Math 1119B: Week 3, Lecture 2

David Thomson

September 29, 2011

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Recap

Linear combinations and vector equations

- As promised: applications

The equation $Ax = b$

Schedule of topics

Completed:

- 1.1 - Systems of linear equations.
- 1.2 - Row reduction (Gaussian/Gauss-Jordan elimination) and echelon forms.
- 2.1 - Matrices and matrix operations.
- Current: 1.3 - Vector equations.

Tenative future schedule:

- 1.3-1.5 - Using matrices to solve linear systems, cont'd.
- 1.7, 2.8, 2.9 - Linear independence, span, basis, space/subspace, rank.
- 1.8-1.10 - Linear transformations, difference equations.
- 2.2-2.3 - Invertible matrices, the invertible matrix theorem.
- 2.6 - Leontief I-O model.
- 3.1-3.3 - Determinants, Cramer's rule.
- 4.9 - Markov chains.

Some things to keep in mind.

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5. Ramping up the speed...

Last class

- Rules of adding/subtracing matrices.
- Scalar multiplication of matrices.
- Special matrices:
  - All-zero matrix,
  - Identity matrix,
  - Transpose matrix (OC Transpose theorem).
- Solving for linear combinations.

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- Solving for linear combinations.

- I rushed through the final example on multiple parameters.
  - ...see your tutorial on Monday

Recall

Definition. Let $c_1, c_2, \ldots, c_p$ be scalars and let $v_1, v_2, \ldots, v_p$ be vectors in $\mathbb{R}^n$. A linear combination of $v_1, v_2, \ldots, v_p$ is any sum of the form

$$y = c_1 v_1 + c_2 v_2 + \cdots + c_p v_p = \sum_{i=1}^{p} c_i v_i.$$

Any combination of vectors in $\mathbb{R}^n$ is again just a vector in $\mathbb{R}^n$. The scalars $c_1, c_2, \ldots, c_p$ in the combination are called the weights of the linear combination.

An example of solving for linear combinations

Let $v_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$, $v_2 = \begin{bmatrix} -2 \\ 0 \\ 3 \end{bmatrix}$, $v_3 = \begin{bmatrix} 4 \\ 12 \\ -3 \end{bmatrix}$ and $b = \begin{bmatrix} 1 \\ -3 \\ 12 \end{bmatrix}$. Are there weights c_1, c_2, c_3 such that b is a linear combination of v_1, v_2, v_3?

Solution.

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$$\begin{bmatrix} 1 & 0 & 6 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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the system is inconsistent and thus there is no linear combination.

A similar example

Let $v_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$, $v_2 = \begin{bmatrix} -2 \\ 0 \\ 3 \end{bmatrix}$, $v_3 = \begin{bmatrix} 4 \\ 12 \\ -3 \end{bmatrix}$, but now let $b = [1, 10, 1]^T$. Are there weights c_1, c_2, c_3 such that b is a linear combination of v_1, v_2, v_3?

Solution. Form the matrix $A = [v_1 \ v_2 \ v_3 \mid b]$ and row-reduce to get

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Let $v_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, v_2 = \begin{bmatrix} -2 \\ 0 \\ 3 \end{bmatrix}, v_3 = \begin{bmatrix} 4 \\ 12 \\ -3 \end{bmatrix}$, but now let $b = [1, 10, 1]^T$. Are there weights c_1, c_2, c_3 such that b is a linear combination of v_1, v_2, v_3?

Solution. Form the matrix $A = [v_1 \;\; v_2 \;\; v_3 \mid b]$ and row-reduce to get

$$\begin{bmatrix} 1 & 0 & 6 & 5 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

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the system is consistent with infinitely many solutions.

The general theorem for vector equations

Theorem. A vector equation

$$x_1\mathbf{v_1} + x_2\mathbf{v_2} + \cdots + x_n\mathbf{v_n} = \mathbf{b}$$

has the same solution set as the linear system whose augmented matrix is

$$\begin{bmatrix}\mathbf{v_1} & \mathbf{v_2} & \cdots & \mathbf{v_n} & \mathbf{b}\end{bmatrix}.$$

In particular, b can be generated as a linear combination of the v_i if and only if there exists a solution to the linear system.

A delicious example

The Clocktower Pub in Ottawa contains their own microbrewery. For each dollar of microbrew they produce, they spend 50 cents on materials, 15 cents on overhead and 5 cents on labour.

For each dollar of jambalaya they produce, they spend 35 cents on materials, 25 cents on overhead, and 15 cents on labour.

Place in two vectors b and j, the cost of manufacture, per dollar,

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Place in two vectors b and j, the cost of manufacture, per dollar, of each of beer and jambalaya: $b = \begin{bmatrix} .50 \\ .15 \\ .05 \end{bmatrix}$, and $j = \begin{bmatrix} .35 \\ .25 \\ .15 \end{bmatrix}$.

(a) If the Clocktower wants to manufacture \$14 of jambalaya and \$6.50 of beer (a decent dinner), give a vector with a break-down of their total costs.

A little more dinner...

Recall: Vectors of manufacture, $\begin{bmatrix} \text{materials} \\ \text{overhead} \\ \text{labour} \end{bmatrix}$ per dollar, of each of beer and jambalaya: $b = \begin{bmatrix} .50 \\ .15 \\ .05 \end{bmatrix}$, and $j = \begin{bmatrix} .35 \\ .25 \\ .15 \end{bmatrix}$.

(b) Suppose the Clocktower wants to provide a \$20 coupon and make 20% profit. They want to split the costs in a 50/20/30 split. How many dollars of beer and jambalaya should they produce per customer?

Solution. To get a 20% profit, the total cost of manufacture should be

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(b) Suppose the Clocktower wants to provide a \$20 coupon and make 20% profit. They want to split the costs in a 50/20/30 split. How many dollars of beer and jambalaya should they produce per customer?

Solution. To get a 20% profit, the total cost of manufacture should be $(1 - 0.2)\$20 = \$16$, materials cost is $1/2(\$16) = \$8$, overhead is $1/5(\$16) = \$3.20$ and labour cost is $3/10(\$16) = \$4.80$.

Solve

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Solve $x_1 b + x_2 j = \begin{bmatrix} 8 \\ 3.20 \\ 4.80 \end{bmatrix}$.

Rephrasing how we look at vector equations

In all of the previous exercises, we have taken linear combinations, and written them in terms of vectors.

For example, if $v_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$, $v_2 = \begin{bmatrix} -2 \\ 0 \\ 3 \end{bmatrix}$, $v_3 = \begin{bmatrix} 4 \\ 12 \\ -3 \end{bmatrix}$, but now let $b = \begin{bmatrix} 1 \\ 10 \\ 1 \end{bmatrix}$, the equation $c_1 v_1 + c_2 v_2 + c_3 v_3 = b$ had infinitely many solutions.

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Let A be the matrix with v_1, v_2, v_3 as its columns, and let $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$. Compute the matrix $A\mathbf{x}$.

Ans.

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Ans. $x_1 v_1 + x_2 v_2 + x_3 v_3$.