, where  $\lambda_1, \dots, \lambda_n$  are the evals of  $A$ .

$[D = P^{-1}AP \Leftrightarrow AP = PD^T \Leftrightarrow A[v_1 \ \dots \ v_n] = [v_1 \ \dots \ v_n] \begin{bmatrix} \lambda_1 & 0 & \\ & \lambda_2 & 0 \\ 0 & & \lambda_n \end{bmatrix}$   $\{v_1, \dots, v_n\}$  basis of  $\mathbb{R}^n$

$\Leftrightarrow [Av_1 \ \dots \ Av_n] = [\lambda_1 v_1 \ \dots \ \lambda_n v_n]$  &  $\{v_1, \dots, v_n\}$  basis of  $\mathbb{R}^n$

$\Leftrightarrow \begin{matrix} Av_1 = \lambda_1 v_1 \\ \vdots \\ Av_n = \lambda_n v_n \end{matrix} \quad \{v_1, \dots, v_n\} \text{ basis of } \mathbb{R}^n$

How can we find <sup>enough</sup>  $\lambda$  evcs? evcs?

- $\lambda$  is an eval of  $A \Leftrightarrow \exists v \in \mathbb{R}^n, v \neq 0$  st  $Av = \lambda v$
- $\Leftrightarrow \exists v \neq 0$  st  $(A - \lambda I)v = 0$
- $\Leftrightarrow (A - \lambda I)$  is not inv
- $\Leftrightarrow \det(A - \lambda I) = 0!$

Thm 2  $\lambda$  is an eval of  $A$  iff  $\det(A - \lambda I) = 0$ .

e.g.  $\begin{vmatrix} 3-\lambda & -1 \\ -2 & 2-\lambda \end{vmatrix} = \lambda^2 - 5\lambda + 6 - 2 = \lambda^2 - 5\lambda + 4 = (\lambda-1)(\lambda-4)$

(80)

$\lambda=1, 4$  are evals.

Find evals: solve  $Av = \lambda v$  for  $\lambda=1$ , then  $\lambda=4$ !

$(A-\lambda I)v = 0$       "

↳ solve  $(A-I)v = 0$       &       $(A-4I)v = 0$

Rank.  $\lambda$  an eval of  $A$ ,  $E_\lambda := \{v \in \mathbb{R}^n \mid Av = \lambda v\}$   
 $= \{v \in \mathbb{R}^n \mid (A-\lambda I)v = 0\} = \ker(A-\lambda I)!$

is the eigenspace corresponding to eval  $\lambda$ ;

$E_1 = \text{span}\{(1,2)\}$ ,  $E_4 = \text{span}\{(1,1)\}$  IN THIS CASE.

e.g.  $J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$        $c_J(\lambda) = \begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 + 1 = 0 \Leftrightarrow \lambda = \pm i$

$J$  has no real evals. Can diagonalize  $J$  but need to use complex vectors... (So,  $J$  not diag'ble over reals)

e.g.  $B = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$        $c_B(\lambda) = \begin{vmatrix} 2-\lambda & 1 \\ 0 & 2-\lambda \end{vmatrix} = (2-\lambda)^2$ ;

only eval is  $\lambda=2$ ; (Rank. evals of triangular (upper or lower) matrices are the diagonal entries!)

Find evals:

$E_2 = \{v \mid (B-2I)v = 0\} = \ker(B-2I)$ ;  $[B-2I \mid 0] = \begin{bmatrix} 0 & 1 & \mid & 0 \\ 0 & 0 & \mid & 0 \end{bmatrix}$   $x=A$   
 $y=0$

$E_2 = \text{span}\{(1,0)\}$ . There isn't a basis of  $\mathbb{R}^2$  consisting of evals for  $B$ !

B is not diag'ble.

e.g.  $A = \begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix}$        $c_A(\lambda) = \begin{vmatrix} 3-\lambda & 2 & 4 \\ 2 & -\lambda & 2 \\ 4 & 2 & 3-\lambda \end{vmatrix} = \begin{vmatrix} 3-\lambda & 2 & 4 \\ 2 & -\lambda & 2 \\ 1+\lambda & 0 & -1-\lambda \end{vmatrix}$

$= \begin{vmatrix} 3-\lambda & 2 & 7-\lambda \\ 2 & -\lambda & 4 \\ 1+\lambda & 0 & -1-\lambda \end{vmatrix} \xrightarrow{R_1 \leftrightarrow R_3} \begin{vmatrix} 1+\lambda & 0 & -1-\lambda \\ 2 & -\lambda & 4 \\ 3-\lambda & 2 & 7-\lambda \end{vmatrix} = (1+\lambda) \begin{vmatrix} 2 & 7-\lambda \\ -\lambda & 4 \end{vmatrix} = \dots = (1+\lambda)^2 (8-\lambda)$       No B Need to factoriz

Find evecs:  $E_{-1} = \ker(A+I)$ ;  $[A+I|0] = \begin{bmatrix} 4 & 2 & 4 & | & 0 \\ 2 & 1 & 2 & | & 0 \\ 4 & 2 & 4 & | & 0 \end{bmatrix}$  (9)

$\sim \dots \begin{bmatrix} 1 & \frac{1}{2} & 1 & | & x = -\frac{1}{2} - t \\ 0 & 0 & 0 & | & y = 0 \\ 0 & 0 & 0 & | & z = t \end{bmatrix}; t \in \mathbb{R} \therefore \{ \overset{v_1}{(-\frac{1}{2}, 1, 0)}, \overset{v_2}{(-1, 0, 1)} \}$  basis for

$E_{-1}$  (2 l.i. evecs so far!)

$E_8 = \{v \mid Av = 8v\} = \ker(A-8I)$ ;  $[A-8I|0] = \begin{bmatrix} -5 & 2 & 4 & | & 0 \\ 2 & -8 & 2 & | & 0 \\ 4 & 2 & -5 & | & 0 \end{bmatrix}$

$\sim \dots \begin{bmatrix} 1 & 0 & -1 & | & 0 \\ 0 & 1 & \frac{1}{2} & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix} \begin{matrix} x = 0 \\ y = +\frac{1}{2} \Delta \\ z = \Delta \end{matrix} \Delta \in \mathbb{R} \therefore \{ \overset{v_3}{(1, \frac{1}{2}, 1)} \}$  basis for  $E_8$

$\{v_1, v_2, v_3\}$  basis of  $\mathbb{R}^3$ ? Set  $P = [v_1 \ v_2 \ v_3]$   $v_j = \text{col}$

$= \begin{bmatrix} -\frac{1}{2} & -1 & 1 \\ 1 & 0 & \frac{1}{2} \\ 0 & 1 & 1 \end{bmatrix}; \det P = \begin{vmatrix} -\frac{1}{2} & -1 & 1 \\ 1 & 0 & \frac{1}{2} \\ 0 & -1 & 0 \end{vmatrix} = - \begin{vmatrix} -\frac{1}{2} & 1 \\ 1 & \frac{1}{2} \end{vmatrix} = -(-\frac{1}{4} - 2) = 2\frac{1}{4} \neq 0$

so  $AP = A[v_1 \ v_2 \ v_3] = [Av_1 \ Av_2 \ Av_3] = [-v_1 \ -v_2 \ 8v_3]$

$= [v_1 \ v_2 \ v_3] \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 8 \end{bmatrix}$  i.e.  $AP = PD$  so  $D = P^{-1}AP$

$D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 8 \end{bmatrix}$

Remark ① (Thm 9, P241) Eigenvectors corresp. to distinct evals are always l.i.

①  $\Rightarrow$  ② (Thm 4, P95) An  $n \times n$  matrix with  $n$  distinct evals is always diagonalizable (each different eval yields l.i. evec).  $\begin{bmatrix} 1 & 6 & 8 \\ 0 & 0 & \pi \\ 0 & 0 & 40 \end{bmatrix}$  diag  
 • ( $\Leftarrow$  "is not true)

①  $\Rightarrow$  ③ If  $\lambda_1, \dots, \lambda_k$  are  $k$  distinct evals of  $A$ , then  $A$  is diagonalizable  $\Leftrightarrow \dim E_{\lambda_1} + \dim E_{\lambda_2} + \dots + \dim E_{\lambda_k} = n$

# Diagonalization Algorithm (P. 96)

- ① Compute  $C_A(\lambda)$  and so find distinct roots of  $A$
- ② Find a basis for each eigenspace
- ③ If there are  $n$  vectors in all,  $A$  is diagonalizable; if not it isn't
- ④ If  $A$  is diagonalizable, let  $P = [v_1 \dots v_n]$  be constructed with vectors from ② as columns; let  $D = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$  have diagonal entries corresponding to the roots assoc to  $v_1, \dots, v_n$  i.e. if  $Av_i = \lambda_i v_i$ , then  $D = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$

## 4.9 Linear Transformations ( $\mathbb{R}^n \rightarrow \mathbb{R}^m$ )

Recall example  $W = \{ (x, y, z) \mid x-z=0 \}$ ; had orthogonal projector  $P = \text{proj}_W(x, y, z) = \frac{1}{2}(x+z, 2y, x+z)$ ;  $P(v) = v$  if  $v = (x, y, z)$

$P = \text{proj}_W : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  (domain  $\mathbb{R}^3$ , range is  $W$ )

$P$  has special properties; let  $v = (x, y, z)$ ,  $w = (x', y', z')$

1)  $P(v+w) = P(x+x', y+y', z+z') = \frac{1}{2}(x+x'+z+z', 2y+y', x+x'+z+z')$   
 $= \frac{1}{2}(x+z, 2y, x+z) + \frac{1}{2}(x'+z', 2y', x'+z')$   
 $= P(v) + P(w) !$

2) if  $k \in \mathbb{R}$ ,  $P(kv) = P(kx, ky, kz) = \frac{1}{2}(kx+kz, 2ky, kx+kz)$   
 $= k \left( \frac{1}{2}(x+z, 2y, x+z) \right)$   
 $= kP(v).$

• another example: Let  $A = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix}$ , define a function

$T_A : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  by  $T_A(v) = Av$  for  $v \in \mathbb{R}^3$  (col.)

Note: If  $v, w \in \mathbb{R}^3$  &  $k \in \mathbb{R}$

83

$$\textcircled{1} T_A(v+w) = A(v+w) = Av + Aw = T_A(v) + T_A(w)$$

$$\textcircled{2} T_A(kv) = A(kv) = k(Av) = k \cdot T_A(v)$$

Defn If  $U, V$  are vector spaces a function

$T: U \rightarrow V$  is a linear transformation

$$\textcircled{1} T(u+u') = T(u) + T(u') \quad \forall u, u' \in U$$

$$\textcircled{2} T(ku) = k \cdot T(u) \quad \forall u \in U, k \in \mathbb{R}$$

e.g.  $P = \text{proj}_w$ ,  $T = T_A$  are l.o.t.

ex  $F: \mathbb{R}^2 \rightarrow \mathbb{R}$  def'd by  $F(x, y) = xy$  is not,

since  $F(1, 0) = 0$ ,  $F(0, 1) = 0$  but

$$F(1, 0) + F(0, 1) = 0 \neq F(1, 1) = 1$$

Remark: They are called "linear" because if  $S: \mathbb{R}^3 \rightarrow \mathbb{R}^3$

is a l.o.t.  $\Leftrightarrow S$  sends through 0 to lines through 0.  
(and other lines to other lines)

e.g.  $P = \text{proj}_w$  as before,  $L = \{(0, s, 2s) \mid s \in \mathbb{R}\} = \text{line through } 0, \text{ dir}^n (0, 1, 2)$

$P(L) = \{\frac{1}{2}(2s, 2s, 2s) \mid s \in \mathbb{R}\} = \{(s, s, s) \mid s \in \mathbb{R}\} = \text{line through } 0, \text{ dir}^n (1, 1, 1)$ .

Thm 1 If  $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$  is a l.o.t., it is completely determined

P258 by what it does to a basis of  $\mathbb{R}^n$ ! In fact if  $\{v_1, \dots, v_n\}$  is

any basis of  $\mathbb{R}^n$  and  $\{w_1, \dots, w_n\}$  are any vectors in  $\mathbb{R}^m$ , there is

a unique l.o.t.  $S: \mathbb{R}^n \rightarrow \mathbb{R}^m$  s.t.  $S(v_i) = w_i$ ,  $i=1, \dots, n$ .

$T$  wasn't so special!

Prop 2 : Let  $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$  be a l.t. Then there is an  $m \times n$  matrix  $A$  s.t.  $T(v) = Av$  for all  $v \in \mathbb{R}^n$ !

(84)

Moreover, if  $\{e_1, \dots, e_n\}$  is the standard ordered basis of  $\mathbb{R}^n$ ,  $A = [T(e_1), \dots, T(e_n)]$  is the matrix. (Standard matrix of  $T$ )

pf. Let  $v = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ ;  $T(v) = T(x_1 e_1 + \dots + x_n e_n)$

$$= x_1 T(e_1) + \dots + x_n T(e_n)$$

&  $A = [T(e_1) \dots T(e_n)]$

$\uparrow$                        $\uparrow$   
 cols

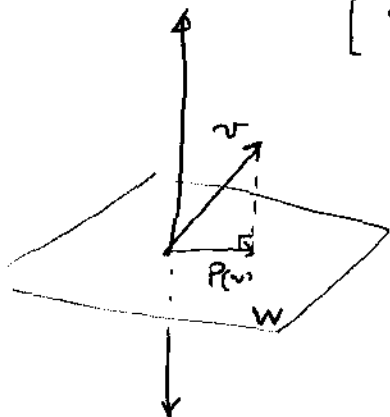
$$= [T(e_1) \dots T(e_n)] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

$$= Av!$$

e.g.  $P = \text{proj}_W$  ; let's find its matrix :

$P(x, y, z) = \frac{1}{2}(x+z, 2y, x+z)$  ;  $P(e_1) = \begin{bmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} \end{bmatrix}$   $P(e_2) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$   $P(e_3) = \begin{bmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} \end{bmatrix}$

$\therefore P(x, y, z) = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{2} \begin{bmatrix} x+z \\ 2y \\ x+z \end{bmatrix} \checkmark$



$P$  sends some vectors to zero (normal line)

$P$  sends all vectors onto  $W$  ; "fills out"  $W$ .

Def<sup>n</sup> (P315) Let  $T: U \rightarrow V$  be a l.t. Then

the kernel of  $T = \ker T = \{u \in U \mid T(u) = 0\}$  - s.s. of  $U$

& the image of  $T = \text{im } T = \{T(u) \in V \mid u \in U\}$  - s.s. of  $V$

eg  $P(v) = \frac{1}{2}(x+3, 2y, x+3)$ ;  $P: \mathbb{R}^3 \rightarrow \mathbb{R}^3$  (88)

$\ker P = \{ v \in \mathbb{R}^3 \mid P(v) = 0 \} = \{ (x, y, z) \mid \frac{1}{2}(x+3, 2y, x+3) = (0, 0, 0) \}$

$= \{ (x, y, z) \mid A \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0 \}$ ;  $A = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$

$\begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} & | & 0 \\ 0 & 1 & 0 & | & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & | & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & | & 0 \\ 0 & 1 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix} \begin{matrix} x = -z \\ y = 0 \\ z = z \end{matrix}$

*matrix of P!*

$\therefore \ker P = \{ (-s, 0, s) \mid s \in \mathbb{R} \}$

$= \text{span} \{ (-1, 0, 1) \}$  (normal line to  $h$ )

$\text{im } P = \{ P(v) \mid v \in \mathbb{R}^3 \}$

;  $P(v) = Av$

$= \{ Av \mid v \in \mathbb{R}^3 \}$

$= \text{col } A!$

$\begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$\therefore \text{im } P = \text{span} \{ (\frac{1}{2}, 0, \frac{1}{2}), (0, 1, 0) \}$

$\stackrel{!}{=} W$  (check)

Note:  $\dim \ker P + \dim \text{im } P$

$= \dim \ker A + \dim \text{col}(A)$

$= \dim \ker A + \text{rank } A = 3$  (= dimension of domain)

$= \dim \mathbb{R}^3$ ,  $P: \mathbb{R}^3 \rightarrow \dots$

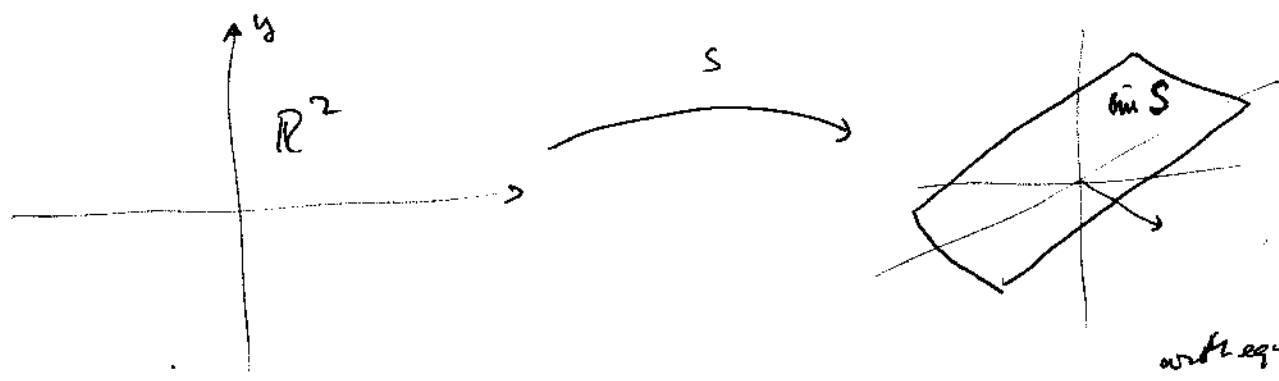
e.g.  $S: \mathbb{R}^2 \rightarrow \mathbb{R}^3$  def'd by  $S(x,y) = (x,y,x+y)$  (8)

note:  $S \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \quad \therefore S(\vec{v}) = A\vec{v}$  so  $S \subseteq$   
a lot.

•  $\ker S = \ker A$ ;  $[A|0] \sim \left[ \begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array} \right] \quad \therefore \ker A = \{ (0,0) \}$

•  $\text{im } S = \text{col } A = \text{span} \{ \overset{v_1}{(1,0,1)}, \overset{v_2}{(0,1,1)} \}$

is plane through 0 with normal  $v_1 \times v_2$   
 $= (1,1,-1)$



$S$  "embeds"  $\mathbb{R}^2$  in  $\mathbb{R}^3$  as plane  $x+y-z=0$  with eq.

$$\begin{aligned} \dim \ker S + \dim \text{im } S &= \dim \ker A + \dim \text{col } A \\ &= \# \text{ col } A = 2 \end{aligned}$$

("Conservation of dimension").