PART 1: Multiple-Choice Questions
Please circle only one answer.

1. **[3 marks]** Let $f(x) = \ln(e^{2x})$. Evaluate $f''(0)$. In other words, find the second derivative of *f* at *x*=0.

   (a) $f''(0) = 1$

   (b) $f''(0) = 2$

   (c) $f''(0) = -1$

   (d) $f''(0) = 0$ $\Leftarrow$ **Answer**

2. **[3 marks]** Let *f*(*x*) = |*x*|. Calculate

$$L = \lim_{h \to 0} \frac{f(h) - f(0)}{h}.$$

   (a) *L*=0

   (b) *L*=1

   (c) *L*=-1

   (d) This limit does not exist $\Leftarrow$ **Answer**

3. **[3 marks]** Let $f(x) = \sqrt{e^{2x} + 2x}$. Evaluate $f'(0)$. In other words, find the derivative of *f* at *x*=0.

   (a) $f'(0) = 1$

   (b) $f'(0) = 4$

   (c) $f'(0) = 2$ $\Leftarrow$ **Answer**

   (d) $f'(0) = -1$

4. **[3 marks]** Let $f(x) = \sin(\cos x)$. Evaluate $f'(x)$. In other words, find the derivative of *f* at *x*.

   (a) $f'(x) = \cos(\cos x)$

   (b) $f'(x) = -\sin x \cdot \cos(\cos x) \Leftarrow \textbf{Answer}$

   (c) $f'(x) = \cos(\sin x)$

   (d) $f'(x) = \cos x$

5. **[3 marks]** Let $f(x) = (\tan x)^{3x}$. Evaluate $f'(x)$. In other words, find the derivative of *f* at *x*.

   (a) $f'(x) = (\tan x)^{3x} \cdot (3x \cdot \ln(\tan x) + 3x)$

   (b) $f'(x) = 3x \cdot (\tan x)^{3x-1} \cdot \sec^2 x$

   (c) $f'(x) = (\tan x)^{3x} \cdot \left(\frac{3x \cdot \sec^2 x}{\tan x} + 3\ln(\tan x)\right) \Leftarrow \textbf{Answer}$

   (d) $f'(x) = \sec^2 3x$

6. **[3 marks]** Let $f(x) = \operatorname{Arccos}\left(\sin x^2\right)$. Calculate $f'(0)$. In other words, find the derivative of *f* at *x*=0.

   (a) $f'(0) = 0 \Leftarrow \textbf{Answer}$

   (b) $f'(0) = 1$

   (c) $f'(0) = -1$

   (d) $f'(0) = -2$

7. **[3 marks]** Evaluate $L = \lim_{x \to 0} \left(\frac{\sin 3x}{\sin 2x} \right)$ using any method.

(a) *L*= -1

(b) *L* = 1.75

(c) *L* = 3

(d) $L = \frac{3}{2}$ $\Longleftarrow$ **Answer**

8. **[3 marks]** Evaluate

$$\lim_{x \to +\infty} \frac{d}{dx} \int_0^{3\sqrt{x}} \frac{3t^4 + 1}{(t-3)(t^2+3)} \, dt$$

(a) $I = 0$

(b) $I = \frac{1}{2}$

(c) This limit does not exist

(d) $I = \frac{27}{2}$ $\Longleftarrow$ **Answer**

9. **[3 marks]** Which of the following expressions gives the **area** of the region bounded by the curves $y = \sqrt{x}$, $y = -x$, and between the lines x=1 and x=4?

(a) $I = \int_1^4 \sqrt{x} \, dx$

(b) $I = \int_1^4 x \, dx$

(c) $I = \int_1^4 (\sqrt{x} - x) \, dx$

(d) $I = \int_1^4 (x + \sqrt{x})\ dx$ $\Longleftarrow$ **Answer**

10. **[3 marks]** Which of the following expressions gives the **volume** of the solid of revolution obtained when the region bounded by the graphs of $y = 2x$ and $y = 4x^2$ is revolved about the ***y*-axis**?

(a) $I = \int_0^1 \pi(2x - 4x^2)\ dx$

(b) $I = \int_0^{\frac{1}{2}} 2\pi x(2x - 4x^2)\ dx$ $\Longleftarrow$ **Answer**

(c) $I = \int_0^1 \pi(y - y^2)\ dy$

(d) $I = \int_0^1 2\pi(\sqrt{y} - y)\ dy$

Subtotal : 30 marks

PART 2
Please show all work here.

11. **[6 marks] (a)** Find the general solution of the differential equation

$$y\frac{dy}{dx} = x^2 e^{2x - y^2},$$

using the method of separation of variables and any method of integration.

Solution: Rewrite this as $ye^{y^2}\frac{dy}{dx} = x^2 e^{2x}.$ Now integrate both sides with respect to *x*. We see that

$$\int ye^{y^2}\frac{dy}{dx}\ dx = \int x^2 e^{2x}\ dx.$$

Now, let $u = y^2(x)$, $du = 2y(x)\ y'(x)\ dx$, on the left and use the Table Method on the right. We get,

$$\int e^u\ \frac{du}{2} = \int x^2 e^{2x}\ dx,$$

or

$$\frac{1}{2}\ e^u = \frac{1}{2}x^2e^{2x} - \frac{1}{2}xe^{2x} + \frac{1}{4}e^{2x} + C,$$

i.e.,

$$\frac{1}{2}e^{y^2} = \frac{1}{2}x^2e^{2x} - \frac{1}{2}xe^{2x} + \frac{1}{4}e^{2x} + C.$$

This is the general solution. **[2 marks] (b)** Find the particular solution of this differential equation which satisfies *y*=1 when *x*=0.

Solution: We simply set *x*=0 and *y*=1 into the general solution and then solve for *C*. This gives us, $\frac{1}{2}e = \frac{1}{4} + C$ or

$$C = \frac{1}{2}e - \frac{1}{4}.$$

The particular solution is then given by

$$\frac{1}{2}e^{y^2} = \frac{1}{2}x^2e^{2x} - \frac{1}{2}xe^{2x} + \frac{1}{4}e^{2x} + \frac{1}{2}e - \frac{1}{4}.$$

12. Evaluate the following integrals using any method.

[4 marks] (a) $\int_0^{\frac{\pi}{6}} e^{-2x} \sin 3x\ dx$

Solution: Write

$$I = \int e^{-2x} \sin 3x\ dx.$$

Integrate by Parts twice or use the "MYCAR" trick! You'll get

$$I = \square\left(-\frac{1}{3}e^{-2x}\cos 3x - \frac{2}{9}e^{-2x}\sin 3x\right),$$

where the entry in the Box is given by $\frac{9}{13}$. See Section 8.3.4 for details. This I is an antiderivative, and so

$$\int_0^{\frac{\pi}{6}} e^{-2x}\sin 3x \; dx = \frac{3}{13} - \frac{2}{13}e^{-\frac{\pi}{3}}.$$

[4 marks] (b) $\int \frac{x^3}{(x-1)(x+1)} \; dx$

Solution: The degree of the numerator exceeds that of the denominator and so we must divide these expressions. Thus

$$\frac{x^3}{x^2-1} = x + \frac{x}{x^2-1}.$$

Next, we use partial fractions on the third integral thus:

$$\begin{aligned} \int \frac{x^3}{x^2-1} \; dx &= \int x \; dx + \int \frac{x}{x^2-1} \; dx \\ &= \frac{x^2}{2} + \frac{1}{2}\ln|x^2-1| + C, \end{aligned}$$

where we let $u = x^2 - 1$, $du = 2x \; dx$, in the second integral (and didn't *have* to use partial fractions).

13. **[10 marks]** Sketch the graph of the function *f* defined by $f(x) = \frac{4}{9+x^2}$ by providing the following information:

[2 marks] (a) Find the critical points of *f*,

[2 marks] (b) Find the intervals where the graph of *f* is increasing and decreasing,

[2 marks] (c) Find the intervals where the graph of *f* is concave up and concave down,

[2 marks] (d) Find all asymptotes,

[2 marks] (e) Sketch the graph of *f*. **Solution:** (a) $f'(x) = -\frac{8x}{(9+x^2)^2}$, so $f'(x) = 0$ only when *x*=0. This is the only critical point. (b) Use the Sign Decomposition Table of f'. You see that *f* is increasing on $(-\infty, 0)$ and decreasing on $(0, \infty)$. (c)

$f''(x) = \frac{24(x^2-3)}{(9+x^2)^4}$. The break-points are $x = \pm\sqrt{3}$. So, the SDT gives that *f* is concave up on $(-\infty, \sqrt{3})$ or $(\sqrt{3}, \infty)$ and concave down on $(-\sqrt{3}, \sqrt{3})$. (d) $\lim_{x\to\pm\infty} \frac{4}{9+x^2} = 0.$ So *y*=0 is a horizontal asymptote. There are no vertical asymptotes since *f*(*x*) is always finite, for finite *x*. (e) the graph of *f* is bell-shaped, dropping down to zero as $x \to \pm\infty$ and its maximum value occurs at *x*=0.

14. **[4 marks]** Plutonium 239 (Pu 239) has a half-life of 24,100 years. This radionuclide is an extremely toxic carcinogen and occurs as a by-product of nuclear activity. How long would it take for a 1 gram sample of Pu 239 to decay to 1 microgram (10^{-6} grams)?

Solution: Use the half-life formula, $N(t) = \frac{N(0)}{2^{t/T}}$. Given that *T* = 24, 000, $N(0) = 1g = 10^6$ micrograms, we get that *N*(*t*) = 1 (what we want) and this forces the equality

$$1 = \frac{1,000,000}{2^{t/24,000}}.$$

Solving this for *t* using natural logarithms gives us

$$t = \frac{24,100 \cdot \ln(1,000,000)}{\ln 2} \approx 480,351$$

years.

in **Subtotal: 30 marks**

Extra Pages for Rough Work: DO NOT UNSTAPLE

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