Math 1119B: Week 10, Lecture 1

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All tests are done

This morning you finished your final test before the final exam.

Recall from the syllabus:
"2. For students who write and pass all four tests, the best three of the four tests will be used to determine the test component of your final mark.

3. Students who fail to achieve a term mark of at least 50% will automatically receive a letter grade of FND (Fail, no deferral) in the course."

The final day to drop the class is December 5. Once you receive your grade for Test 4, you will almost know you mark for the term grade. If it is less than 50%, then email me if you have any doubts.

Recap

A look at the previous lecture:

1. Leontief Input-Output Model
2. ... and that's it.

The 2×2 case

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Remember when computing inverses in the 2×2 case, we introduced the determinant of the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ to be $ad - bc$.

The inverse of the matrix exists only if $ad - bc \neq 0$, and the inverse of the matrix is given by $\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

The determinant of a matrix is only a number, or property, of the matrix. It is no different than my height (no jokes) or weight (no jokes), the area of a circle, the length of a board, etc.

Where does $ad - bc$ come from?

Let $A = \begin{bmatrix} a & b \\ c' & d' \end{bmatrix}$, you can rewrite A as $A = \begin{bmatrix} a & b \\ ca & da \end{bmatrix}$.
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Row-reduce and get

$$\begin{bmatrix} a & b \\ 0 & da - cb \end{bmatrix}.$$

The quantity in the bottom right gives the determinant of the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

If $ad - bc = 0$, then the matrix A is singular, and A is non-singular in every other case.

Extending determinants

A formula for the determinant of any matrix is theoretically possible. But it gets ugly fast. Let $B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{33} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$, the determinant of B is given by

$$\det(B) = b_{11} \det \begin{bmatrix} b_{22} & b_{23} \\ b_{32} & b_{33} \end{bmatrix} - b_{12} \det \begin{bmatrix} b_{21} & b_{23} \\ b_{31} & b_{33} \end{bmatrix} + a_{13} \det \begin{bmatrix} b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix},$$

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and for brevity (which is the soul of wit, so shall I be brief), we write

$$\det(B) = b_{11} \det(B_{11}) - b_{12} \det(B_{12}) + b_{13} \det(B_{13}).$$

This class is full of: Minors

In the previous case (3×3), the 2×2 matrices B_{ij} are called the minors of B.

Definition. For any $n \times n$ matrix A, the (i, j)-minor of A is denoted A_{ij} and is given by the $(n-1) \times (n-1)$ submatrix of A where the ith row and jth column of A is removed.

Example. Let $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$. Find the $(1, j)$ minors, $j = 1, 2, 3, 4$.

Cofactor expansion of the determinant

For any $n \times n$ matrix A, we can calculate $\det(A)$ by taking the cofactor expansion across any row or column.

When $A = [a_{ij}]$, the determinant is given by the sum of n terms of the form $\pm a_{ij} \det(A_{ij})$. As a formula, we have:

$$\begin{aligned}\det(A) &= a_{1j} \det(A_{1j}) - a_{2j} \det(A_{2j}) + \cdots + (-1)^{i+n} a_{1n} \det(A_{1n}) \\ &= a_{i1} \det(A_{i1}) - a_{i2} \det(A_{i2}) + \cdots + (-1)^{n+j} a_{in} \det(A_{in})\end{aligned}$$

Definition. The (i,j)-cofactor is given by the term $(-1)^{i+j} a_{ij} \det(A_{ij})$, where A_{ij} is the (i,j)-minor of A.

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Sometimes it is easier to see with an example:

Example. Find $\det \begin{bmatrix} 1 & 1 & 0 \\ 2 & 0 & -1 \\ 0 & -3 & 2 \end{bmatrix}$ by performing cofactor expansion across the first row (and third column).

Ans. $|A| = -7$

Do not forget the alternating sign!!

Remember in the 3×3 case, there are 3 sums with alternating sign on each minor coming from the $(-1)^{i+j}$.

This is the sort of diagram that can help you remember:

$$\begin{bmatrix} + & - \\ - & + \end{bmatrix} \quad \begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix} \quad \begin{bmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{bmatrix}$$

The easiest way (for me) to remember is to count across the first row, alternating sign until reaching the proper column, and then moving down that column until reaching the appropriate entry.

Zeroes are your friend

Very often, we perform cofactor expansion across the first row, since psychologically this seems easier. However, this is not always the case. We should always prefer the row or column with the most zeroes.

Perform cofactor expansion of the matrix

$$C = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

by performing co-factor expansion across the first row. Then, perform the expansion across the first column and the fourth row.

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Ans. $\det(C) = 1$.

Why was that example so easy?

The previous example was simple because it was an upper triangular matrix. An $n \times n$ upper triangular matrix is any matrix of the form:

$$\begin{bmatrix} * & * & \cdots & * \\ 0 & * & \cdots & * \\ 0 & 0 & \ddots & * \\ 0 & 0 & \cdots & * \end{bmatrix}.$$

We can similarly define a lower triangular matrix.

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Theorem. The determinant of an upper (or lower) triangular matrix is the product of the entries on the diagonal.

If an $n \times n$ triangular matrix has a 0 on its diagonal then it is not invertible. If the diagonal is all non-zero, then it is invertible Why? (there is a pivot in every column).

Thus, the determinant of a triangular matrix is nonzero if it is invertible, and 0 if it is not invertible.

Use cofactor expansion to find the determinant of the following matrices:

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & 3 \\ 6 & 0 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 0 & 2 & 1 \\ 0 & 1 & -2 & 3 \\ -2 & 2 & 3 & -1 \end{bmatrix}.$$

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Ans. $|A| = 8, \quad |B| = 2, \quad |C| =$

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Ans. $|A| = 8, \quad |B| = 2, \quad |C| = 13.$

An interesting observation

In the Leontief example, I mentioned that Leontief used a matrix relating approximately 40 sectors.

At each step, a cofactor expansion requires (forgetting zeroes) summing the determinants of n $(n-1) \times (n-1)$ minors. Each of these minors requires summing the determinants of $n-1$ $(n-2) \times (n-2)$ minors, and so on.

A 25×25 matrix requires the product of roughly of $25 \cdot 24 \cdot 23 \cdots 2 \cdot 1 \approx 1.5 \times 10^{25}$ numbers.

If my laptop can do a trillion (10^9) multiplications per second, it would need to run for approximately $500,000$ years to complete calculating the determinant. Obviously a faster way is needed (and possible!)