Math 1119B: Week 11, Lecture 1

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Recap

A look at the previous lecture:

1. Determinants by row-reducing (3 rules):
   1.1 **Flipping rows.** If two rows of A are interchanged to produce a matrix B, then $\det(B) = -\det(A)$.
   1.2 **Scaling.** If one row of A is multiplied by a scalar k to produce a matrix B, then $\det(B) = k \cdot \det(A)$.
   1.3 **Combination.** If a multiple of one row of A is added to another row to produce a matrix B, then $\det(B) = \det(A)$.

A disclaimer

The following trick does work, however you are not allowed to use it in this class.

So why am I teaching it to you?

Because it is useful to know, and you are able to check your answers by using it.

The trick

Write a 3×3 matrix as follows:

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Example. Find the determinant of

$$\begin{bmatrix} 3 & 4 & -1 \\ -2 & 5 & -3 \\ 4 & -1 & 2 \end{bmatrix}$$

Ans.

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What is it?

Cramer's rule can be used to study how the solution of $Ax = b$ is affected by changes in the entries in b. One example of this could be:

I run a farm and can track my expenses (supplies, processing, admin) per chicken, per litre of milk and per bale of hay, maybe by the following expression

$$\begin{bmatrix} \$1.25 & \$1.50 & \$3.50 \\ \$2.40 & \$2.80 & \$5.60 \\ \$4.40 & \$8.45 & \$1.00 \end{bmatrix} \begin{bmatrix} c \\ m \\ h \end{bmatrix} = \begin{bmatrix} \text{supplies} \\ \text{processing} \\ \text{admin} \end{bmatrix}.$$

I want to know how the amount of product I make changes as my final costs vary.

So, what is it?

Definition. Let $A = [a_1 \quad a_2 \quad \cdots \quad a_n]$ be an invertible $n \times n$ matrix. We form the matrix $A_i(b)$ by replacing column i with b, that is,

$$A_i(b) = [a_1 \quad a_2 \quad \cdots \quad \underbrace{b}_{\text{Col } i} \quad \cdots a_n].$$

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Cramer's rule. Let A be an invertible $n \times n$ matrix. For any $b \in \mathbb{R}^n$ the unique solution $x = [x_1, x_2, \ldots, x_n]^T$ of $Ax = b$ has entries

$$x_i = \frac{\det(A_i(b))}{\det(A)}, \quad i = 1, 2, \ldots, n.$$

An example

Example. Use Cramer's rule to solve the system

$$\begin{aligned} 3x_1 + 2x_2 &= 2 \\ -x_1 + 3x_2 &= 4 \end{aligned}$$

Ans.

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Ans. $\begin{bmatrix} -2/11 \\ 14/11 \end{bmatrix}$.

When to, and when not to

Cramer's rule is an extremely inefficient method to solve an entire system. However, recall my farm:

I run a farm and can track my expenses by the following expression:

$$\begin{bmatrix} \$1.25 & \$1.50 & \$3.50 \\ \$2.40 & \$2.80 & \$5.60 \\ \$4.40 & \$8.45 & \$1.00 \end{bmatrix} \begin{bmatrix} c \\ m \\ h \end{bmatrix} = \begin{bmatrix} \text{supplies} \\ \text{processing} \\ \text{admin} \end{bmatrix}.$$

If I limit my expenses to $\begin{bmatrix} \$200,000 \\ \$100,000 \\ \$30,000 \end{bmatrix}$, how many bales of hay do I need to produce?

In this case, the numbers are ugly. I computed $\det(A) = \$5.57$ and $\det(A_3(b)) = \$1,192,750$. Thus $h = 214138$.

An example with numbers we can use

Use Cramer's rule to solve for x_1 and x_3 in the equation

$$Ax = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ -1 \end{bmatrix}.$$

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Ans. $\det(A) = -2$ and $x_1 = 1/2, x_3 = -1$.