misprint Q6: Should be $\nabla f = \vec{F}$

Solutions

School of Mathematics and Statistics
~~Carleton University~~
Math. 2004A, Fall 2013
TEST 5

Any non-programmable calculator permitted, 1 blank sheet permitted for roughs

Print Name :

Student Number:

Tutorial Section (A1, A4, ...):

PART I: Multiple Choice Questions
(Choose and CIRCLE only ONE answer - No part marks here.)

1. **[3 marks]** Find the Jacobian $\frac{\partial(x,y)}{\partial(r,\theta)}$ of the transformation $x = 3r\cos\theta$, $y = 2r\sin\theta$.
   (a) $2r$, (b) r, (c) $6r$, (d) 0, (e) none of these.

2. **[3 marks]** Find the divergence of the vector field defined by $\mathbf{F}(x,y,z) = x^2y^3\,\mathbf{i} + xz^2\,\mathbf{k}$
   (a) 0, (b) $6xyz + xy$, (c) $3xy^2 - xz$, (d) $2xy^3 + 2xz$, (e) none of these.

3. **[3 marks]** Find the curl of the vector field defined by $\mathbf{F}(x,y,z) = e^{-x}\cos y\,\mathbf{i} + e^{-x}\sin y\,\mathbf{j} + \ln z\,\mathbf{k}$
   (a) $e^{-x}\,\mathbf{i} + \sin y\,\mathbf{j} + z\,\mathbf{k}$, (b) $e^{-z}\sin y\,\mathbf{i} + e^{-x}\cos y\,\mathbf{j} + z^2\,\mathbf{k}$, (c) $e^{-x}\sin y\,\mathbf{i} + e^{-x}\cos y\,\mathbf{j}$, (d) $\mathbf{0}$.

4. **[3 marks]** The Jacobian $\frac{\partial(x,y,z)}{\partial(u,v,w)}$ of the transformation $x = 2u + w$, $y = u^2 - v^2$, $z = u + v^2 - 2w^2$ is given by
   (a) $uv^2 - vw + 6$, (b) $4uv + 16vw + 2v$, (c) $4w^2 - 7uvw + uw$, (d) uvw, (e) none of these

5. **[3 marks]** The vector field defined by $\mathbf{F}(x,y) = (x\cos y + \sin y)\,\mathbf{i} + (\cos y - x\sin y)\,\mathbf{j}$ is conservative.
   (a) True, (b) False, (c) none of these

PART II: Show all work here and give details.
No additional pages will be accepted

misprint

6. **[7+8 marks]** a) Determine whether the vector field $\mathbf{F}(x,y,z) = yz\,\mathbf{i} + xz\,\mathbf{j} + xy\,\mathbf{k}$ is conservative and, if so, find a scalar function f such that ~~$\nabla F = f$~~ $\nabla f = \vec{F}$

   b) Evaluate the double integral $\iint_{\mathcal{R}} 2xy\,dA$, where $\mathcal{R}$ is the region in the first quadrant defined by the ellipse $4x^2 + 9y^2 = 36$ using the change of variable $x = 3u$, $y = 2v$.

a) yes ← (1) because curl $\vec{F} = \vec{0}$ ← (1)

Now $\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) = (yz, xz, xy)$ ← (1)

$\therefore \frac{\partial f}{\partial x} = yz \Rightarrow f = xyz + g(y,z)$ ← (1)

and $xz = \frac{\partial f}{\partial y} = xz + \frac{\partial g}{\partial y} \Rightarrow \frac{\partial g}{\partial y} = 0$ ← (1)

$\frac{\partial g}{\partial y} = 0 \Rightarrow g(y,z) = h(z)$

$\therefore f = xyz + h(z)$

$xy = \frac{\partial f}{\partial z} = xy + h'(z) \Rightarrow h'(z) = 0$ ← (1)

$\Rightarrow h(z) =$ constant.

$\therefore \boxed{f(x,y,z) = xyz + C}$ ← (1)

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b) $x = 3u$, $y = 2v$ Jacobian $\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 3 & 0 \\ 0 & 2 \end{vmatrix} = 6$. ← (1)

& the ellipse $4x^2 + 9y^2 = 36$ goes into the region $\mathcal{S}$ defined by $4(3u)^2 + 9(2v)^2 = 36 \Leftrightarrow u^2 + v^2 = 1$, the unit circle in the uv-plane. (1)

verso →