MAT 1332, Winter 2014, Assignment 1
Due Friday January 17 by 3:00pm.
Late assignments will not be accepted; nor will unstapled assignments.
Professors in the math department will not lend you a stapler; do not ask for one.

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Student Name ____________________________ Student Number ____________________________

By signing below, you declare that this work was your own and that you have not copied from any other individual or other source.

Signature ____________________________

QUESTION 1. Find the indefinite integral

$$\int \arcsin(x)dx.$$

First Solution: Begin with integration by parts and use substitution afterwards. For integration by parts, we set $f(x) = \arcsin(x)$ and $g'(x) = 1$. Then $f'(x) = (1 - x^2)^{-1/2}$ and $g(x) = x$. Hence

$$\int \arcsin(x)dx = x\arcsin(x) - \int \frac{x}{\sqrt{1-x^2}}dx.$$

Now we substitute $u(x) = 1 - x^2$ so that $du = -2xdx$. Then we find

$$-\int \frac{x}{\sqrt{1-x^2}}dx = \int \frac{du}{2\sqrt{u}} = \sqrt{u} + C.$$

Altogether, we obtain

$$\int \arcsin(x)dx = x\arcsin(x) + \sqrt{1-x^2} + C.$$

Second Solution: We can also use substitution first and integration by parts afterwards. Substitute $y(x) = \arcsin(x)$, or $\sin(y) = x$. Typically, we differentiate the first form to find the differential, but here it is advantageous to differentiate the second and obtain $\cos(y)dy = dx$. Then the integral becomes

$$\int \arcsin(x)dx = \int y\cos(y)dy.$$

The latter integral is clearly solves by integration by parts with $f(y) = y$, $g'(y) = \cos(y)$ so that

$$\int y\cos(y)dy = y\sin(y) - \int \sin(y)dy = y\sin(y) + \cos(y) + C.$$

Now we substitute back and find the same solution as above. Note

$$\cos(y) = \sqrt{1-\sin^2(y)} = \sqrt{1-x^2}.$$

Question 2. Find the definite integral

$$\int_0^2 x^5 e^{-x^3} dx.$$

First we substitute $y = x^3$. Then $dy = 3x^2 dx$ and $x = 0$ becomes $y = 0$, while $x = 2$ becomes $y = 8$. We get

$$\int_0^2 x^5 e^{-x^3} dx = \int_0^8 \frac{y}{3} e^{-y} dy$$

The latter integral is solved via integration by parts as

$$\int_0^8 \frac{y}{3} e^{-y} dy = -\frac{y}{3} e^{-y}|_0^8 + \int_0^8 \frac{1}{3} e^{-y} dy = -\frac{y}{3} e^{-y}|_0^8 - \frac{1}{3} e^{-y}|_0^8.$$

Finally, we evaluate this expression as

$$\int_0^2 x^5 e^{-x^3} dx = -\frac{1}{3}(y+1)e^{-y}|_0^8 = \frac{1}{3} - 3e^{-8}.$$

Question 3. Consider the definite integral

$$\int_1^2 \ln(x) dx.$$

(a) Find the value of L_4, the Riemann sum approximation of the integral with four subintervals and function evaluation at the left hand endpoint of each subinterval.

$L_4 =$

$$\frac{1}{4}\left[\ln(1) + \ln(1.25) + \ln(1.5) + \ln(1.75)\right] \approx 0.297$$

(b) Find the value of L_{10}.

$L_{10} =$

$$\frac{1}{10}\left[\ln(1) + \ln(1.1) + \ln(1.2) + \ln(1.3) + \cdots + \ln(1.9)\right] \approx 0.351$$

(c) Use the fundamental theorem to calculate the definite integral and compare its value with the two approximations.

Answer:

$$\int_1^2 \ln(x)dx = x\ln(x)|_1^2 - \int_1^2 dx = [x\ln(x) - x]|_1^2 = 2\ln(2) - 1 \approx 0.386$$

Both Riemann sum approximations underestimate the true value. The second one is closer. For your information, $L_{100} = 0.383$ still only captures the first two decimals. In other words, it takes a long time to evaluate integrals in this form with high accuracy.

QUESTION 4. The number of new cases per day (i.e. change in the total number of cases) during a disease outbreak usually increases at first and decreases later on. One function that describes this behavior is $f(t) = 100te^{-t/10}$, where t is the time in days since the beginning.

(a) On which day does the number of new cases peak?

Answer: Since $f'(t) = 100e^{-t/10}(1 - t/10)$, we find $f'(t) = 0$ if $t = 10$. Since $f(0) = 0$ and $\lim_{t\to\infty} f(t) = 0$ and $f \geq 0$, the critical point must correspond to a maximum. The number of new cases peaks at day 10.

(b) How many total cases have accumulated by day 10?

Answer: The total number of cases is the definite integral $\int_0^{10} f(t)dt$. Using integration by parts (see question 2), we find

$$\int_0^{10} f(t)dt = 100\left[-10te^{-t/10}|_0^{10} + \int_0^{10} 10e^{-t/10}dt\right] = -1000[t+10]e^{-t/10}|_0^{10}$$

Evaluation gives the answer $10000 - 20000e^{-2} \approx 7293$.

(c) Find the function that describes the total number of cases on day T.

Answer: The total number of cases up to day T is the definite integral $N(T) = \int_0^T f(t)dt$. Using hte same calculation as above, we find

$$N(T) = 10000 - 1000[T+10]e^{-T/10}$$

(d) How many cases will there be in total after a very long time?

Answer:

$$\lim_{T\to\infty} N(T) = 10000$$