MATH 3705A
Test 2 Solutions

February 11, 2010

[Marks] Questions 1-6 are multiple choice. Circle the correct answer. Only the answer will be marked.

[3] 1. The general solution of $x^2y'' - 2xy' + 3y = 0$ for $x \neq 0$ is
(a) $c_1|x|^{\frac{3}{2}} + c_2|x|^{\frac{\sqrt{3}}{2}}$ (b) $c_1|x|^{\frac{3}{2}+\frac{\sqrt{3}}{2}} + c_2|x|^{\frac{\sqrt{3}}{2}-\frac{\sqrt{3}}{2}}$ (c) $|x|^{\frac{3}{2}+\frac{\sqrt{3}}{2}}\left[c_1 + c_2 \ln|x|\right]$
(d) $|x|^{\frac{3}{2}}\left[c_1 \cos\left(\frac{\sqrt{3}}{2}\ln|x|\right) + c_2 \sin\left(\frac{\sqrt{3}}{2}\ln|x|\right)\right]$ (e) None of the above
Solution: (d)

[3] 2. The general solution of $x^2y'' - 3xy' + 4y = 0$ for $x \neq 0$ is
(a) $|x|^2\left[c_1 + c_2\ln|x|\right]$ (b) $c_1|x|^2 + c_2|x|^2$ (c) $|x|^2\left[c_1\cos(\ln|x|) + c_2\sin(\ln|x|)\right]$
(d) $|x|^2\left[c_1\cos(2\ln|x|) + c_2\sin(2\ln|x|)\right]$ (e) None of the above
Solution: (a)

[3] 3. The general solution of $2x^2y'' - xy' + y = 0$ for $x \neq 0$ is
(a) $c_1|x|^{\frac{1}{2}} + c_2|x|$ (b) $|x|^{\frac{1}{2}}\left[c_1\cos\left(\frac{\sqrt{7}}{2}\ln|x|\right) + c_2\sin\left(\frac{\sqrt{7}}{2}\ln|x|\right)\right]$
(c) $|x|^2\left[c_1\cos(2\ln|x|) + c_2\sin(2\ln|x|)\right]$ (d) $|x|^2\left[c_1\cos(\ln|x|) + c_2\sin(\ln|x|)\right]$
(e) None of the above
Solution: (a)

[2] 4. The general solution of $x^2y'' + xy' + (5x^2 - 9)y = 0$ for $x > 0$ is
(a) $c_1J_3(\sqrt{5}\,x) + c_2J_{-3}(\sqrt{5}\,x)$ (b) $c_1J_3(\sqrt{5}\,x) + c_2Y_3(\sqrt{5}\,x)$
(c) $c_1J_{\sqrt{5}}(3x) + c_2J_{-\sqrt{5}}(3x)$ (d) $c_1J_{\sqrt{5}}(3x) + c_2Y_{\sqrt{5}}(3x)$ (e) None of the above
Solution: (b)

[2] 5. The general solution of $x^2y'' + xy' + (4x^2 - 3)y = 0$ for $x > 0$ is
(a) $c_1J_2(\sqrt{3}\,x) + c_2J_{-2}(\sqrt{3}\,x)$ (b) $c_1J_{\sqrt{3}}(2x) + c_2J_{-\sqrt{3}}(2x)\ln(x)$
(c) $c_1J_{\sqrt{3}}(2x) + c_2J_{-\sqrt{3}}(2x)$ (d) $c_1J_2(\sqrt{3}\,x) + c_2Y_2(\sqrt{3}\,x)$ (e) None of the above
Solution: (c)

[2] 6. The singular points of the equation $(x-1)y'' + xy' + \frac{1}{x+2}y = 0$ are
(a) 0, 1 and -2 (b) 0 and 1 (c) 0 and -2 (d) 1 and -2 (e) -2
Solution: (d)

[15] 7. Find one (nonzero) series solution y_1 of the equation $x^2y'' - xy' + (1-2x)y = 0$ valid for $x > 0$ near $x_0 = 0$.
Solution:
$p(x) = -\frac{1}{x}$, $xp(x) = -1 \Rightarrow p_0 = -1$, $q(x) = \frac{1-2x}{x^2}$, $x^2q(x) = 1 - 2x \Rightarrow q_0 = 1$.
$r^2 + (p_0 - 1)r + q_0 = r^2 - 2r + 1 = (r-1)^2 = 0 \Rightarrow r_1 = r_2 = 1$. One solution is given by $y_1 = \sum_{n=0}^{\infty} a_n x^{n+r_1} = \sum_{n=0}^{\infty} a_n x^{n+1}$. Then

$$y_1' = \sum_{n=0}^{\infty}(n+1)a_n x^n,\ \ y_1'' = \sum_{n=0}^{\infty} n(n+1)a_n x^{n-1} \ \Rightarrow$$

$$\sum_{n=0}^{\infty} n(n+1)a_n x^{n+1} - \sum_{n=0}^{\infty}(n+1)a_n x^{n+1} + \sum_{n=0}^{\infty} a_n x^{n+1} - \sum_{n=0}^{\infty} 2a_n x^{n+2} = 0 \ \Rightarrow$$

$$\sum_{n=0}^{\infty} n^2 a_n x^{n+1} - \sum_{n=0}^{\infty} 2a_n x^{n+2} = 0 \ \Rightarrow \ \sum_{n=0}^{\infty} \left[(n+1)^2 a_{n+1} - 2a_n\right] x^{n+2} = 0$$

$$\Rightarrow \ (n+1)^2 a_{n+1} - 2a_n = 0,\ n \geq 0 \ \Rightarrow$$

$$a_{n+1} = \frac{2a_n}{(n+1)^2},\ n \geq 0.$$

$$n = 0 \ \Rightarrow \ a_1 = \frac{2a_0}{1^2},$$

$$n = 1 \ \Rightarrow \ a_2 = \frac{2a_1}{2^2} = \frac{2^2 a_0}{1^2 2^2},$$

$n = 2 \ \Rightarrow \ a_3 = \dfrac{2a_2}{3^2} = \dfrac{2^3 a_0}{1^2 2^2 3^2}$, etc., so $a_n = \dfrac{2^n a_0}{(n!)^2},\ n \geq 0$, and one solution is

$$y_1 = \sum_{n=0}^{\infty} \frac{2^n}{(n!)^2} x^{n+1}.$$