Math 1119B: Week 8, Lecture 2

David Thomson

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What not to write your professor

Dear Prof.,

I hope you are having a good time in Brazil. I missed your test on Monday due to unforeseen circumstances. I did not do well in the first two tests, so I hope you can make this up to me, and I can write the test any time this week.

Safe travels,

-Student

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Last day

- Transformations,
- Domain, co-domain and range,
- the concept of linear,
- matrices $\leftrightarrow$ tranformation rules.

Another matrix example

Let T be a linear transformation from $\mathbb{R}^3 \to \mathbb{R}^3$. Find the associated matrix A if $T(x) = 14x$.

Ans.

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Ans. $A = 14I_3$.

Example. Let $R\colon \mathbb{R}^4 \to \mathbb{R}^2$ be a linear transformation with the following relations:

$$R(e_1) = \begin{bmatrix} 2 \\ 2 \end{bmatrix}, R(e_2) = \begin{bmatrix} 3 \\ -1 \end{bmatrix}, R(e_3) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, R(e_4) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Find $R \begin{bmatrix} -2 \\ 1 \\ 12 \\ 0 \end{bmatrix}$.

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Find $R \begin{bmatrix} -2 \\ 1 \\ 12 \\ 0 \end{bmatrix}$.

Ans. $\begin{bmatrix} 1 \\ -5 \end{bmatrix}$

Putting it together.

Example. Let $R(x) = Bx$, where $B = \begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix}$. Is $\begin{bmatrix} 3 \\ -5 \end{bmatrix}$ in the range of R?

Example. Let $T \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$ and $T \begin{bmatrix} 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \end{bmatrix}$.
Find $T \begin{bmatrix} -1 \\ -4 \end{bmatrix}$.

Ans. Only for those that attend class.

Difference equations

Difference equations are used in many areas such as economics and engineering to map a changing system (such as population) that, when measured at discrete time intervals (say, each year), follows a prescribed pattern.

Suppose the vector v_1 is the initial state of the system, v_2 is the state of the system at the second measurement, ..., v_k is the state at the kth measurement.

Definition. A linear difference equation is given when there is a matrix A such that

$$v_1 = Av_0, \quad v_2 = Av_1, \quad \dots \quad , v_{k+1} = Av_k.$$

Population movement

We can model population movement from one group to another as a difference equation.

Suppose I am concerned about population migration for university students moving to/from residence. Let v_0 be a vector which gives the population of student living both on and off of residence in 2008, say

$$v_0 = \begin{bmatrix} r_0 \\ n_0 \end{bmatrix}.$$

Then v_1 gives the populations in the year 2009, v_2 in 2010, and so on.

Modelling migration

Suppose 20% of students on-campus move off-campus each term and 5% of student off-campus move on-campus in a given year (demographic study).

After one term, the number r_0 of students from residence are distributed in the following way:

$$\begin{bmatrix} 0.8r_0 \\ 0.2r_0 \end{bmatrix} = r_0 \begin{bmatrix} 0.8 \\ 0.2 \end{bmatrix},$$

and the number s_0 of students off-campus are given by

$$\begin{bmatrix} 0.05n_0 \\ 0.95n_0 \end{bmatrix} = n_0 \begin{bmatrix} 0.05 \\ 0.95 \end{bmatrix}.$$

The distribution of people in the next term is given by the sum:

$$r_0 \begin{bmatrix} 0.8 \\ 0.2 \end{bmatrix} + n_0 \begin{bmatrix} 0.05 \\ 0.95 \end{bmatrix} =$$

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Let $M = \begin{bmatrix} 0.8 & 0.05 \\ 0.2 & 0.95 \end{bmatrix}$. Then M is called the migration matrix.

If the migration percentages remain constant, then we have $v_2 = Mv_1, v_3 = Mv_2$, and so on.

The sequence of vectors $v_0, v_1, \ldots$ describes the population of residence and non-residence students over a period of terms.

City of Ottawa

Example. The city of Ottawa can be split into three regions: Ottawa, Kanata and Orléans. If $x_0 = \begin{bmatrix} 400,000 \\ 200,000 \\ 120,000 \end{bmatrix}$ is a population vector for the year 2004 and $O = \begin{bmatrix} 0.8 & 0.2 & 0.1 \\ 0.15 & 0.65 & 0.35 \\ 0.05 & 0.15 & 0.55 \end{bmatrix}$ is the migration matrix of the city.

(a) Describe in words what O means.

(b) Write an expression for the population vector in the year 2010.

Ans.

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Ans.

$$x_6 = O^6 \begin{bmatrix} 400,000 \\ 200,000 \\ 120,000 \end{bmatrix}.$$

What is an invertible matrix?

We made it to the end of Chapter 1!!

The idea. Almost every number has an inverse. For example, $5^{-1} = 1/5$, that is the inverse of a number is its reciprocal, and $5 \cdot 5^{-1} = 5 \cdot 1/5 = 1$ and also $5^{-1} \cdot 5 = 1/5 \cdot 5 = 1$.

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The only number that has no inverse is 0. This means that there is no number a such that $0a = a0 = 1$.

Definition. An $n \times n$ matrix A is invertible if there is an $n \times n$ matrix B such that $AB = BA = I_n$, the $n \times n$ identity matrix. In this case, we call B the inverse of A and usually denote $B = A^{-1}$.

Theorem. If A is invertible, then its inverse is unique.

More inverses

Definition. (Just to confuse you) A matrix that is not invertible is sometimes called singular.

A matrix that is invertible is sometimes called non-singular.

Properties of inverses

1. If A is an invertible matrix, then A^{-1} is an invertible matrix and $(A^{-1})^{-1} = A$.
2. If O and C are $n \times n$ invertible matrices, then so is OC and the inverse of OC is given by $(OC)^{-1} =$

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1. If A is an invertible matrix, then A^{-1} is an invertible matrix and $(A^{-1})^{-1} = A$.
2. If O and C are $n \times n$ invertible matrices, then so is OC and the inverse of OC is given by $(OC)^{-1} = C^{-1}O^{-1}$ (recall the OC-transpose theorem).
3. If A is an invertible matrix, then so is A^T and $(A^T)^{-1} = (A^{-1})^T$.