Math 1119B: Week 8, Lecture 1

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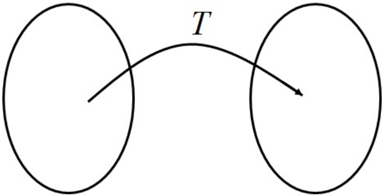
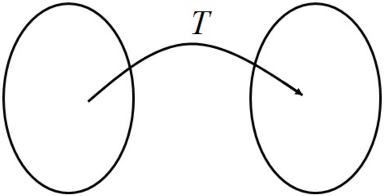
Transformations

- At first, matrices were introduced in order to solve systems of linear equations. In the past two weeks, we dealt with matrices as their own objects.
- A further use of matrices is as a linear transformation from one space to another.



Definition. A transformation $T\colon \mathbb{R}^m \to \mathbb{R}^n$ is a rule (or mapping or function) that uniquely assigns, for every vector $u \in \mathbb{R}^m$, a certain value in $\mathbb{R}^n$. That is, $T(u) = v$ for some $v \in \mathbb{R}^m$.

Relating transformation from what we already know

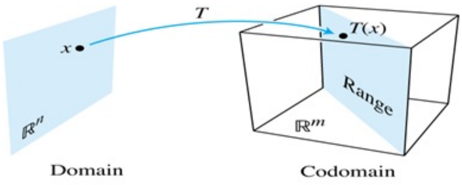


In calculus, or in high school, a transformation from $\mathbb{R}^1 \to \mathbb{R}^1$ is given by, for example $f(x) = x^2$. That is, the number x is uniquely assigned, under f, to the number x^2.

Examples. Two more examples: Let T be a transformation $\mathbb{R}^n \to \mathbb{R}^n$ such that $T(v) = v$. Then T is called the identity transformation.

Let S be a transformation $\mathbb{R}^n \to \mathbb{R}^m$ with the rule $S(u) = 0$ for all $u \in \mathbb{R}^n$. Then S is called the trivial transformation, sometimes denoted 0.

Domain, co-domain, range



Domain, codomain, and range of $T: \mathbb{R}^n \to \mathbb{R}^m$.

Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a transformation.

Definition. The space that T assigns values to (that is, the space on the left), $\mathbb{R}^n$, is called the domain of T.

Definition. The space containing the images of T, $\mathbb{R}^m$, is called the co-domain of T.

Definition. The set of images of T, that is $\{T(u) : u \in \mathbb{R}^n\}$, is called the range of the transformation T.

How we declare the transformation

To describe a transformation $T : \mathbb{R}^n \to \mathbb{R}^m$, we must secify the vector $T(x) \in \mathbb{R}^m$ associated to every single vector $x \in \mathbb{R}^n$.

There are usually an infinite number of such vectors (so writing them out would take a long time).

So we often (like in calculus) use a formula to describe the transformation.

A couple of examples

Example. Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be given by $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \end{bmatrix}$ is called the projection of $\mathbb{R}^2$ onto the first coordinate.

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The co-domain of T is $\mathbb{R}^2$.

The range of T is $\left\{ \begin{bmatrix} a \\ 0 \end{bmatrix} : a \in \mathbb{R} \right\}$.

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The co-domain of T is $\mathbb{R}^2$.

The range of T is $\left\{ \begin{bmatrix} a \\ 0 \end{bmatrix} : a \in \mathbb{R} \right\}$. This is not the same as $\mathbb{R}^1$, though there is a simple transformation between the two.

Another quick example

Example. Let $S\colon \mathbb{R}^3 \to \mathbb{R}^2$ be given by $S\left(\begin{bmatrix}0\\0\\0\end{bmatrix}\right) = \begin{bmatrix}0\\0\end{bmatrix}$.

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The domain of S is $\mathbb{R}^3$.

The co-domain of S is $\mathbb{R}^2$.

The range of S is $\{0\}$.

In general it is not always easy to find the range of a transformation.

One more for the road

Example. Take the transformation $W \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix} = \begin{bmatrix} w_1 + 2w_4 \\ -w_2 + 3w_3 \end{bmatrix}.$

(a) W defines a transformation $\mathbb{R}^n \to \mathbb{R}^m$. Determine the values of n and m.

Ans.

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(a) W defines a transformation $\mathbb{R}^n \to \mathbb{R}^m$. Determine the values of n and m.

Ans. $n = 4, m = 2$.

(b)

$$W \begin{bmatrix} 1 \\ 0 \\ -1 \\ 2 \end{bmatrix} =$$

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$$W \begin{bmatrix} 1 \\ 0 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ -3 \end{bmatrix}.$$

Transformations to linear transformations

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Instead of using a formula, like before, a certain type of transformation, linear transformations (or matrix transformations) can be described by a... matrix. The transformation is given by left-multiplying vectors by a matrix.

Example. Show that the previous example can be represented by multiplying on the left by the matrix $\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & -1 & 3 & 0 \end{bmatrix}$.

Matrix transformations

In general if A is a $m \times n$ matrix, left-multiplication by A gives a transformation

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Matrix transformations

In general if A is a $m \times n$ matrix, left-multiplication by A gives a transformation

$$T : \mathbb{R}^n \to \mathbb{R}^m, \text{defined by } T(x) = Ax$$

for each $x \in \mathbb{R}^n$.

We call T the matrix transformation induced by A.

Example. Describe the matrix transformation induced by the matrix I_n.

A couple of more examples

Describe the matrix transformation induced by the following matrices:

(a) $\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$

(b) $\begin{bmatrix} 1/2 & 1/2 & 1/2 \end{bmatrix}$

(c) $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

So why linear?

Definition. A transformation $T\colon \mathbb{R}^n \to \mathbb{R}^m$ is called a linear transformation if it satisfies the following two properties for all vectors x and y in $\mathbb{R}^n$:

1. $T(x+y) = T(x) + T(y)$
2. $T(ax) = aT(x)$ for all scalars a.

More generally, we can transform any linear combination of vectors $v_1, v_2, \ldots, v_p \in \mathbb{R}^n$ in the following way:

Theorem. If $T\colon \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation and $b = c_1 v_1 + c_2 v_2 + \cdots + c_m v_p$, then

$$T(b) = c_1 T(v_1) + c_2 T(v_2) + \cdots + c_p T(v_p).$$

Linear versus nonlinear

Example. Determine if the transformation $T\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}x_1+x_2\\x_1-x_3\end{bmatrix}$ is linear.

Linear versus nonlinear

Example. Determine if the transformation $T\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ x_1 - x_3 \end{bmatrix}$ is linear. Yes!

Example. Determine if the transformation $T\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_1 x_2 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ is linear.

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Example. Determine if the transformation $T\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_1 x_2 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ is linear. No.

The matrix of a linear transformation

Every linear transformation can be realized as a matrix transformation. To determine the matrix of a linear transformation, recall the standard basis vectors of $\mathbb{R}^n$:

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \ldots, e_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}.$$

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Theorem. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation, then T has a corresponding $m \times n$ matrix A, given by

$$A = \begin{bmatrix} T(e_1) & T(e_2) & \cdots & T(e_n) \end{bmatrix},$$

such that $T(x) = Ax$.

Finding the matrix A

Example. Consider the linear transformation

$$S\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}3x_1+4x_2\\x_1+7x_2\\x_1\end{bmatrix}.$$

(A) What is the domain and codomain of S?
Ans.

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Ans. $\mathbb{R}^2, \mathbb{R}^3$.

(B) What are the dimensions of the matrix that induces S?
Ans.

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Ans.3×2.

(C) Find the matrix A.

Ans. $\begin{bmatrix} 3 & 4 \\ 1 & 7 \\ 1 & 0 \end{bmatrix}$

Another matrix example

Let T be a linear transformation from $\mathbb{R}^3 \to \mathbb{R}^3$. Find the associated matrix A if $T(x) = 14x$.

Ans.

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Ans. $A = 14I_3$.

Example. Let $R\colon \mathbb{R}^4 \to \mathbb{R}^2$ be a linear transformation with the following relations:

$$R(e_1) = \begin{bmatrix} 2 \\ 2 \end{bmatrix}, R(e_2) = \begin{bmatrix} 3 \\ -1 \end{bmatrix}, R(e_3) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, R(e_4) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Find $R \begin{bmatrix} -2 \\ 1 \\ 12 \\ 0 \end{bmatrix}$.

Ans.

Another matrix example

Let T be a linear transformation from $\mathbb{R}^3 \to \mathbb{R}^3$. Find the associated matrix A if $T(x) = 14x$.

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Find $R \begin{bmatrix} -2 \\ 1 \\ 12 \\ 0 \end{bmatrix}$.

Ans. $\begin{bmatrix} 1 \\ -5 \end{bmatrix}$

Putting it together.

Example. Let $T\begin{bmatrix}1\\-1\end{bmatrix}=\begin{bmatrix}-3\\1\end{bmatrix}$ and $T\begin{bmatrix}4\\1\end{bmatrix}=\begin{bmatrix}0\\-2\end{bmatrix}$.
Find $T\begin{bmatrix}-1\\-4\end{bmatrix}$.

Ans. Only for those that attend class.