Math 1119B: Week 9, Lecture 1

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Recap

Finding A^{-1} (Section 2.2)

The invertible matrix theorem (Section 2.3)

Invertible linear transformations

Recap

A look at the previous lecture:

1. Overview of linear transformations,
2. What is linear? Solving for linearity,
   - If showing linear then CANNOT use numbers,
   - If showing a transformation is not linear, a counter-example (i.e., with numbers) is sufficient.
3. The matrix of a linear transformation,
4. Exploiting the linearity of transformations.
5. Difference equations and migration matrices.
6. Brief intro to invertible matrices.

More inverses

Definition. (Just to confuse you) A matrix that is not invertible is sometimes called singular.

A matrix that is invertible is sometimes called non-singular.

Properties of inverses

1. If A is an invertible matrix, then A^{-1} is an invertible matrix and $(A^{-1})^{-1} = A$.
2. If O and C are $n \times n$ invertible matrices, then so is OC and the inverse of OC is given by $(OC)^{-1} =$

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2. If O and C are $n \times n$ invertible matrices, then so is OC and the inverse of OC is given by $(OC)^{-1} = C^{-1}O^{-1}$ (recall the OC-transpose theorem).
3. If A is an invertible matrix, then so is A^T and $(A^T)^{-1} = (A^{-1})^T$.

How to solve for A^{-1}

Given a matrix A, solving for A^{-1} is fairly simple:

If A is a $n \times n$ matrix, form the augmented matrix

$$[A|I_n]$$

and perform Gauss-Jordan elmination. If A is invertible, what remains is

$$\left[I_n|A^{-1}\right].$$

If A is not invertible, then it will not row-reduce to the identity matrix.

Example. Find the inverse of $A = \begin{bmatrix} 3 & 6 \\ 4 & 7 \end{bmatrix}, B = \begin{bmatrix} 1 & 2 & -1 \\ -4 & -7 & 3 \\ -1 & -6 & 5 \end{bmatrix}$ if it exists.

Ans.

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Ans. $\begin{bmatrix} -7/3 & 2 \\ 4/3 & -1 \end{bmatrix}$, Inverse does not exist.

Using inverse matrices

- An $n \times n$ invertible matrix A row-reduces to the identity matrix
- so it has n pivot columns,
- its rank is n,
- for any $b \in \mathbb{R}^n$, the equation $Ax = b$

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Example. Let $A = \begin{bmatrix} 3 & 6 \\ 4 & 7 \end{bmatrix}$ and let $b = \begin{bmatrix} 3 \\ -5 \end{bmatrix}$. Solve the matrix equation $Ax = b$ by: (a) row-reducing, (b) inversion.

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Ans. We know $A^{-1} = \begin{bmatrix} -7/3 & 2 \\ 4/3 & -1 \end{bmatrix}$, so $x = A^{-1}b = \begin{bmatrix} -17 \\ 9 \end{bmatrix}$.

A formula for the inverse of a 2×2 matrix

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a 2×2 matrix. Define the determinant of A to be the number $ad - bc$, and denote it $\det(A)$.

Theorem. The 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible if and only if $ad - bc \neq 0$.

Theorem. If the 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible, its inverse is given by

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Some examples

Let $A = \begin{bmatrix} 1 & 3 \\ -2 & 1 \end{bmatrix}$.

(a) Is A invertible?

(b) If $b = \begin{bmatrix} -3 \\ -2 \end{bmatrix}$, find the solution to $Ax = b$.

Ans. (a)

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Ans. (a) Yes! $ad - bc = 7$. (b) $x = \begin{bmatrix} 3/7 \\ -8/7 \end{bmatrix}$.

A huge theorem

Section 2.3 deals with the proof of the Invertible Matrix Theorem, which states that a matrix A being invertible is equivalent to 11 other things. We have actually shown most of these already.

Theorem. Let A be an $n \times n$ square matrix. The following are equivalent:

1. A is an invertible matrix.
2. There is a matrix B such that $AB = BA = I_n$.
3. The transpose A^T is invertible.
4. A is row-equivalent to the identity matrix.
5. A has n pivot positions.
6. The equation $Ax = 0$ has only the trivial solution.

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5. A has n pivot positions.
6. The equation $Ax = 0$ has only the trivial solution.
7. The columns of A are linearly independent.
8. The equation $Ax = b$ has a solution for all $b \in \mathbb{R}^n$.
9. The columns of A span $\mathbb{R}^n$.

So, what?

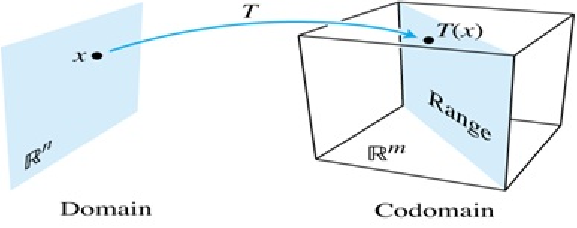
The main point in the previous theorem is that invertible matrices are row-equivalent to the identity matrix (that is, their RREF is the identity).

All of the other points are simply equivalent statements.

For this class, that's as much of the proof as we need.

Recall linear transformations

Remember this photo?



Domain, codomain, and range of $T : \mathbb{R}^n \to \mathbb{R}^m$.

Recall that a linear transformation $T(x) \colon \mathbb{R}^n \to \mathbb{R}^m$ has an associated matrix A such that $T(x) = Ax$.

If $n = m$ (that is, if A is an $n \times n$ matrix) and A is invertible then

1. A has a pivot in every row/column
2. For every $b \in \mathbb{R}^n$, the equation $Ax = b$ has

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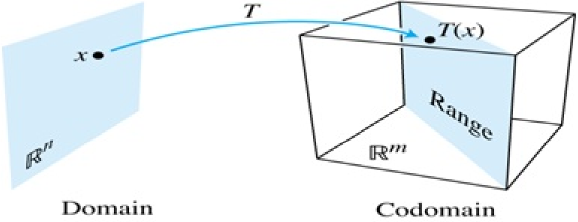

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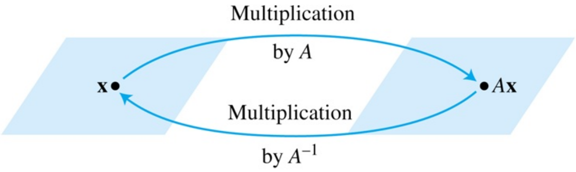
1. A has a pivot in every row/column
2. For every $b \in \mathbb{R}^n$, the equation $Ax = b$ has a unique solution.
3. Therefore, the range of T is equal to all of $\mathbb{R}^n$.

Invertible linear transformations

Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation with corresponding matrix A and suppose A is an invertible matrix (equivalently, A is non-singular).

The matrix A^-1 is an $n \times n$ matrix and also defines a linear transformation S which maps in the opposite direction.



A^{-1} transforms $A\mathbf{x}$ back to $\mathbf{x}$.

We usually refer to the inverse transformation corresponding to the matrix A^{-1} as $T^{-1}(x) = A^{-1}x$.

An example

Example. Let $T(x) = Ax$, where $A = \begin{bmatrix} 1 & 3 \\ -2 & -4 \end{bmatrix}$.

(a) Find the matrix corresponding to the inverse transformation T^{-1}.

(b) Calculate $T^{-1}\left(A\begin{bmatrix} 1 \\ -4 \end{bmatrix}\right)$.

(c) Calculate $T^{-1}\left(\begin{bmatrix} 1 \\ -4 \end{bmatrix}\right)$.

(d) Calculate $A^{-1} \cdot T\left(\begin{bmatrix} -2 \\ 0 \end{bmatrix}\right)$.

Ans. (a)

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Ans. (a) Use formula (b) $\begin{bmatrix} 1 \\ -4 \end{bmatrix}$

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