Math 1119B: Week 4, Lecture 2

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I mentioned a Nobel prize winner: Leontief

The Leontief Input-Output (or production) model is a system of 500 equations in 500 variables. We will solve this later, in more detail.

The next example will be an "exchange" model, also proposed by Leontief.

- An economy is divided into sectors.
- We know the total output of a sector for one year.
- We know how the output is divided (or exchanged) between the other sectors.
- The dollar value of a sector's output is the price of the output.

There are equilibrium prices such that the total outputs of the sectors completely balance their inputs. (I call it the economy's sweet spot).

A homogeneous system in economics

Outline. The economy of Dave-land contains three sectors: Tourism, Sports and Medical (to fix all the sports injuries). The output of each sector is distributed among the other sectors as follows:

Tourism	Sports	Medical	Purchased by:
.2	.6	.4	Tourism
.4	.2	.5	Sports
.4	.2	.1	Medical

Since all of the output must be taken into account, the sum across each column must be 1.
Denote Tourism by T, sports by S and Medical by M.

Continuing the exchange

Looking down a column to see where the output goes, and across a row to see what the input is.

For example: Tourism receives (and pays for) 20% of all tourism, 60% of all sports, and %40 of all medical costs. Thus, reading across the row, $T = .2T + .6S + .4M$. Doing this for all three sectors gives the system of equations,

$$T = .2T + .6S + .4M \quad (1)$$
$$S = .4T + .2S + .5M \quad (2)$$
$$M = .4T + .2S + .1M \quad (3)$$

Re-arrange, and solve to find the equilibrium point of the exchange. Ans.

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Re-arrange, and solve to find the equilibrium point of the exchange.
Ans.

$$\begin{bmatrix} T \\ S \\ M \end{bmatrix} = M \begin{bmatrix} 31/20 \\ 7/5 \\ 1 \end{bmatrix}$$

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Any non-negative choice for M results in an equilibrium price (where net input equals net-output; the sweet spot of the economy).

The value of M determines the price of medicine from previous data. For example, in 2010 we know that Dave-land spent $60 million dollars on medicin, so

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The general method

Solving Leontief equilibrium point follows the same steps each time:

1. Write a table of input and output for the sectors. For now, I will label the sectors $S_1, S_2, \ldots, S_n$. The columns represent the output of a sector, and the rows represent the purchases of a sector.

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2. As a check, make sure the sum of the entries in each column is 1.
3. Let Row $i = (t_1, t_2, \ldots, t_n)$. Construct the equation $S_i = t_1 S_1 + t_2 S_2 + \cdots + t_n S_n$ for each row.
4. Re-arrange the equations, pushing everything to the left-side, and ending with a homogeneous system (i.e., zeroes on the right-hand-side).
5. Note that every entry on the diagonal will be positive and every other entry will be negative.
6. Solve the resulting system. (Hint: multiply each row by 10 to make it easier).

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- An alternative process: Let $S = \begin{bmatrix} S_1 & S_2 & \cdots & S_n \end{bmatrix}$. That is, let S be the table of inputs and outputs.
- Construct the matrix $I_n - S$, where I_n is the $n \times n$ identity matrix.

This is exactly the same process as before, but skips the step of writing out every equation separately. Choose your favorite process.

If there is a non-trivial equilibrium point, the system will contain a free variable. If not, then the economy can never be in equilibrium.

To analyze the data: If you have a free-variable, S_j, consult the last year's output for that sector. If S_j's price was \$50, then input \$50 for the value of that parameter.

Another example

- See the tutorial for a small example

Another example, removing the word-problem. Let
$S = \begin{bmatrix} .2 & .6 & .5 \\ .4 & .3 & .2 \\ .4 & .1 & .3 \end{bmatrix}$ be a table of inputs and output of an economy.

1. Compute

$$I_n - S = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} .2 & .6 & .5 \\ .4 & .3 & .2 \\ .4 & .1 & .3 \end{bmatrix} =$$

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2. Multiply by 10 to make your life easier. Solve.

Ans. $\begin{bmatrix} S_1 \\ S_2 \\ S_3 \end{bmatrix} = S_3 \begin{bmatrix} 47/32 \\ 9/8 \\ 1 \end{bmatrix}$.

If the price of Sector 3 was \$17.4 billion in 2010, what was the equilibrium output?