CONCORDIA UNIVERSITY

DEPARTMENT OF COMPUTER SCIENCE AND SOFTWARE ENGINEERING

COMP232 MATHEMATICS FOR COMPUTER SCIENCE

Solutions to ASSIGNMENT 1 FALL 2013

1. (Not marked) For each of the following statements use a truth table to determine whether it is a tautology, a contradiction, or a contingency.
   (a) $((p \wedge r) \wedge (q \vee r)) \rightarrow ((p \wedge q) \vee r)$
   Soln. Tautology.
   (b) $(p \oplus q) \wedge (p \oplus \neg q)$
   Soln. Contradiction.
   (c) $(p \rightarrow q) \vee (\neg p \rightarrow q)$
   Soln. Tautology.
   (d) $((p \vee r) \wedge (p \rightarrow q) \wedge (r \rightarrow q)) \rightarrow q$
   Soln. Tautology.

2. (4 marks) Prove the following logical equivalences *without* using truth tables.
   (a) $\neg(p \rightarrow q) \equiv \neg q \wedge (p \vee q)$
   Soln. $\neg q \wedge (p \vee q)$
   $\equiv (\neg q \wedge p) \vee (\neg q \wedge q)$ distributive law
   $\equiv (\neg q \wedge p) \vee F$ negation law
   $\equiv (\neg q \wedge p)$ identity law
   $\equiv \neg(q \vee \neg p)$ de Morgan's law
   $\equiv \neg(p \rightarrow q)$ disjunctive form of the conditional
   (b) $(p \vee q) \rightarrow r \equiv (p \rightarrow r) \wedge (q \rightarrow r)$
   Soln. $(p \vee q) \rightarrow r$
   $\equiv \neg(p \vee q) \vee r$ disjunctive form of the conditional
   $\equiv (\neg p \wedge \neg q) \vee r$ de Morgan's law
   $\equiv (\neg p \vee r) \wedge (\neg q \vee r)$ distributive law
   $\equiv (p \rightarrow r) \wedge (q \rightarrow r)$ disjunctive form of the conditional

3. (12 marks) For each of the statements below, write it down in the form "if p then q", then write down the converse statement, and finally the contrapositive.
   (a) A positive integer is a prime only if it has no divisors other than 1 and itself.
   Soln. If a positive integer is a prime, then it has no divisors other than 1 and itself.
   Converse: If a positive integer has no divisors other than 1 and itself, then it is prime.
   Contrapositive: If a positive integer has divisors other than 1 and itself, then it is not prime.
   (b) To get an A in this class, it is necessary to do all the assignments.
   Soln. If you want to get an A in this class, then you have to do all the assignments.
   Converse: If you do all the assignments, then you will get an A in this class.
   Contrapositive: If you don't do all the assignments, then you cannot get an A in this class.

(c) Being born in Canada is a sufficient condition for Canadian citizenship.
Soln. If you are born in Canada, then you can get Canadian citizenship.
Converse: If you can get Canadian citizenship, then you were born in Canada.
Contrapositive: If you cannot get Canadian citizenship, then you were not born in Canada.

(d) You will reach the summit unless you begin your climb too late.
Soln. If you do not begin your climb too late, then you will reach the summit.
Converse: If you reached the summit, then you did not begin your climb too late.
Contrapostitive: If you did not reach the summit, then you began your climb too late.

4. (Not marked) This is a problem about an island in which the inhabitants are all either knights or knaves. Knights always tell the truth and knaves always lie. According to this old problem, three of the inhabitants of this island - A, B, and C - were standing together in a garden. A stranger passed by and asked A, "How many knights are among you?" A answered, but rather indistinctly, so the stranger could not make out what she said. The stranger then asked B, " What did A say?" B replied, "A said that there is one knight among us." At this point, the third person C said, "Don't believe B; he is lying."

   The question is: What are B and C, knights or knaves? Explain your answer.

   Soln:: First we observe that there is exactly one knight between B and C. If C is a knight, then he speaks the truth, which means that B was lying. If B was lying, B must be a knave. If instead C is a knave, then C is lying, which means B was telling the truth, which means B is a knight.

   Let S denote the statement "There is one knight among A, B, and C." Next, we argue that regardless of whether she is a knight or a knave, A could not have said S. Suppose A was a knight. Then together with the one knight among B and C stipulated in the previous paragraph, there are two knights between A, B, and C. Thus, saying S would be a lie, and since A is a knight by assumption, he could not have said S. Suppose instead that A was a knave. Then there is exactly one knight between A, B, and C; in other words, S is true. Since A must lie, A could not have said S.

   Finally, since A could not have said S as claimed by B, it follows that B was lying, and is therefore a knave. Also, C was telling the truth, and is therefore a knight. The conclusion is: B is a knave and C is a knight.

5. (8 marks) Write the following statements in predicate form, using logical operators $\wedge$, $\vee$, $\neg$, quantifiers $\forall$, $\exists$, and arithmetic operators. Assume the domain consists of all integers.

   (a) The square of a positive integer is always bigger than the integer.
   Soln. $\forall x \ x > 0 \rightarrow x^2 > x$

   (b) There is no integer solution to the equation $x = x + 1$.
   Soln. $\forall x \ x \neq x + 1$

   (c) The absolute value of an integer is not necessarily positive.
   Soln. $\exists x \ |x| \leq 0$

   (d) The absolute value of the sum of two integers does not exceed the sum of the absolute values of those integers.
   Soln. $\forall x \, \forall y \ |x + y| \leq |x| + |y|$

6. (8 marks) Let $f(x, y, z) = x^2 - y + z^3$, where $x, y, z \in Z$. For each of the following determine its truth value. Justify your answers.

   (a) $\exists x, y, z$: $f(x, y, z) = 0$
   Soln. True, take $x = y = z = 0$.

   (b) $\forall y \exists x, z$: $f(x, y, z) > 0$
   Soln. True, for any y, take $z = 1, x = y$.

   (c) $\forall y, z \exists x$: $f(x, y, z) < 0$
   Soln. False. Take $y = z = 0$. Then there is no $x \in Z$ such that $x^2 < y - z^3$.

   (d) $\exists z, x \forall y$: $f(x, y, z) \neq 0$
   Soln. False. For any choice of x, z, $f(x, y, z) = 0$ when $y = x^2 + z^3$.

7. (8 marks) Let the domain for x be the set of all students in this class and the domain for y be the set of all countries in the world. Let $P(x, y)$ denote "student x has visited country y" and $Q(x, y)$ denote "student x has a friend in country y." Express each of the following using logical operations and quantifiers, and the propositional functions $P(x, y)$ and $Q(x, y)$.

   (a) Carlos has visited Bulgaria.
   Soln. $P(\textit{Carlos, Bulgaria})$

   (b) Every student in this class has visited the United States.
   Soln. $\forall x \ P(x, \textit{United States})$

   (c) Every student in this class has visited some country in the world.
   Soln. $\forall x \, \exists y \ P(x, y)$

   (d) There is no country that every student in this class has visited.
   Soln. $\forall y \, \exists x \ \neg P(x, y)$

   (e) There are two students in this class, who between them, have a friend in every country in the world.
   Soln. $\exists x \, \exists y \ x \neq y \wedge \forall z \ Q(x, z) \vee Q(y, z)$

   (f) Nobody in this class has visited a country in which they did not have a friend.
   Soln. $\forall x \, \forall y \ P(x, y) \rightarrow Q(x, y)$

   (g) There is a country in which nobody in this class has a friend.
   Soln. $\exists y \, \forall x \ \neg Q(x, y)$

   (h) Everyone in this class has visited every country in which they have a friend.
   Soln. $\forall x \, \forall y \ Q(x, y) \rightarrow P(x, y)$

8. (Not marked) For each of the following equivalences, determine if it is valid for all predicates P and Q. If yes then give a full explanation. If not then provide a counterexample.

   (a) $\forall x(P(x) \vee Q(x)) \ \equiv \ \forall x P(x) \vee \forall x Q(x)$
   Soln. False. Let the universe of discourse be $\{a, b\}$, with $P(a) = Q(b) = \textit{True}$ while $P(b) = Q(a) = \textit{False}$. Then $\forall x(P(x) \vee Q(x))$ is true while $\forall x P(x) \vee \forall x Q(x)$ is false.

(b) $\forall x(P(x) \wedge Q(x)) \equiv \forall x P(x) \wedge \forall x Q(x)$

Soln. True. If $\forall x(P(x) \wedge Q(x))$ is true, then for every x in the domain both $P(x)$ and $Q(x)$ are true. That is, $P(x)$ is true for every x and $Q(x)$ is true for every x. Therefore, $\forall x P(x) \wedge \forall x Q(x)$ is true. On the other hand, if $\forall x P(x) \wedge \forall x Q(x)$ is true, then $P(x)$ is true for every x in the domain, and $Q(x)$ is also true for every x in the domain. Therefore $\forall x(P(x) \wedge Q(x))$ is true.

(c) $\exists x, y(P(x) \wedge Q(y)) \equiv \exists x P(x) \wedge \exists x Q(x)$

Soln. True. Suppose $\exists x, y(P(x) \wedge Q(y))$ is true. Then $P(a)$ and $Q(b)$ are true for some a, b in the universe of discourse. It follows that $\exists x P(x)$ is true and $\exists x Q(x)$ is true. Therefore, the RHS is true. Conversely, suppose $\exists x P(x) \wedge \exists x Q(y)$ is true. Then $\exists x P(x)$ is true and also $\exists x Q(x)$ is true. Since $\exists x P(x)$ is true, we know $P(a)$ is true for some a, and similarly, since $\exists x Q(x)$ is true, we know that $Q(b)$ is true for some b. Therefore $P(a) \wedge Q(b)$ is true. It follows that $\exists x, y(P(x) \wedge Q(y))$ is true.

(d) $\exists x, y(P(x) \vee Q(y)) \equiv \exists x P(x) \vee \exists y Q(y)$

Soln. True. Suppose $\exists x P(x) \vee \exists y Q(y)$ is true. Then either $\exists x P(x)$ is true or $\exists y Q(y)$ is true. In the first case, $P(a)$ is true for some a. Then $P(a) \vee Q(y)$ is true for every y in the universe of discourse. Therefore $\exists x, y(P(x) \vee Q(y))$ is true. A similar argument for the second case (when $\exists y Q(y)$ is true). Conversely, suppose $\exists x P(x) \vee \exists y Q(y)$ is false. Then both $\exists x P(x)$ and $\exists x Q(y)$ are false. That is $P(x)$ and $Q(x)$ are both false for every x in the universe of discourse. It follows that $\exists x, y(P(x) \vee Q(y)$ is false.