Math 1119B: Week 5

David Thomson

October 12, 2011

Table of Contents

Recap

- Some previous test recap.
- Homogeneous systems.
- What is the trivial solution of a homogeneous systems?
- Leontief's exchange problem. (And re-done today).

Two dimensions

Solve the following system of equations

$$\begin{array}{rcrcl} 3x_1 & + & 3x_2 & = & 9 \\ x_1 & + & x_2 & = & 3. \end{array}$$

Ans.:

Two dimensions

Solve the following system of equations

$$\begin{array}{rcrcl} 3x_1 & + & 3x_2 & = & 9 \\ x_1 & + & x_2 & = & 3. \end{array}$$

Ans.:

$$\begin{bmatrix} 3 \\ 0 \end{bmatrix} + v \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Let's draw what this means:

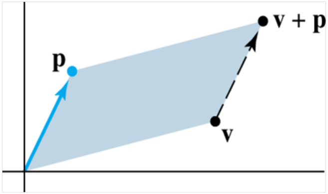
Two dimensions

Solve the following system of equations

$$\begin{array}{ccccc} 3x_1 & + & 3x_2 & = & 9 \\ x_1 & + & x_2 & = & 3. \end{array}$$

Ans.:

$$\begin{bmatrix} 3 \\ 0 \end{bmatrix} + v \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Let's draw what this means: 

The span of one vector

The span of a vector v is simply every scalar multiple of that vector.

For example, if $v = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, then $\mathrm{Span}(v) =$

The span of one vector

The span of a vector v is simply every scalar multiple of that vector.

For example, if $v = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, then $\mathrm{Span}(v) = a \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $a \in \mathbb{R}$.

Skill testing question: How many vectors are in $\{v\}$?:

The span of one vector

The span of a vector v is simply every scalar multiple of that vector.

For example, if $v = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, then $\mathrm{Span}(v) = a \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $a \in \mathbb{R}$.

Skill testing question: How many vectors are in $\{v\}$?: 1.
How many vectors are in $\mathrm{Span}(v)$?:

The span of one vector

The span of a vector v is simply every scalar multiple of that vector.

For example, if $v = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, then $\mathrm{Span}(v) = a \begin{bmatrix} 1 \\ 1 \end{bmatrix}, a \in \mathbb{R}$.

Skill testing question: How many vectors are in $\{v\}$?: 1.
How many vectors are in $\mathrm{Span}(v)$?: Infinitely many.

The span of many vectors

Definition. The span of a set of vectors $v_1, v_2, \ldots, v_n$ is the set of all linear combinations of $v_1, v_2, \ldots, v_n$. That is,

$$c_1 v_1 + c_2 v_2 + \cdots + c_n v_n, \quad c_1, c_2, \ldots, c_n \in \mathbb{R}.$$

Skill testing question: How many vectors are in $\{v_1, v_2, \ldots, v_n\}$?:

The span of many vectors

Definition. The span of a set of vectors $v_1, v_2, \ldots, v_n$ is the set of all linear combinations of $v_1, v_2, \ldots, v_n$. That is,

$$c_1 v_1 + c_2 v_2 + \cdots + c_n v_n, \quad c_1, c_2, \ldots, c_n \in \mathbb{R}.$$

Skill testing question: How many vectors are in $\{v_1, v_2, \ldots, v_n\}$?:
n.
How many vectors are in $\mathrm{Span}(v)$?:

The span of many vectors

Definition. The span of a set of vectors $v_1, v_2, \dots, v_n$ is the set of all linear combinations of $v_1, v_2, \dots, v_n$. That is,

$$c_1 v_1 + c_2 v_2 + \cdots + c_n v_n, \quad c_1, c_2, \dots, c_n \in \mathbb{R}.$$

Skill testing question: How many vectors are in $\{v_1, v_2, \dots, v_n\}$?:
n.
How many vectors are in $\mathrm{Span}(v)$?: Infinitely many.

Span of two vectors – how many ways can we do this?

Blackboard. Suppose we have any two vectors u and v. How many different pictures can we draw for $\mathrm{Span}(u, v)$?

Span of two vectors – how many ways can we do this?

Blackboard. Suppose we have any two vectors u and v. How many different pictures can we draw for $\mathrm{Span}(u, v)$?

- $u = v = 0$.

Span of two vectors – how many ways can we do this?

Blackboard. Suppose we have any two vectors u and v. How many different pictures can we draw for $\mathrm{Span}(u, v)$?

- $u = v = 0$.
- $u = cv$.

Span of two vectors – how many ways can we do this?

Blackboard. Suppose we have any two vectors u and v. How many different pictures can we draw for $\mathrm{Span}(u, v)$?

- $u = v = 0$.
- $u = cv$.
- $u \neq cv$.

Note: In general, we have a set of vectors is either:

1. All zero.

Span of two vectors – how many ways can we do this?

Blackboard. Suppose we have any two vectors u and v. How many different pictures can we draw for $\mathrm{Span}(u, v)$?

- $u = v = 0$.
- $u = cv$.
- $u \neq cv$.

Note: In general, we have a set of vectors is either:

1. All zero.
2. One vector can be expressed as a linear combination of the others.

Span of two vectors – how many ways can we do this?

Blackboard. Suppose we have any two vectors u and v. How many different pictures can we draw for $\mathrm{Span}(u, v)$?

- $u = v = 0$.
- $u = cv$.
- $u \neq cv$.

Note: In general, we have a set of vectors is either:

1. All zero.
2. One vector can be expressed as a linear combination of the others.
3. No vectors can be given as linear combination of the others.

Some examples

Let $v_1 = \begin{bmatrix} 4 \\ 5 \\ -6 \end{bmatrix}$, $v_2 = \begin{bmatrix} -3 \\ -2 \\ 2 \end{bmatrix}$ and $v_3 = \begin{bmatrix} 1 \\ 5 \\ -3 \end{bmatrix}$. Set $b = \begin{bmatrix} -7 \\ -3 \\ 10 \end{bmatrix}$.

(i) How many vectors are in $\{v_1, v_2, v_3\}$?

(ii) How many vectors are in $\mathrm{Span}(v_1, v_2, v_3)$?

(iii) Is $b \in \mathrm{Span}(v_1, v_2, v_3)$?

Ans.

Some examples

Let $v_1 = \begin{bmatrix} 4 \\ 5 \\ -6 \end{bmatrix}$, $v_2 = \begin{bmatrix} -3 \\ -2 \\ 2 \end{bmatrix}$ and $v_3 = \begin{bmatrix} 1 \\ 5 \\ -3 \end{bmatrix}$. Set $b = \begin{bmatrix} -7 \\ -3 \\ 10 \end{bmatrix}$.

(i) How many vectors are in $\{v_1, v_2, v_3\}$?

(ii) How many vectors are in $\mathrm{Span}(v_1, v_2, v_3)$?

(iii) Is $b \in \mathrm{Span}(v_1, v_2, v_3)$?

Ans. (i) 3

Some examples

Let $v_1 = \begin{bmatrix} 4 \\ 5 \\ -6 \end{bmatrix}$, $v_2 = \begin{bmatrix} -3 \\ -2 \\ 2 \end{bmatrix}$ and $v_3 = \begin{bmatrix} 1 \\ 5 \\ -3 \end{bmatrix}$. Set $b = \begin{bmatrix} -7 \\ -3 \\ 10 \end{bmatrix}$.

(i) How many vectors are in $\{v_1, v_2, v_3\}$?
(ii) How many vectors are in $\mathrm{Span}(v_1, v_2, v_3)$?
(iii) Is $b \in \mathrm{Span}(v_1, v_2, v_3)$?

Ans. (i) 3 (ii) Infinitely many

Some examples

Let $v_1 = \begin{bmatrix} 4 \\ 5 \\ -6 \end{bmatrix}$, $v_2 = \begin{bmatrix} -3 \\ -2 \\ 2 \end{bmatrix}$ and $v_3 = \begin{bmatrix} 1 \\ 5 \\ -3 \end{bmatrix}$. Set $b = \begin{bmatrix} -7 \\ -3 \\ 10 \end{bmatrix}$.

(i) How many vectors are in $\{v_1, v_2, v_3\}$?

(ii) How many vectors are in $\mathrm{Span}(v_1, v_2, v_3)$?

(iii) Is $b \in \mathrm{Span}(v_1, v_2, v_3)$?

Ans. (i) 3 (ii) Infinitely many

(iii) Set $A = \begin{bmatrix} v_1 & v_2 & v_3 & b \end{bmatrix}$ and row reduce to get

Some examples

Let $v_1 = \begin{bmatrix} 4 \\ 5 \\ -6 \end{bmatrix}$, $v_2 = \begin{bmatrix} -3 \\ -2 \\ 2 \end{bmatrix}$ and $v_3 = \begin{bmatrix} 1 \\ 5 \\ -3 \end{bmatrix}$. Set $b = \begin{bmatrix} -7 \\ -3 \\ 10 \end{bmatrix}$.

(i) How many vectors are in $\{v_1, v_2, v_3\}$?

(ii) How many vectors are in $\mathrm{Span}(v_1, v_2, v_3)$?

(iii) Is $b \in \mathrm{Span}(v_1, v_2, v_3)$?

Ans. (i) 3 (ii) Infinitely many

(iii) Set $A = \begin{bmatrix} v_1 & v_2 & v_3 & b \end{bmatrix}$ and row reduce to get

$$\begin{bmatrix} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix}.$$

Thus, b is a linear combination of v_1, v_2, v_3 (what are the scalars?) and so $b \in \mathrm{Span}(v_1, v_2, v_3)$.

More examples

Let $w_1 = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$, $w_2 = \begin{bmatrix} 7 \\ 3 \end{bmatrix}$ and $w_3 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ and $d = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.
Determine if d is in $\mathrm{Span}(v_1, v_2, v_3)$.

Ans. Form the matrix $\begin{bmatrix} w_1 & w_2 & w_3 & d \end{bmatrix}$ and row-reduce to get

$$\begin{bmatrix} 1 & 0 & -13/23 & 0 \\ 0 & 1 & -1/23 & 0 \end{bmatrix}.$$

The system is consistent and so d is in the span of v_1, v_2 and v_3.

Note!! We didn't need to reduce all the way to RREF! If the question asks only "is d in the span...?", then all we care about is existence of solutions. Thus, row-echelon form is sufficient.

Test topics

- Solving systems of equations with multiple parameters,
- Vector equations,
- Matrix equations,
- Linear combinations,
- Leontief exchange protocol,
- Span.

The format of the test will be mostly the same as the previous test. There will be some skill-testing questions (true/false style) and some long answer.

Definition

A set of vectors $v_1, v_2, \dots, v_n \in \mathbb{R}^m$ is called linearly independent if there is only the trivial solution to the vector equation $c_1 v_1 + c_2 v_2 + \cdots + c_n v_n = 0$. If a set of vectors is not linearly independent, it is linearly dependent.

I often write l.i for "linearly independent" and l.d for "linearly dependent".

You can solve for linear independence in the following way: Let $A = \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix}$, and solve $Ax = 0$. If the only solution is $x = 0$, then the vectors are linearly independent.

If there is a non-trivial solution to a homogeneous solution, then there are free variables!

When are vectors linearly independent?

Theorem. The following are all equivalent:

1. A set of vectors $v_1, v_2, \ldots, v_n \in \mathbb{R}^m$ are linearly independent.
2. The vector equation $c_1 v_1 + c_2 v_2 + \cdots + c_n v_n = 0$ has only the trivial solution.
3. If $A = \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix}$, the matrix equation $Ax = 0$ has only the trivial solution.
4. The matrix A has a pivot in every column.

Linear independence in $\mathbb{R}^2$

There were 3 ways to draw a set of 2 vectors in $\mathbb{R}^2$

1. Both were the 0 vector,
2. They were both on the same line ($u = cv$),
3. Ther were not on the same line ($u \neq cv$).

Let's look at the solutions to $Ax = b$ in the previous cases:

(1) $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, so $Ax = 0$ has

Linear independence in $\mathbb{R}^2$

There were 3 ways to draw a set of 2 vectors in $\mathbb{R}^2$

1. Both were the 0 vector,
2. They were both on the same line ($u = cv$),
3. Ther were not on the same line ($u \neq cv$).

Let's look at the solutions to $Ax = b$ in the previous cases:

(1) $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, so $Ax = 0$ has 2 free variables, no pivots.

(2) $A = \begin{bmatrix} v & cv \end{bmatrix}$ so $Ax = 0$ has

Linear independence in $\mathbb{R}^2$

There were 3 ways to draw a set of 2 vectors in $\mathbb{R}^2$

1. Both were the 0 vector,
2. They were both on the same line ($u = cv$),
3. Ther were not on the same line ($u \neq cv$).

Let's look at the solutions to $Ax = b$ in the previous cases:

(1) $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, so $Ax = 0$ has 2 free variables, no pivots.
(2) $A = \begin{bmatrix} v & cv \end{bmatrix}$ so $Ax = 0$ has a free variable in the solution.
(3) (Example.) $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, so $Ax = 0$ has no free variables (only trivial solution).