Math 1119B: Week 11, Lecture 2

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Recap

A look at the previous lecture:

1. Trick for computing 3×3 determinants,
2. Cramer's rule.
3. Test 4 recap.

What is a Markov chain

- Markov chains are used as mathematical models in biology, chemistry, engineering, physics, business, geography and elsewhere.
- The model is used to describe measurements that are performed many times in the same way.
- In this model, the outcome of the subsequent trial depends only on the previous trial.
- We have already seen a similar example as Markov chains using migration matrices.

From Test 4

The city of Toronto can be thought of as broken into 3 areas: Downtown, Scarborough and Mississauga. The population travels between areas at a constant rate over the years, 90% of each area's population remains in place and 5% of the population spreads to each of the neighboring areas.

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The migration matrix is given by $\begin{bmatrix} .9 & .05 & .05 \\ .05 & .9 & .05 \\ .05 & .05 & .9 \end{bmatrix}$.

Suppose the population of Toronto in 2008 is given by the vector

$$v_0 = \begin{bmatrix} \text{Downtown} \\ \text{Scarborough} \\ \text{Mississauga} \end{bmatrix} = \begin{bmatrix} 8,000,000 \\ 2,000,000 \\ 4,000,000 \end{bmatrix}.$$

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In 2009, the the population in each sector is given by

$$v_1 = Mv_0 = \begin{bmatrix} 7,500,000 \\ 2,400,000 \\ 4,100,000 \end{bmatrix}.$$

The difference equation model

In general, we have the state for the $(k+1)$th year is given by the equation

$$v_{k+1} = Mv_k.$$

In particular, the $(k+1)$th year of the population depends only on the population in the kth year. Furthermore, in the year 2024, we have $v_{16} = M^{16}v_0$

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In general, $v_{k+1} = M^{k+1}v_0$.

Definition. In population examples, we refer to M as a migration matrix. In general, when M is a matrix that gives constant transition percentages between states, that is a square matrix whose columns add to 1, we call M a stochastic matrix.

Definiton. Let x be a vector whose entries add to 1. Then x is called a probability vector.

Definition. A Markov chain is a sequence of probability vectors $v_0, v_1, \ldots,$ together with a stochastic matrix M such that

$$v_1 = Mv_0, \quad v_2 = Mv_1, \quad \ldots$$

That is, the Markov chain is modeled by the difference equation

$$v_{k+1} = M^{k+1}v_0.$$

The difference between this and the general difference equation model is that the vectors now indicate probabilities. We often refer to the vectors v_k as state vectors at time k.

Changing the previous model into a Markov chain

In the previous example, the migration matrix between burroughs of Toronto (which is a stochastic matrix) was given by

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The population of Toronto in 2008 is given by $x_0 = \begin{bmatrix} \text{Downtown} \\ \text{Scarborough} \\ \text{Mississauga} \end{bmatrix} = \begin{bmatrix} 8,000,000 \\ 2,000,000 \\ 4,000,000 \end{bmatrix}$. Dividing by the total population gives the population distribution vector

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$$v_0 = \begin{bmatrix} 4/7 \\ 1/7 \\ 2/7 \end{bmatrix} \approx \begin{bmatrix} .57 \\ .14 \\ .29 \end{bmatrix}.$$

Finishing the previous example

The Markov chain is given by

$$\begin{aligned} v_1 &= Mv_0 = \begin{bmatrix} .9 & .05 & .05 \\ .05 & .9 & .05 \\ .05 & .05 & .9 \end{bmatrix} \begin{bmatrix} .57 \\ .14 \\ .29 \end{bmatrix} \approx \begin{bmatrix} .54 \\ .17 \\ .29 \end{bmatrix}, \\ v_2 &= Mv_1, \\ v_3 &= Mv_2, \\ &\dots \end{aligned}$$

Interpreting. This Markov chain gives how the proportion of the population is distributed (the state of the system) after k years.

Predicting the future

Let's see what happens after a few more years:

$$v_2 = \begin{bmatrix} .51 \\ .20 \\ .29 \end{bmatrix}, v_3 = \begin{bmatrix} .48 \\ .22 \\ .30 \end{bmatrix}, v_4 = \begin{bmatrix} .46 \\ .23 \\ .31 \end{bmatrix}$$

$$v_{10} = \begin{bmatrix} .38 \\ .30 \\ .32 \end{bmatrix}, v_{20} = \begin{bmatrix} .34 \\ .33 \\ .33 \end{bmatrix}, v_{30} = \begin{bmatrix} .34 \\ .33 \\ .33 \end{bmatrix}$$

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It looks like the probabilities are converging to $\begin{bmatrix} .33 \\ .33 \\ .33 \end{bmatrix}$.

That is, $\begin{bmatrix} .33 \\ .33 \\ .33 \end{bmatrix} = M \begin{bmatrix} .33 \\ .33 \\ .33 \end{bmatrix}$.

The steady-state vector

Definition. If P is a stochastic matrix, then a steady-state vector for P is a probability vector q such that $Pq = q$.

Note. Every stochastic matrix has a unique steady-state vector determined solely by the matrix.

Solving the steady-state

Solving the steady-state is done by a simple process: given a stochastic matrix P, we must find a vector q such that $Pq = q$.

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Re-writing means we need to solve

$$(I - P)q = 0.$$

Exercise. (a) Find the steady-state vector for the stochastic matrix $P = \begin{bmatrix} .88 & .04 \\ .12 & .96 \end{bmatrix}$.

(b) Find the steady-state vector for $P = \begin{bmatrix} .8 & .4 \\ .2 & .6 \end{bmatrix}$.

Thanks to Dr. Horn for this brilliant example

The weather in Ottawa is either good, “so-so”, or bad. If the weather is good today, there is a 60% chance the weather will be good tomorrow, a 10% chance it will be “so-so’, and a 30% chance it will be bad. If the weather is “so-so” today, it will be good, “so-so” or bad tomorr with respective probabilities .2, .5 and .3. If the weather is bad today, it will be good, “so-so” or bad tomorrow with respective probabilities .1, .3 and .6.

(a) What is the stochastic matrix P for this situation?

(b) What is the probability the weather will be good, “so-so” and bad on Friday?

(c) In the long run, what proportion of days have good, “so-so” and bad weather?

...and that's a wrap

- That signifies the end of the new content of this course.
- The following 3 lectures will be review.
- For next lecture, I will review the topics of the course, and then we can go to the textbook for some examples. If there are any specific questions from the book, tests or tutorials, I will take on all questions at the subsequent lecture.
- On Monday, December 5 if I do not receive requests for review material, I will treat the class as an office hour (check WebCT!). It's to your benefit to come prepared.
- What I will not give is a more detailed break-down of the final than I have already mentioned, nor give clues as to which questions will appear there, beyond what I have already mentioned. The entire course is fair game.

The end, my friends

- At the end of class on Wednesday, November 30, I will be going to Oliver's (or Mike's) for a celebratory adult beverage. All are welcome to join me.
- I will hold office hours on Thursday, December 15 from 11:00-14:00. See WebCT for the place and for any updates.