Math 1119B: Week 2, Lecture 2

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September 21, 2011

Table of Contents

Last class

- Row reduction into:
  - row-echelon form;
  - reduced echelon form.

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- Row reduction into:
  - row-echelon form;
  - reduced echelon form.
- Comparison of row-echelon and reduced row-echelon forms.
- Parametric solutions of equations (infinite solutions)
- The row-reduction algorithm.
- Tall and wide examples
- Pivots.

The example explaining pivots

Row reduce the matrix A below to echelon form, and find the pivot positions.

$$A = \begin{bmatrix} 0 & -6 & -12 & 8 & 18 \\ -1 & -2 & -1 & 3 & 1 \\ -3 & -5 & -1 & 6 & 0 \\ 0 & 2 & 4 & -6 & -6 \end{bmatrix}$$

Ans.

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Ans.

$$\begin{bmatrix} \color{red}{1} & 4 & 5 & -9 & -7 \\ 0 & \color{red}{2} & 4 & -6 & -6 \\ 0 & 0 & 0 & \color{red}{-5} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Why do we care about pivots?

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2. A pivot allows us to further reduce the matrix into RREF (i.e., we can remove all entries in the same column in the rows above the pivot).
3. The variables corresponding to pivot columns are the basic variables.
4. The variables in non-pivot column are the free variables.
5. If there is a pivot in the final column of an augmented matrix, the corresponding system of equations has no solution.

Working with matrices

We have seen matrices in the context of systems of equations, but we can manipulate matrices themselves.

Some matrix operations

1. addition and subtraction

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Some matrix operations

1. addition and subtraction
2. multiplication by a scalar
3. multiplying matrices together

Addition of matrices

- Addition and subtraction of matrices is easy.
- Simply add/subtract the matrices term-by-term.
  - You can only add/subtract matrices of the same size!!!!!!!

Example Suppose $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} 7 & 8 & 9 \\ 10 & 11 & 12 \end{bmatrix}$, then

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$$A - B = \begin{bmatrix} -6 & -6 & -6 \\ -6 & -6 & -6 \end{bmatrix}.$$

Another matrix additon example

Suppose $A = \begin{bmatrix} 1 & 1 \\ 2 & 4 \\ 5 & 12 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 2 & 3 \\ 4 & 7 & 9 \\ 13 & 8 & 11 \\ 0 & 0 & 1 \end{bmatrix}$.

Then,

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Another matrix additon example

Suppose $A = \begin{bmatrix} 1 & 1 \\ 2 & 4 \\ 5 & 12 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 2 & 3 \\ 4 & 7 & 9 \\ 13 & 8 & 11 \\ 0 & 0 & 1 \end{bmatrix}$.

Then,

$$A + B =$$

NOTHING. The matrices A and B have different sizes, and so addition/subtraction doesn't mean anything!

Scalar multiples

- Taking scalar multiples of matrices is even easier!!
- A scalar times a matrix is simply scaling the entries.
- You can multiply any scalar times any matrix of any size

Examples.

Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 3 \end{bmatrix}, B = \begin{bmatrix} 5 \end{bmatrix}, C = \begin{bmatrix} 2 \\ 4 \\ 5 \end{bmatrix}, D = \begin{bmatrix} 2 & 4 & 5 \end{bmatrix}$.

What is $2A, \quad 0B, \quad 3B, \quad 5C, \quad 1/2D$?

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What is $2A$, $0B$, $3B$, $5C$, $1/2D$?

Skill testing question: What is the size of each matrix? If E is an $m \times n$ matrix, and c is any number, what is the size of cE?

Vectors are just matrices!

Look at the (column) vector $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$.

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Look at the (column) vector $\begin{bmatrix}1\\2\\3\\4\end{bmatrix}$. This (column) vector is just a 4×1 matrix.

Then addition/subtraction/scaling of vectors is the exact same thing as addition/subtraction/scaling of matrices.

- We can only add/subtract vectors of the same size.
- We can scale any vector of any size by any number.

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Motivation - put down your pencils for a second. We know that matrices can represent systems of linear equations. This is not the only way to look at matrices.

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Another application of matrices (we will see this in a couple of weeks) is a transformation/mapping/function.

Using the idea of a matrix as a function, it takes vectors and maps them to other vectors. The idea is: if B is a matrix and x is a vector, then $B \circ x \to (Bx)$, where (Bx) is another vector. If A is another matrix, $A \circ (Bx) \to (AB)x$

Matrix multiplication done right.

Matrix multiplication may look strange at first, but it is designed to make sense under the last slide's notion of "composition of mappings".

Pick up your pens again.

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- Matrix multiplication is performed in a special way, and is only valid on certain sizes of matrices

Let A be an $m \times n$ matrix, and let B be an $n \times p$ matrix. Then the product of A and B has size like:

$$A_{m \times n} \cdot B_{n \times p} = (AB)_{m \times p}.$$

So how do I do it?

Matrices are multiplied "row-by-column". Let's look at an easy example.

Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ and let $B = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$.

First. Check the sizes: A is 3×3, B is 3×1, and so matrix multiplication is defined.

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Row-by-column. In the $(1, 1)$ position, we take Row 1 of A by Column 1 of B like:

$$\begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = 1 \cdot 1 + 1 \cdot 2 + 1 \cdot 2 = 5.$$

Keep going...

Finish it off,

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ \\ \end{bmatrix}$$

Keep going...

Finish it off,

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \\ 3 \end{bmatrix}$$

In general, in the (i,j) position of AB we take Row i of A "multiplied" by Column j of B.

Don't worry. It gets easy with practice. A few more examples.

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 4 & 6 \\ 2 & 3 & 5 \end{bmatrix}$$

Find AB, AC, BC.

An unfortunate consequence

Take the matrices B and C from the previous slide:

$$B = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 4 & 6 \\ 2 & 3 & 5 \end{bmatrix}.$$

The product BC does not exist. What about CB?

Ans. $CB = \begin{bmatrix} 27 \\ 23 \end{bmatrix}$.

Try the same thing with $D = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}, E = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

Ans.

$$DE =$$

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$$DE = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}, \quad ED = \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix}.$$

In general, matrix multiplication is not commutative!

What have we done so far?

You tell me.

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1. Systems of linear equations.
2. Expressing systems of linear equations as matrices.
    - Coefficient matrices.
    - Augmented matrices.
3. What are solutions of systems of linear equations?
4. Types of solutions of linear equations:
    - Unique solutions (consistent)
    - Infinite Solutions (consistent)
    - No solutions (inconsistent)
5. Row-echelon form.
6. Reduced row-echelon form.
7. Existence and uniqueness of solutions.
8. Expressing parametric solutions.
9. Pivots.