Math 1119B: Week 6, Lecture 1

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Table of Contents

Recap

- Test this morning.
- Leontief exchange re-done.
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- We need to ramp up the speed of the course a little bit.

- Wednesday and Monday's lectures will be covered by Gary Bazdell. Office hours are cancelled this week but will be held and made up next week. Tutorials and TA office hours will run. Class is running as normal.

Determining if something is in the Span

For which values of h is the vector $\begin{bmatrix} -1 \\ h \\ 3 \end{bmatrix}$ in $\mathrm{Span}\left(\begin{bmatrix} 2 \\ 8 \\ -6 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \\ 0 \end{bmatrix}\right)$?

Ans. We want solutions to the vector equation
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We can solve this by inspection, too.

- Since there are only 2 variables, we can have at most 2 pivots.
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- So the coefficient matrix will have

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- The vectors in the span are not scalar multiples.
- So the coefficient matrix will have 2 pivots.
- Thus, we have a unique solution.
- No value of h gives a scalar multiple with the second vector in the span.
- The value $h = -4$ gives a scalar multiple.
- Since the solution is unique, this is the only value of h.

Definition

A set of vectors $v_1, v_2, \ldots, v_n \in \mathbb{R}^m$ is called linearly independent if there is only the trivial solution to the vector equation $c_1 v_1 + c_2 v_2 + \cdots + c_n v_n = 0$. If a set of vectors is not linearly independent, it is linearly dependent.

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You can solve for linear independence in the following way: Let $A = \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix}$, and solve $Ax = 0$. If the only solution is $x = 0$, then the vectors are linearly independent.

If there is a non-trivial solution to a homogeneous solution, then there are free variables!

Examples of linear (in)dependence

1. Let $A = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 5 & 10 \\ 3 & 3 & 8 \end{bmatrix}$, are the columns of A independent?

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2. Let $v_1 = (1,2,1)^T, v_2 = (-1,4,3)^T, v_3 = (-1,10,7)^T$.

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Ans. Let $B = \begin{bmatrix} 1 & -1 & -1 \\ 2 & 4 & 10 \\ 1 & 3 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$. Since B has a free variable, there are non-trivial solutions to $Ax = 0$.

When are vectors linearly independent?

Theorem. The following are all equivalent:

1. A set of vectors $v_1, v_2, \ldots, v_n \in \mathbb{R}^m$ are linearly independent.
2. The vector equation $c_1 v_1 + c_2 v_2 + \cdots + c_n v_n = 0$ has only the trivial solution.
3. If $A = \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix}$, the matrix equation $Ax = 0$ has only the trivial solution.
4. The matrix A has a pivot in every column.

Linear independence in $\mathbb{R}^2$

There were 3 ways to draw a set of 2 vectors in $\mathbb{R}^2$

1. Both were the 0 vector,
2. They were both on the same line ($u = cv$),
3. They were not on the same line ($u \neq cv$).

Let's look at the solutions to $Ax = 0$ in the previous cases:

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(2) $A = \begin{bmatrix} v & cv \end{bmatrix}$ so $Ax = 0$ has a free variable in the solution.
(3) (Example.) $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, so $Ax = 0$ has no free variables (only trivial solution).

Too many vectors.

The case of 1 vector is trivial: 1 (nonzero) vector is always linearly independent!

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- The matrix A has size 2×3.
- In order for v_1, v_2, v_3 to be linearly independent, A must have a pivot in every

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- The matrix A has size 2×3.
- In order for v_1, v_2, v_3 to be linearly independent, A must have a pivot in every column.
- Since there are only 2 rows, there can be at most 2 pivots!!
- Therefore any three vectors in $\mathbb{R}^2$ are linearly independent.

Too-many vectors (cont'd)

Was there anything special about 3 vectors in $\mathbb{R}^2$?

Too-many vectors (cont'd)

Was there anything special about 3 vectors in $\mathbb{R}^2$? No.

Too many vectors theorem. Let $n > m$. Any n vectors in $\mathbb{R}^m$ are linearly dependent.

The reasoning is the same as before – there isn't enough room to fit all of the pivots.

The zero vector makes things dependent

Let $v = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, then $5v = 3v = 1v = 0v = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$. Thus, the set $\{v\}$ itself is linearly dependent

Adding vectors to a linearly dependent set can never make a set independent.

Theorem. Any set of vectors including the zero vector is linearly dependent.