Math 1119B: Week 10, Lecture 2

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Recap

A look at the previous lecture:

1. Determinants,
2. Minors, cofactors, cofactor expansion,
3. Triangular matrices, and how cofactor expansion is easy when you get a triangular matrix.

The only way to get good at it is to practice

Use cofactor expansion to find the determinant of the following matrices:

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & 3 \\ 6 & 0 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 0 & 2 & 1 \\ 0 & 1 & -2 & 3 \\ -2 & 2 & 3 & -1 \end{bmatrix}.$$

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Ans. $|A| = 8, \quad |B| = 2, \quad |C| = 13.$

Row reduction so that cofactor expansion is feasible!

Recall. By itself, cofactor expansion is difficult when

- The matrix is large
- and does not contain many zeroes.

This lecture will give some rules for determinants of larger matrices when the cofactor expansion becomes difficult.

What does row-reducing do to my determinant?

- We have seen that matrices which reduce to the identity are invertible.
- We know that the determinant of upper-triangular matrices is given by the product of the terms on the diagonal.
- If there is a 0 on the diagonal, the determinant is 0 and the matrix is not-invertible.

We can generalize this to any matrix by analyzing what row-reduction does to the determinant of a matrix.

Some rules for row-reducing to find determinants

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3. **Combination.** If a multiple of one row of A is added to another row to produce a matrix B, then $\det(B) = \det(A)$.

Since we know that the determinant of a triangular matrix can be read by multiplying across the diagonal (by cofactor expansion!!), we can use row-reduction into row-echelon form and keep track of our steps in order to find the deteminant of an original matrix.

An example we've already seen:

Let

$$C = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 0 & 2 & 1 \\ 0 & 1 & -2 & 3 \\ -2 & 2 & 3 & -1 \end{bmatrix}.$$

Use row-reduction to find the determinant of C.

Ans.

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Ans.As before, $\det(C) = 13$.

From last class: the worst-case analysis of solving determinants gave that cofactor expansion of a 25×25 matrix required about 1.5 trillion operations. Using the combination of row-reduction and cofactor requires about $10,000$ operations, which can be done in a fraction of a second.

Some things to be careful of:

Be careful: The operation of taking a combination must be done in the proper way:

$$R_i \leftarrow R_i + cR_j.$$

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The second line is actually two basic row-operations done at once:

$$R_i \leftarrow cR_i \tag{1}$$

$$\text{New } R_i \leftarrow R_i + R_j. \tag{2}$$

Another thing to be careful of:

Be careful: Keep in mind the rules for determinants: Suppose I perform the following row operation to a matrix A to produce a matrix B:

1. Interchange - $\det(B) = -\det(A)$;
2. Scaling by k - $\det(B) = k \cdot \det(A)$;
3. Combination - no change.

This gives the determinant of the reduced matrix in terms of the original matrix. However, when we use row-reduction, we want to use the determinant of the reduced matrix in order to find the determinant of the original matrix.

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The lesson. The lesson is: when you keep track of row-operations, particularly when scaling a row by k, you want to keep the factor of $1/k$.

Now you finally know why:

I have verbally said it many times, but now you have the following: Let A be an $n \times n$ matrix,

1. A invertible if and only if it row-reduces to the identity.
2. A reduces to the identity if and only if it has a pivot in every column.
3. A has a pivot in every column if and only if the diagonal entries in row-echelon form are all non-zero.
4. The diagonal entries in row-echelon form are all non-zero if and only if A is invertible.

A couple of examples:

Find the determinant of the following matrices:

Example. $|A| = \begin{vmatrix} 1 & 0 & 0 & 1 \\ 2 & 1 & -1 & 2 \\ 0 & 1 & -1 & 0 \\ 3 & 1 & 4 & 2 \end{vmatrix} =$

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Example. $|B| = \begin{vmatrix} 1 & 2 & -1 & 3 & 2 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 2 & 2 & -1 & 1 & 1 \\ 0 & 0 & 2 & 1 & 3 \end{vmatrix} = 9$

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What is the determinant of $2A$? Ans.

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What is the determinant of $2A$? Ans. -24.

A couple of more properties

Let A and B be a square matrices,

1. $\det(A^T) = \det(A)$.
2. $\det(AB) = \det(A)\det(B)$, therefore AB is invertible if and only if A and B are both invertible.

Corollary/Example. Suppose A is invertible. Find an expression for A^{-1}.

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Ans.

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Ans. -1.