Multiple Choice Questions:

1. [2 points] Use the linear approximation of $f(x) = \sqrt{1+x}$ at $a = 0$ to estimate $\sqrt{0.95}$.

(A) 0.9502 (B) 0.9750 (C) 0.9747 (D) 0.9942 (E) None of them.

Solution: $f(0) = 1$ and $f'(0) = 1/2$. Hence $L_0(x) = x/2+1$. Now $\sqrt{0.95} = \sqrt{1+(-0.05)} \approx L_0(-0.05) = 0.9750$. Answer: (B)

2. [2 points] Find the maximum value of $f(x) = x - 2\ln x$ on the interval $[1, 4]$.

(A) 1 (B) 0.6317 (C) 1.2274 (D) 1.4472 (E) None of them.

Solution: $f(1) = 1$, $f(4) = 4 - 2\ln(4) \approx 1.2274$, $f'(x) = 1 - 2/x$. Hence the critical point is $x = 2$. $f(2) = 2 - 2\ln(2) \approx 0.6137$. Answer: (C)

3. [2 points] Find the equation of the tangent line to the curve $x^3y + 4x^2y^2 - 5y^3 = 0$ at the point $(1, 1)$.

(A) $y = \frac{2}{3}x + \frac{1}{3}$ (B) $y = \frac{11}{6}x - \frac{5}{6}$ (C) $y = \frac{14}{9}x - \frac{5}{9}$ (D) $y = 3x - 2$

(E) None of them.

Solution: The implicit derivation gives $3x^2y+x^3y'+8xy^2+8x^2yy'-15y^2y' = 0$. Evaluated at $(1, 1)$ it gives $3 + y' + 8 + 8y' - 15y' = 0$, equivalently $6y' = 11$, i.e., $y' = 11/6$. Now $y = 11x/y + b$ evaluated at $(1, 1)$ one gets $b = -5/6$. Answer: (B)

4. [2 points] We use Newton's Method to estimate the solution of the equation $x^3 + 6x - 5 = 0$. If $x_1 = 1$, what is the value of x_2?

(A) $\frac{7}{9}$ (B) $\frac{1}{9}$ (C) $\frac{1}{3}$ (D) $\frac{1}{2}$ (E) None of them.

Solution: $x_2 = 1 - f(1)/f'(1) = 1 - (1 + 6 - 5)/(3 + 6) = 7/9$. Answer: (A)

5. [2 points] Find the limit $\lim_{x\to 0} \frac{\sin x - xe^x}{x^2}$.

(A) 0 (B) Undefined (C) 1 (D) -1 (E) None of them.

Solution: $\lim_{x\to 0} \frac{\sin x - xe^x}{x^2} = \lim_{x\to 0} \frac{\cos(x) - e^x - xe^x}{2x} = \lim_{x\to 0} \frac{-\sin(x) - e^x - e^x - xe^x}{2} = -1.$
Answer: (D)

6. [2 points] Which one of these statements in relation to the function $f(x) = x + \frac{3}{x}$ is FALSE?

(A) f is decreasing on the interval $]-\infty, 0[$.

(B) f is increasing on the interval $]3, \infty[$.

(C) The graph of f admits a vertical asymptote at $x = 0$.

(D) The graph of f is concave down on $]-\infty, 0[$.

(E) The graph of f is concave up on $]0, \infty[$.

Solution: $f'(x) = 1 - 3/x^2$. Hence on $]-\infty, -\sqrt{3}[$ $f'(x) > 0$, i.e., f is increasing on that interval. Answer: (A)

7. [2 points] If 2700 cm^2 of cardboard is available to make a box with an open top and a square base, what is the largest possible volume of the box?

(A) 13 500 cm^3 (B) 20 784 cm^3 (C) 27 000 cm^3 (D) 9500 cm^3

(E) None of them.

Solution: If a is the length of the sides of the bottom square and h is the height of the box, then $V = a^2 h$ and $S = a^2 + 4ah$. Thus

$$V = a^2 \left(\frac{2700 - a^2}{4a} \right) = \frac{1}{4}(2700a - a^3).$$

Thus $V'(a) = \frac{1}{4}(2700 - 3a^2)$, i.e., $a = 30$ cm. Hence $V(30) = 13\,500$ cm^3. Answer: (A)

8. [2 points] If a snowball melts so that its surface area decreases at the rate of 1 cm^2/min, find the rate at which the diameter decreases when it reaches 10 cm. Recall that the surface of a sphere is $4\pi r^2$.

(A) $\dfrac{-1}{20\pi}$ (B) $\dfrac{-1}{10\pi}$ (C) $\dfrac{-1}{30\pi}$ (D) $\dfrac{-1}{5\pi}$ (E) None of them.

Solution: $S = 4\pi r^2 = 4\pi(D/2)^2 = \pi D^2$. Hence $S' = 2\pi D D'$, i.e.,

$$D' = \frac{S'}{2\pi D} = \frac{-1}{20\pi}.$$

Answer: (A)

Problems:

9. [2 points]

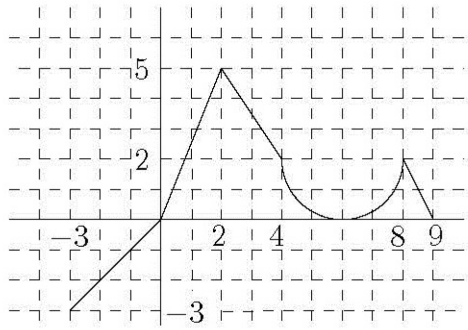
(a) Compute the derivative of $f(x) = \arctan(\cos(x^2 + 1))$.

Answer: $f'(x) = \dfrac{1}{1 + \cos^2(x^2 + 1)} \cdot - \sin(x^2 + 1) \cdot 2x.$

(b) Compute $\displaystyle\int 3x^4 + \frac{6}{\sqrt{1 - x^2}} - \frac{4}{x}\, dx.$

Answer: $\dfrac{3}{5}x^5 + 6 \arcsin(x) - 4 \ln |x| + C.$

10. [2 points] Here is the graph of a function f.



$y = f(x)$

Compute

(a) $\int_{2}^{-3} f(x)dx$

Answer: $\int_{2}^{-3} f(x)dx = -\int_{-3}^{2} f(x)dx = -\int_{-3}^{0} f(x)dx - \int_{0}^{2} f(x)dx = \frac{9}{2} - 5 = -\frac{1}{2}.$

(b) $\int_{2}^{8} f(x)dx$

Answer: $\int_{2}^{8} f(x)dx = \int_{2}^{4} f(x)dx + \int_{4}^{8} f(x)dx = 2 \cdot 2 + \frac{2 \cdot 3}{2} + 4 \cdot 2 - \frac{\pi \cdot 2^2}{2} = 15 - 2\pi.$