Math 1119B: Week 9, Lecture 2

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Recap

A look at the previous lecture:

1. Review of invertible matrices,
2. Invertible matrix theorem,
3. Solving for invertible matrices,
4. Invertible linear transformations,

Expanding the Leontief Exchange Model

- Recall: The Leontief Exchange model dealt with sectors of an economy, all of whom share (or exchange) the gross product of the economy.
- In the more general Input-Output model, there is another part of the economy, the open sector which does not produce goods and only consumes them.
- Let $x \in \mathbb{R}^n$ be a production vector which lists the output of each sector for a given year.
- Let $d \in \mathbb{R}^n$ be the final demand vector which lists the values of goods and services from the sectors by the open sector. The vector d can be thought of as consumer demand, government consumption (government being unproductive? NEVER), surplus production, export, etc.

Leontief Input-Output further explained

- As the sectors produce goods, the producers create additional intermediate demand. For example, manufacturing requires energy; tourism requires goods and services, etc.
- Leontief: Is there a production level such that the amounts produced will balance the total demand for the production.
- That is,
  Amount produced = Intermediate demand + final demand.
- Vectors are given as amount of input needed per unit of output. I think of this like a cost-percentage.

Continuing the exchange

Example. Suppose I have 3 sectors in my economy: tourism, sports and medical (yes, we're in Daveland).
For each unit of output, Tourism consumes .2 units of input from Tourism, .4 units from Sports and .4 units from Medicine.
Continuing like this, we get the following table:

	Input per unit of output:		
Purchased from:	Tourism	Sports	Medicine
Tourism	.2	.6	.4
Sports	.4	.2	.5
Medicine	.4	.2	.1

What amounts will be consumed by Sports if it outputs 50 units? (Say, 50 games)?

Ans.

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What amounts will be consumed by Sports if it outputs 50 units? (Say, 50 games)?

Ans. $50 \begin{bmatrix} .6 \\ .2 \\ .2 \end{bmatrix} = \begin{bmatrix} 30 \\ 10 \\ 10 \end{bmatrix}.$

Consumption matrix

Each column of the table

Purchased from:	Input per unit of output: Tourism	Sports	Medicine
Tourism	.2	.6	.4
Sports	.4	.2	.5
Medicine	.4	.2	.1

corresponds to the consumption of goods per unit out output per vector.

Suppose we want to produce x_1 units of Tourism (say, hotels), x_2 units of Sports (say, games) and x_3 units of Medicine (say, surguries). If we label the columns of the table as v_1, v_2, v_3 the intermediate demand of production is

$$x_1 v_1 + x_2 v_2 + x_3 v_3.$$

Consumption matrix

So if $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, the intermediate demand is given by

$$\begin{bmatrix} .2 & .6 & .4 \\ .4 & .2 & .5 \\ .4 & .2 & .1 \end{bmatrix} \cdot x.$$

Definition. The consumption matrix is given by the coefficients of the input-output table.

Putting together Leontief IO:

Recall: The Leontief input-output model gives the following equation:

Amount produced = intermediate demand + final demand

Now, we have an equation for the intermediate demand. Putting it together, if x is the amount produced and d is the final demand, we get:

$$x = Cx + d,$$

and rearranging gives

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$$(I - C)x = d.$$

Solving this system for x gives the amount produced so that the output will precisely balance the demand.

This looks familiar

- The way to solve the Leontief Input-Output model is precisely the same as how to solve the Leontief exchange model.
- The difference between the two models is that in the Leontief I-O model, there is a final demand vector which represents the open sector of the economy (consumption without production).
- Solving the system by row-reduction can be ugly. Now we show how (and when!) to use invertible matrices to be able to solve the system in a nicer way.

A key difference

Recall in the Leontief exchange model, the sum of the column entries must be equal to 1. This is because there was no final demand, and the sum of the column represented the total output of the economy.

In the Leontief Input-Output model, the column sum of the consumption matrix need not be equal to 1. In fact, this is usually the case: a sector should require less than 1 unit of input to produce one unit of output.

We can solve for x when $(I - C)x = d$ by using $(I - C)^{-1}$ only if it exists.

The matrix $(I - C)$

The following theorem states that, under normal circumstances, the Leontief Input-Output problem is solvable.

Theorem. Let C be the consumption matrix for an economy with final demand vector d. If C and d contain only non-negative entries and if each column sum of C is less than 1, then $(I - C)^{-1}$ exists and the production vector

$$x = (I - C)^{-1} d$$

has non-negative entries.

An example

Example. My economy has 2 sectors: Government and Academia. The total gross output of Government is 40 units and Government consumes 32 input units from Government and 4 input units of Academia for the total output. Academia produces 200 units of output, consuming 140 units from Academia and 30 units from Government. The open sector demands 20 units of Government and 400 units of Academia.

The results form the following table:

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The results form the following table:

	Government	Academia	Open Sector
Government	32	30	30
Academia	4	140	400
Total Gross Output	40	200	

Example, cont'd

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Example, cont'd

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So we convert the table to give input consumed per unit output.

	Government	Academia
Government	.8	.15
Academia	.1	.7

Solve the system. (Why do you know you have a unique solution?).

Interpreting $(I - C)^{-1}$

The entries in $(I - C)^{-1}$ are important because they predict how the production x will have to change when the final demand d changes.

The entries in the jth column of C are the amount that the sectors will have to increase with an increase of 1 unit in the final demand for sector j.

Example. In the previous example,

	Government	Academia
Government	.8	.15
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And $(I - C)^{-1} = \frac{1}{0.045}\begin{bmatrix} .3 & .15 \\ .1 & .2 \end{bmatrix}$. Determine the increase in production if the final demand of Academia increases by 300 units.

Ans.

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Ans. $\begin{bmatrix} 1000 \\ 1333 \end{bmatrix}$.

Recap of topics

If you want to see a larger example... Look at Example #13 in Section 2.6. This gives a 7×7 consumption matrix based on data from the US economy in 1958. However, the numbers would take a long time to work out, so please do not waste your time solving this by hand. Question 14 continues the same idea, but contrasts with 1964 numbers.

Recap of topics:

1. Linear transformations
2. Difference equations/migration matrices
3. Invertible matrices
4. Leontief Input/Output Model

Say we have time...

The economy of MATH 1119 has 3 sectors: Dave, Wayne and Mathieu. A corresponding consumption matrix is the following:

$$\begin{bmatrix} .1 & .2 & .2 \\ .2 & .5 & .2 \\ .2 & .2 & .5 \end{bmatrix}.$$

Interpret the matrix, show why a solution exists and then solve the Leontief Input-Output problem if the final demand vector is

$$\begin{bmatrix} 2000 \\ 500 \\ 500 \end{bmatrix}.$$

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Interpret the matrix, show why a solution exists and then solve the Leontief Input-Output problem if the final demand vector is
$$\begin{bmatrix} 2000 \\ 500 \\ 500 \end{bmatrix}.$$

Once you get tired of fractions,
$$(I - C)^{-1} = \begin{bmatrix} 30/19 & 20/19 & 20/10 \\ 20/19 & 410/133 & 220/133 \\ 20/19 & 220/133 & 410/133 \end{bmatrix}.$$