Graphical Model: Homework 1

Question 1 *In the "Burglar-Earthquake-Alarm" model, we will let B, E and A to represent random variables BURGLAR, EARTHQUAKE and ALARM respectively, each taking value from $\{0, 1\}$ (where "1" means the event occurring). We define the following conditions:*

- **No-False-Alarm Condition**: *the ALARM will ring only if either Earthquake or Burglar occurs (or both occur). Notice that with this condition, we do not guarantee that the ALARM will ring as long as either EARTHQUAKE or BURGLAR occurs.*
- **Sensitive-Alarm Condition**: *the ALARM will ring if either Earthquake or Burglar occurs (or both occur). Notice that with this condition, we do not guarantee that there is no false alarm.*
- **Positive-Prior Condition**: *$P(B)$ and $P(E)$ are both strictly positive functions.*

1. Let
$$P(B = 1) = 0.3, P(B = 0) = 0.7;$$
$$P(E = 1) = 0.1, P(E = 0) = 0.9;$$
Assume that both the No-False-Alarm Condition and the Sensitive-Alarm Condition hold. Show that conditioned upon ALARM, EARTHQUAKE and BURGLAR are dependent.

2. Assume that the No-False-Alarm Condition, Sensitive-Alarm Condition and Positive-Prior Condition all hold. Can you construct a set of distributions and conditional distributions, $P(E), P(B), P(A|B, E)$ such that conditioned on ALARM, EARTHQUAKE and BURGLAR are independent? Do so if you can, otherwise prove that it is not possible.

3. Assume that both the No-False-Alarm Condition and the Sensitive-Alarm Condition hold, but the Positive-Prior Condition does not. Can you construct a set of distributions and conditional distributions, $P(E), P(B), P(A|B, E)$ such that conditioned on ALARM, EARTHQUAKE and BURGLAR are independent? Do so if you can, otherwise prove that it is not possible.

4. Assume that both the No-False-Alarm Condition and the Positive-Prior Condition hold, but the Sensitive-Alarm Condition does not. Can you construct a set of distributions and conditional distributions, $P(E), P(B), P(A|B, E)$ such that conditioned on ALARM, EARTHQUAKE and BURGLAR are independent? Do so if you can, otherwise prove that it is not possible.

5. Assume that both the Sensitive-Alarm Condition and the Positive-Prior Condition hold, but the No-False-Alarm Condition does not. Can you construct a set of distributions and conditional distributions, $P(E), P(B), P(A|B, E)$ such that conditioned on ALARM, EARTHQUAKE and BURGLAR are independent? Do so if you can, otherwise prove that it is not possible.

Question 2 *For random variables X, Y and Z, suppose that $P(x|y, z)$ is some positive function, say $f(x, y)$, involving only variables x and y.*

Prove that $X \perp\!\!\!\perp Z|Y$ and that $P(x|y) = f(x, y)$.

Question 3 *A man has two umbrellas. Every day he leaves home for his office in the morning, and comes back home in the evening. In the morning, he will take an umbrella with him to go to work if and only if*

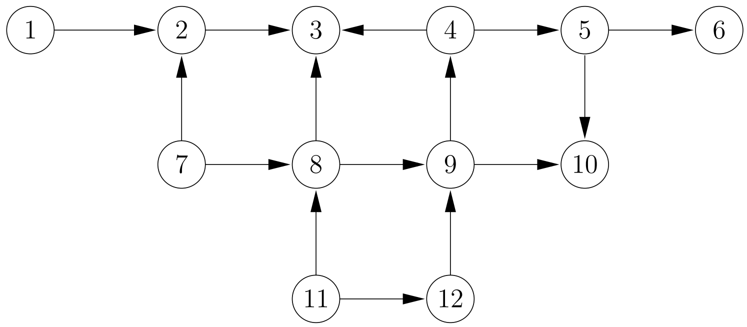

Figure 1: A Bayesian network.

1. it rains and

2. he has an umbrella at home.

Similarly, in the evening, he will take an umbrella with him to go home if and only if

1. it rains and

2. he has an umbrella in his office.

The probabilities of raining in the morning and in the evening are both 0.1. *It is also assumed that raining in the morning and raining in the evening are independent events; further, across days, whether it rains in the morning or evening is also independent.*

On the morning of day 1 before he leaves home, the man has two umbrellas at home.

1. What is the probability that he will get wet in the evening of day 1 (i.e., going back home without umbrella while it is raining)?
2. Denote by X_i the number of umbrellas the man has at home in the evening of day i after he comes back from work. Find the pmf for X_2.
3. Derive the expression of the pmf of X_{10}.

Question 4 *Claim: In a Bayesian network, let A, B, C and S be four disjoint subsets of vertices. If S neither d-separates A from B nor d-separates B from C, then S does not d-separate A from C.*

Is this claim correct? Prove or disprove it.

Question 5 *Consider the Bayesian network in Figure 1, where vertex i represents random variable X_i.*

1. For the following choices of subsets A, B and S, verify whether S d-separates A from B.
   (a) $A = \{1\}, B = \{6\}, S = \{2, 8, 11\}$;
   (b) $A = \{11\}, B = \{6\}, S = \{3, 8, 12\}$;
   (c) $A = \{12\}, B = \{8\}, S = \{6, 11\}$.
2. Let $A = \{2\}$ and $B = \{12\}$. Find the set S with fewest vertices that d-separates A and B.