Math 1119B: Week 4, Lecture 1

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Table of Contents

Paul Menton Centre Request

The Paul Menton Centre for Students with Disabilities is looking for a Volunteer Notetaker for this class. The criteria for being a Volunteer Notetaker include:

- Attending all assigned classes.
- Taking complete and legible notes.
- Typing up lecture notes and submitting them electronically within 48-hours after each Class (math & other formula heavy notes can be scanned at the Notetaking Office).
- Maintaining confidentiality.

Interested students should email the volunteer notetaking coordinator (`volunteer_notetaking@carleton.ca`) or stop by the Notetaking Office located in 501 University Centre (Paul Menton Centre). Volunteers can request to have their volunteer hours recognized through Carleton Universities Co-curricular Record.

Recap

- Linear combinations.
- Solving equations $c_1 v_1 + c_2 v_2 + c_3 v_3 = b$.
- Using matrices to solve vector equations.
- Some word problems.

Rephrasing how we look at vector equations

In all of the previous exercises, we have taken linear combinations, and written them in terms of vectors.

For example, if $v_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, v_2 = \begin{bmatrix} -2 \\ 0 \\ 3 \end{bmatrix}, v_3 = \begin{bmatrix} 4 \\ 12 \\ -3 \end{bmatrix}$, but now let $b = \begin{bmatrix} 1 \\ 10 \\ 1 \end{bmatrix}$, the equation $c_1 v_1 + c_2 v_2 + c_3 v_3 = b$ had infinitely many solutions.

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Let A be the matrix with v_1, v_2, v_3 as its columns, and let $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$. Compute the matrix $A\mathbf{x}$.

Ans.

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Ans. $x_1 v_1 + x_2 v_2 + x_3 v_3$.

The equation $Ax = b$

In general, if A is an $m \times n$ matrix (with n columns labelled $v_1, v_2, \ldots, v_n$), and $\mathbf{x} = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}^T \in \mathbb{R}^n$, then the product $Ax =$

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For example, let $v_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, v_2 = \begin{bmatrix} -2 \\ 0 \\ 3 \end{bmatrix}, v_3 = \begin{bmatrix} 4 \\ 12 \\ -3 \end{bmatrix}$, form the matrix A and find

$$A \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} =$$

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In general, if A is an $m \times n$ matrix (with n columns labelled $v_1, v_2, \ldots, v_n$), and $\mathbf{x} = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}^T \in \mathbb{R}^n$, then the product $Ax = x_1 v_1 + x_2 v_2 + \cdots + x_n v_n$.

For example, let $v_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, v_2 = \begin{bmatrix} -2 \\ 0 \\ 3 \end{bmatrix}, v_3 = \begin{bmatrix} 4 \\ 12 \\ -3 \end{bmatrix}$, form the matrix A and find

$$A \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 10 \\ 1 \end{bmatrix}.$$

Sound familiar?

Solving $Ax = b$

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Solving the matrix equation $Ax = b$ is the same as solving a system of linear equations by placing the system into the augmented matrix

$$\left[A \quad | \quad b\right]$$

and solving normally.

Remember: in the equation $Ax = b$, we are solving for x, not b!!

A couple of examples

Section 1.4, Exercises 6 Write the following matrix equation as a vector equation, or vice versa.

i)

$$\begin{bmatrix} 2 & -3 \\ 3 & 2 \\ 8 & -5 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} -3 \\ 5 \end{bmatrix} = \begin{bmatrix} -21 \\ 1 \\ -49 \\ 11 \end{bmatrix}.$$

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$$-3 \begin{bmatrix} 2 \\ 3 \\ 8 \\ -2 \end{bmatrix} + 5 \begin{bmatrix} -3 \\ 2 \\ -5 \\ 1 \end{bmatrix} = \begin{bmatrix} -21 \\ 1 \\ -49 \\ 11 \end{bmatrix}.$$

1.4 - Exercise 8

ii)

$$z_1 \begin{bmatrix} 2 \\ -4 \end{bmatrix} + z_2 \begin{bmatrix} -1 \\ 5 \end{bmatrix} + z_3 \begin{bmatrix} -4 \\ 3 \end{bmatrix} + z_4 \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 12 \end{bmatrix}.$$

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ii)

$$z_1 \begin{bmatrix} 2 \\ -4 \end{bmatrix} + z_2 \begin{bmatrix} -1 \\ 5 \end{bmatrix} + z_3 \begin{bmatrix} -4 \\ 3 \end{bmatrix} + z_4 \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 12 \end{bmatrix}.$$

Ans.:

$$\begin{bmatrix} 2 & -1 & -4 & 0 \\ -4 & 5 & 3 & 2 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} = \begin{bmatrix} 5 \\ 12 \end{bmatrix}.$$

Existence of solutions

We know when solutions of systems of linear equations exist by performing Gaussian elimination (row-reducing) an augmented matrix. If there is a pivot in the last column of the augmented matrix, there is no solution.

If there is a solution, then there is a vector

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

such that $Ax = x_1 v_1 + x_2 v_2 + \cdots + x_n v_n = b$, where $v_1, v_2, \ldots, v_n$ are the columns of A.

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such that $Ax = x_1 v_1 + x_2 v_2 + \cdots + x_n v_n = b$, where $v_1, v_2, \ldots, v_n$ are the columns of A.

Theorem. The equation $Ax = b$ has a solution if and only if b is a linear combination of the columns of A.

Existence of solutions of $Ax = b$

Important In the equation $Ax = b$, the matrix A is a coefficient matrix.

To ensure (for any b) the augmented matrix $[A \quad b]$ has a solution, we need a pivot position in every row of A. This is to ensure that there is no pivot position in the final column of the augmented matrix.

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Important In the equation $Ax = b$, the matrix A is a coefficient matrix.

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Theorem. Let A be an $m \times n$ matrix. The following statements are either all true or all false:

1. For each $b \in \mathbb{R}^m$, the equation $Ax = b$ has a solution.
2. Each $b \in \mathbb{R}^m$ is a linear combination of the columns of A.
3. The matrix A has a pivot position in every row.

Examples showing existence of solutions

Let $A = \begin{bmatrix} 1 & -2 & -1 \\ -2 & 2 & 0 \\ 4 & -1 & 3 \end{bmatrix}$ and let $b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$. Show that the equation $Ax = b$ does not have a solution for all possible b, and describe the set of b for which $Ax = b$ does have a solution.

Ans.:

Examples showing existence of solutions

Let $A = \begin{bmatrix} 1 & -2 & -1 \\ -2 & 2 & 0 \\ 4 & -1 & 3 \end{bmatrix}$ and let $b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$. Show that the equation $Ax = b$ does not have a solution for all possible b, and describe the set of b for which $Ax = b$ does have a solution.

Ans.: Row-reduce the augmented matrix to get

$$\begin{bmatrix} 1 & -2 & -1 & b_1 \\ 0 & -2 & -2 & b_2 + 2b_1 \\ 0 & 0 & 0 & 3b_1 + (7/2)b_2 + b_3 \end{bmatrix}.$$

There are pivots in only two rows, and thus there is no solution unless $3b_1 + (7/2)b_2 + b_3 = 0$ (Remember from high school: this is a plane through the origin with norm vector $(6, 7, 2)$.

The equation $Ax = 0$

Definition. A system of linear equations is homogeneous if it can be written in the form $Ax = 0$, where A is an $m \times n$ matrix and 0 is the zero vector in $\mathbb{R}^m$. You can notice homogeneous equations since each equation has 0 as a constant term.

Homogeneous linear systems always have at least one solution

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Homogeneous linear systems always have at least one solution $x = 0_n$, where 0_n is the zero vector in $\mathbb{R}^n$. The zero solution is called the trivial solution.

For any equation $Ax = 0$, the important question is whether there exists a non-trivial solution.

Interesting homogeneous linear systems

- Suppose A has a pivot in every column. Then A is the coefficient matrix of a system of equations with a unique solution.

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- Suppose A has a pivot in every column. Then A is the coefficient matrix of a system of equations with a unique solution.
- The solution set of $Ax = 0$ is then only the zero-vector!
- If A has a column which is a non-pivot column, this corresponds to a free variable.

So,

Theorem. The homogeneous equation $Ax = 0$ has a non-trivial solution if and only if the equation has at least one free variable.

Test 1, Question 1

(a) Place the following system of equations into an augmented matrix and then row reduce to *reduced* row-echelon form. Circle the pivots and indicate the pivot columns in the *reduced* matrix. Indicate whether the system is consistent or inconsistent, and list the number of solutions of the system (0, 1 or infinite).

$$\begin{array}{rcrcrcl} 3x_1 & - & 2x_2 & + & 4x_3 & = & 0 \\ 9x_1 & - & 6x_2 & + & 13x_3 & = & 0 \\ -6x_1 & + & 4x_2 & - & 8x_3 & = & 0 \end{array}$$

(b) If the solution is unique, check your solution in one of the equations. If you have infinitely many solutions, indicate which variables are basic, and which are free. Express your solution as a column vector $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \cdots$.