Math 1119B: Week 6, Lecture 2

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Showing dependence by inspection

Determine if the given vectors are linearly independent.

1. Let $v_1 = \begin{bmatrix} 2 \\ 2 \\ -3 \\ 4 \end{bmatrix}, v_2 = \begin{bmatrix} -1 \\ -1 \\ 3/2 \\ 2 \end{bmatrix}$,

2. Let $w_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, w_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, w_3 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$,

3. The columns of the matrix $\begin{bmatrix} 1 & 3 & 5 \\ 2 & 3 & 8 \end{bmatrix}$,

4. The rows of the matrix $\begin{bmatrix} 1 & -1 \\ 2 & -4 \\ 3 & -2 \end{bmatrix}$.

Another definition! Rank

Definition. The rank of a matrix, denoted $\mathrm{rk}(A) = \mathrm{rank}(A)$, is the number of linearly independent columns of A.

An alternate definition is the rank of a matrix is the number of pivot columns of a matrix.

Solve for the rank by row-reducing!!

Some things about ranks

Theorem. The rank of a matrix A is the same as the rank of any echelon form of A.

This is obvious: we do not know the number of pivot columns until we row-reduce, but they are still pivot columns!

Easy example. What is the rank of $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$?

Ans.

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Ans. 3!

In general, if any $m \times m$ matrix A row-reduces to the identity matrix, $\mathrm{rk}(A) = m$.

Some friendly advice

I strongly suggest that you try the practice problem at the end of Section 1.7. When you're finished, check the solution after the exercises.

In fact, if you haven't been doing them, to date, you should do ALL of those practice problems (and check your solutions after).

Spanning $\mathbb{R}^m$

Geometrically, we can think of $\mathbb{R} = \mathbb{R}^1$ as the real line.

We think of $\mathbb{R}^2$ as a plane (think: the blackboard spread infinitely in both directions).

We think of $\mathbb{R}^3$ as the world.

Forget badly explained television shows on the Discovery network. The "fourth dimension" is not time. It is the exact same as the other three dimensions – the difference is we only see three dimensions, so we can't visualize it.

Algebraically, we can represent the space $\mathbb{R}^m$ as vectors of length m.

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A space is a "self-contained body" (Dave: my wording, sorry). In our cases we deal with multiple copies of the real numbers. Any space V (in our case, think $V = \mathbb{R}^m$) is defined by three properties:

1. V contains the zero vector.
2. If u and v are two vectors of V, so is $u + v$.
3. If u is any vector of V and c is any scalar, then cv is also in V.

Examples of spaces.

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Examples of spaces.

1. $\mathbb{R}^m$
2. $\mathrm{Span}\left(\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}\right)$.

Back to $\mathbb{R}^m$

Suppose u and v are linearly independent vectors in $\mathbb{R}^2$ (for example, $u = (1, 0)$ and $v = (0, 1)$). Let

$$A = \begin{bmatrix} u & v \end{bmatrix}$$

and let b be any vector in $\mathbb{R}^2$. Since A contains 2 pivots, the matrix equation $Ax = b$ is

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and let b be any vector in $\mathbb{R}^2$. Since A contains 2 pivots, the matrix equation $Ax = b$ is consistent for any choice of b.

What this means is that any vector $b \in \mathbb{R}^2$ can be given as a linear combination of u and v.

Again, nothing here is special about $\mathbb{R}^2$!

Spanning $\mathbb{R}^m$

Let $v_1, v_2, \ldots, v_m$ be m linearly independent vectors in $\mathbb{R}^m$ and let

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Since A has m pivots, A row reduces to

$$A \sim I_m$$

and thus there is a unique solution for the matrix equation $Ax = b$, for any $b \in \mathbb{R}^m$.

This is the definition of b being in the Span!

Spanning $\mathbb{R}^m$, as a theorem

Theorem. Let $v_1, v_2, \ldots, v_m$ be m linearly independent vectors in $\mathbb{R}^m$. Then $v_1, v_2, \ldots, v_m$ span R^m.

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Let $w_1 = (1,0,0)^T, w_2 = (0,1,0)^T, w_3 = (0,0,1)^T \in \mathbb{R}^3$, etc.

Definition. Vectors of the form

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \dots, v_m = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

are called the canonical basis of $\mathbb{R}^m$.

Bases

Let $v_1 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, v_2 = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}, v_3 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$. I can write any vector b in $\mathbb{R}^3$, $b = (b_1, b_2, b_3)^T$, as the linear combination

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$$b = b_1 v_1 + b_2 v_2 + b_3 v_3.$$

So v_1, v_2 and v_3 span $\mathbb{R}^3$. And no set of two vectors can span $\mathbb{R}^3$, and no set of four vectors can be linearly independent in $\mathbb{R}^3$.

Definition. Let V be a space. A linearly independent set which spans V is called a basis of V.

Clearly, any set of m independent vectors in $\mathbb{R}^m$ is a basis of $\mathbb{R}^m$.

Dimensions

Definition. Let V be a space. The dimension of V, called $\dim(V)$, is the number of basis vectors of V.

Example. Let $V = \mathrm{Span}(\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix})$. These vectors are linearly

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Skill testing question. If $\dim(V) = 2$, then does $V = \mathbb{R}^2$?

Example. Let $A = \begin{bmatrix}1 & -1\\2 & -4\\3 & -2\end{bmatrix}$. Determine a basis for the span of the columns of A (and so what is the dimension of that space)?

Ans.

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Ans. Any two columns of A.

The row/column space

The span of any number of vectors form a space. Until now, we have taken vectors, placed them into a matrix and row-reduced to determine which are linearly independent. However, we can begin with the matrices, themselves.

Definition. The column space of A, denoted $\mathrm{Col}(A)$, is given by the Span of the columns of A.

Definition. The row-space of A, denoted $\mathrm{Row}(A)$, is given by the Span of the rows of A.

Finding a basis of the column space of A

A basis of the $\mathrm{Col}(A)$ is given by the linearly independent columns of A. The linearly independent columns of A are given by

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A basis of the $\mathrm{Col}(A)$ is given by the linearly independent columns of A. The linearly independent columns of A are given by the pivot columns (alternatively, the columns given by the basic variables – the pivot columns) of the matrix A (not of the reduced matrix).

By the definition, $\dim \mathrm{Col}(A) = \mathrm{rk}(A)$!

Example. Let $A = \begin{bmatrix} 3 & -6 & 9 & -2 \\ 6 & 5 & 1 & 12 \\ 3 & 4 & 8 & -3 \end{bmatrix} \sim$

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The pivots are in Column 1 and Column 3, therefore these 2 columns form a basis of $\mathrm{Col}(A)$. The dimension of $\mathrm{Col}(A)$ is

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The pivots are in Column 1 and Column 3, therefore these 2 columns form a basis of $\mathrm{Col}(A)$. The dimension of $\mathrm{Col}(A)$ is 2.

Finding a basis of the row space of A

A basis of $\mathrm{Row}(A)$ is given by the linearly independent rows of A. The linearly independent rows of A are given by both the pivot rows of the original matrix A and the rows of the reduced matrix of A.

Example. Let

$$A = \begin{bmatrix} 3 & -1 & -3 & -1 & 8 \\ 3 & 1 & 3 & 0 & 2 \\ 0 & 3 & 9 & -1 & -4 \\ 6 & 3 & 9 & -2 & 6 \end{bmatrix} \sim \begin{bmatrix} 3 & -1 & -3 & 0 & 6 \\ 0 & 2 & 6 & 0 & -4 \\ 0 & 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

There are pivots in Rows

Finding a basis of the row space of A

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$$A = \begin{bmatrix} 3 & -1 & -3 & -1 & 8 \\ 3 & 1 & 3 & 0 & 2 \\ 0 & 3 & 9 & -1 & -4 \\ 6 & 3 & 9 & -2 & 6 \end{bmatrix} \sim \begin{bmatrix} 3 & -1 & -3 & 0 & 6 \\ 0 & 2 & 6 & 0 & -4 \\ 0 & 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

There are pivots in Rows $1, 2, 3$ and so Rows $1, 2, 3$ give a basis of the row space of A.

Note. The non-zero rows of any echelon form of A give a basis for $\mathrm{Row}(A)$.

A bigger example

Example. Let

$$A = \begin{bmatrix} 1 & -2 & 6 & 3 & -2 \\ 3 & -4 & -3 & -1 & 1 \\ -1 & 0 & 15 & 7 & -5 \end{bmatrix}.$$

Find (i) Vectors which Span $\mathrm{Row}(A)$. (ii) A basis of $\mathrm{Col}(A)$.
(iii) The dimension of $\mathrm{Row}(A)$.
(iiii) A basis of $\mathrm{Row}(A)$ where the first entry in the second vector is 0.

Ans.

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Ans. (i) By definition, the Rows of A Span $\mathrm{Row}(A)$.

ii) $A \sim \begin{bmatrix} 1 & 0-15 & -7 & -5 & \\ 0 & 2 & -21 & -5 & 7 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$, there are pivots in columns

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Ans. (i) By definition, the Rows of A Span $\mathrm{Row}(A)$.

ii) $A \sim \begin{bmatrix} 1 & 0-15 & -7 & -5 & \\ 0 & 2 & -21 & -5 & 7 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$, there are pivots in columns 1 and 2, so Columns 1 and 2 form a basis of $\mathrm{Col}(A)$.

Answer continued

Ans. $A \sim \begin{bmatrix} 1 & 0-15 & -7 & -5 & \\ 0 & 2 & -21 & -5 & 7 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

(iii) There are pivots in Rows 1 and 2, and so $\left\{ \begin{bmatrix} 1 \\ 0 \\ -15 \\ -7 \\ -5 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ -21 \\ -5 \\ 7 \end{bmatrix} \right\}$ forms a basis of $\mathrm{Row}(A)$. Thus $\dim(\mathrm{Row}(A)) = 2$.

(iiii) The nonzero rows of the reduced matrix give a basis of $\mathrm{Row}(A)$. Thus, $\{\begin{bmatrix} 1 & 0-15 & -7 & -5 \end{bmatrix}^T, \begin{bmatrix} 0 & 2 & -21 & -5 & 7 \end{bmatrix}^T\}$ is a basis of $\mathrm{Row}(A)$ where the second vector begins with a 0.

The Null space: Calculating $Ax = 0$

Defintion. The null space of a matrix A, denoted $\mathrm{null}(A)$, is the set of all solutions of $Ax = 0$.

Solving for the null space of A is the exact same as solving the matrix equation $Ax = 0$, simply row-reduce the matrix A (or row-reduce the augmented matrix $[A \mid 0]$.

If the matrix A contains a pivot in every column, the null space is

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Dimension of the null space

A basis for the null space is given by the vectors in the parametric-vector solution of $Ax = 0$

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Example. Let $A = \begin{bmatrix} 1 & 0 & -1 \\ 2 & -1 & 3 \\ -6 & 4 & -14 \end{bmatrix}$. Find a basis for $\mathrm{null}(A)$.

Ans. $A \sim$

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Ans. $A \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -5 \\ 0 & 0 & 0 \end{bmatrix}$ and the solution is parametric vector form is $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = s \begin{bmatrix} 1 \\ 5 \\ 1 \end{bmatrix}$, $s \in \mathbb{R}$. Thus, a basis of $\mathrm{null}(A)$ is $\left\{ \begin{bmatrix} 1 \\ 5 \\ 1 \end{bmatrix} \right\}$.

The Rank Theorem

A very important theorem in linear algebra is called the Rank Theorem. We have actually gone through the entire derivation of the theorem:

- The number of pivot columns in a matrix is the dimension of the column space of a matrix.
- The dimension of the column space is the rank of the matrix.
- The number of free variables in a matrix is the dimension of the null space of the matrix.
- The columns of a matrix correspond to basic variables, all non-pivot columns correspond to free variables.

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- The number of free variables in a matrix is the dimension of the null space of the matrix.
- The columns of a matrix correspond to basic variables, all non-pivot columns correspond to free variables.

The Rank Theorem Let A be an $m \times n$ matrix. Then

$$\mathrm{rk}(A) + \dim \mathrm{null}(A) = n.$$

Recap

- A set of vectors is linearly independent when the vector equation $c_1 v_1 + c_2 v_2 + \cdots + c_n v_n = 0$ contains only the trivial solution $c_1 = c_2 = \cdots = c_n$.
- Solve linear independence of vectors by placing them into a matrix and row-reduce. The pivot columns indicate linearly-independent vectors.
- A space is a set that contains the zero vector and is closed under (i) addition and (ii) scalar multiplication.
- A basis is a linear independent set of vectors that spans a space.
- The dimension of a space is the number of basis vectors of the space.
- The dimension of $\mathbb{R}^m$ is m.
- The rank of a matrix is the number of pivot rows/columns in the matrix.

More recap

- The column space of a matrix is spanned by the pivot columns of the original matrix.
- The row space of a matrix is spanned by the pivot rows of any echelon form of a matrix.
- The null space of a matrix is given by the set of solutions of $Ax = 0$
- If a $m \times n$ matrix has n pivot columns, the null space is $\{0\}$.
- The dimension of the null space of a matrix is the number of free variables.
- A basis of the null space is given by the parameter vectors in the solutions of $Ax = 0$.