Math 1119B: Week 3, Lecture 1

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Table of Contents

Last class

- Introduction to vectors and matrices:
  - Addition and subtraction of matrices.
  - Multiplication of a matrix by a scalar.
  - Multiplication of two matrices.
  - Matrix multiplication is not commutative.
- Brief test review.

An omission on my part.

Until now, I have simply referred to row-reducing as the row-reduction algorithm. However, there is a proper name for this (and I will specify this more throughout the term. Row reduction to echelon form is called Gaussian elimination, and further reduction to reduced echelon form (RREF) is called Gauss-Jordan elimination.

This terminology did not show up on your test, but may be on the final exam.

Rules of addition and scaling matrices

Theorem. Let A, B and C be matrices of the same size, and let r and s be scalars (i.e., $r \in \mathbb{R}$ and $s \in \mathbb{R}$).

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5. $(r + s)A = rA + sA$

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6. $r(sA) = (rs)A$ (order of scaling doesn't matter).

Rules of multiplying matrices

Theorem. Let A, B, C be matrices and let r be any scalar. Suppose that A, B, C all have appropriate size so that the following multiplications are defined. Then,

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The all-zero matrix

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Similarly, let Z_2 be a $q \times m$ matrix. Z_2A is a $q \times m$ matrix with every entry equal to zero.

We denote the $m \times m$ all-zero matrix by 0_m.

The identity matrix

Definition. Let I_n be the $n \times n$ matrix with 1s on the diagonal and 0s elsewhere, for example

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The matrix I_n is called the $n \times n$ identity matrix.

Multiplying the identity matrix

Example. Let $A = \begin{bmatrix} 1 & 1 & 2 \\ -1 & -3 & 4 \\ 0 & 1 & -2 \end{bmatrix}$. Let's compute $I_3 A$:

$$I_3 A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ -1 & -3 & 4 \\ 0 & 1 & -2 \end{bmatrix} =$$

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An easy identity

If A is any $n \times n$ matrix, we have the following crucial identity:

$$AI_n = I_nA = A.$$

Let $A = \begin{bmatrix} 1 & 1 & 2 \\ -1 & -3 & 4 \\ 0 & 1 & -2 \end{bmatrix}$ and let $B = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$. Compute AB and BA (i) directly and (ii) using rules of matrices.

Ans.

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$$\begin{bmatrix} 3 & 3 & 6 \\ -3 & -9 & 12 \\ 0 & 3 & -6 \end{bmatrix}.$$

The transpose of a matrix

Definition. Given any $m \times n$ matrix A, the transpose of A, denoted by A^T, is given by writing the rows of A as the columns of A^T.

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Theorem. Let A and B be matrices of the same size. Then,

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2. $(A+B)^T = A^T + B^T$
3. for any scalar r, $(rA)^T = rA^T$

Theorem.

se

Let O be an $m \times n$ matrix and let C be an $n \times p$ matrix (so that the product OC is defined). Then $(OC)^T =$

Theorem.

OC Transpose

Let O be an $m \times n$ matrix and let C be an $n \times p$ matrix (so that the product OC is defined). Then $(OC)^T = C^T O^T$.

A preview of later work

Example. Let $X = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ and let $Y = \frac{-1}{3}\begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix}$. Compute XY and YX.

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Ans. I_2.

Definition. If A is a $n \times n$ matrix, and B is an $n \times n$ matrix such that

$$AB = BA = I_n,$$

then we say B is the inverse of A, and we often denote $B = A^{-1}$.

Key points.

- Not every matrix has an inverse (think O_m!).
- Determining when a matrix has an inverse, and solving for the inverse will be most of the content of chapter 2 and 3.

Remember: (column) vectors are $n \times 1$ matrices!

Definition. A (column) vector in $\mathbb{R}^n$ is given by an $n \times 1$ matrix (a column with n entries).

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} \in \mathbb{R}^2, \quad \begin{bmatrix} 3 \\ 0 \\ 5 \end{bmatrix} \in \mathbb{R}^3, \quad \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_s \end{bmatrix} \in \mathbb{R}^s.$$

- Geometrically, $\mathbb{R}^1 = \mathbb{R}$ is simply the real number line. We can think of $\mathbb{R}^2$ as a plane (for example, the blackboard extended in all directions) and $\mathbb{R}^3$ as being the 3-dimensional universe.

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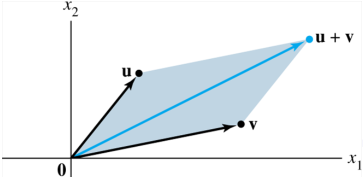
- Geometrically, we can add vectors like the following (but we will not focus on that in this course).



Life remains easy...

- All the rules of working with matrices hold when working with vectors.
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If $u = \begin{bmatrix} 3 \\ 1 \\ -1 \\ 0 \end{bmatrix}$ and $v = \begin{bmatrix} 2 \\ 0 \\ -1 \\ 1 \end{bmatrix}$, then

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If $u = \begin{bmatrix} 3 \\ 1 \\ -1 \\ 0 \end{bmatrix}$ and $v = \begin{bmatrix} 2 \\ 0 \\ -1 \\ 1 \end{bmatrix}$, then

$$3u - 2v = \begin{bmatrix} 3(3) - 2(2) \\ 3(1) - 2(0) \\ 3(-1) - 2(-1) \\ 3(0) - 2(1) \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \\ -1 \\ -2 \end{bmatrix}.$$

Since vectors are of size $n \times 1$, for some n, multiplying vectors is not defined for vectors larger then 1×1.

Linear combinations

Definition. Given vectors $v_1, v_2, \ldots, v_p \in \mathbb{R}^n$, and scalars $c_1, c_2, \ldots, c_p$, a linear combination of $v_1, v_2, \ldots, v_p$ with weights $c_1, c_2, \ldots, c_p$ is given by

$$y = c_1 v_1 + c_2 v_2 + \cdots + c_p v_p = \sum_{i=1}^{p} c_i v_i .$$

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The vector y is an element of $\mathbb{R}^n$.

The scalars in a linear combination can be any number including zero.

The equation $Ax = b$

Example. Let $v_1 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$, $v_2 = \begin{bmatrix} 2 \\ 0 \\ -3 \end{bmatrix}$ and $b = \begin{bmatrix} 4 \\ -4 \\ -1 \end{bmatrix}$. Determine if there exist scalars c_1 and c_2 such that $c_1 v_1 + c_2 v_2 = b$.

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This reduces to the equation:

$$c_1 v_1 + c_2 v_2 = \begin{bmatrix} c_1 \\ -2c_1 \\ 1c_1 \end{bmatrix} + \begin{bmatrix} 2c_2 \\ 0 \\ -3c_2 \end{bmatrix} = \begin{bmatrix} 4 \\ -4 \\ -1 \end{bmatrix}$$

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Re-write the system of equations into the augmented matrix

$$\left[\begin{array}{cc|c} 1 & 2 & 4 \\ -2 & 0 & -4 \\ 1 & -3 & -1 \end{array}\right]$$

and solve to get

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and solve to get $c_1 = 2, c_2 = 1$.

Multiple parameters

Let

$$\begin{bmatrix} 1 & -2 & -1 & 3 & 0 \\ -2 & 4 & 5 & -8 & 3 \\ 3 & -6 & -6 & 10 & -3 \end{bmatrix}$$

be the augmented matrix of a system of linear equations. Find the solution to the system.

Ans. On the tutorial.