Math 1119B: Week 2, Lecture 1

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- Row-reduction is quicker than elimination.
- Row-reducing "square" systems using the three valid row-operations:
  - Interchange,
  - scaling,
  - combination.
- Existence and uniqueness of solutions.

Introduction to parametric solutions

- On Tutorial 1, I gave two "advanced" problems. The first one said:

0. This is an *augmented* matrix of a system of linear equations that is already fully reduced:

$$\begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & 4 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

i. What is the size of the augmented matrix? The coefficient matrix?

ii. How many equations does the system contain? How many unknowns/variables?

iii. Convince yourself, as a group, why the system has an infinite number of solutions.

Advanced: Can you tell me what all of the solutions look like?

Introduction to non-"square" systems

For which h and k does the following system have:

i. No solution?

ii. Infinite solutions?

iii. Unique solution?

$$\begin{bmatrix} 1 & 2 & 4 \\ 0 & 2 & 6 \\ 0 & 2h & 3k \end{bmatrix}$$

Ans.:

Introduction to non-“square” systems

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Ans.: (i.) $k \neq 2h$

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Ans.: (i.) $k \neq 2h$ (ii.) Never (iii.) $k = 2k$.

Row-echelon form

- Here, we formalize some of the techniques we used to reduce matrices from the previous lectures.

Definition. A matrix is in echelon form (or row-echelon form) if

1. All non-zero rows are above any zero rows

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1. All non-zero rows are above any zero rows
2. Each leading entry of a row is in a column to the right of the leading entry of the row above it.
3. All entries in a column below a leading entry are zeroes.

I tend to say that matrices in echelon form have the staircase property.

An example of an echelon matrix

Recall from last class:

$$\begin{array}{rcrcrcr} x_1 & & & - & 7x_3 & = & 8 \\ -4x_1 & + & 7x_2 & + & x_3 & = & -1 \\ -2x_1 & + & x_2 & + & 11x_3 & = & -9 \end{array}$$

Ans.: Put the system into a matrix and row-reduce to get:

$$\begin{bmatrix} 1 & 0 & -7 & 8 \\ 0 & 1 & -3 & 7 \\ 0 & 0 & -6 & -18 \end{bmatrix}$$

Check the points:

1. Any zero rows?

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3. Are all entries in a column below the leading entry zero? Yes.

Thus, this is in row-echelon form.

A wide example

Reduce the matrix

$$\begin{bmatrix} 0 & 1 & 1 & 4 & 3 \\ 1 & -1 & -3 & 8 & 1 \\ 0 & 1 & 0 & 2 & 5 \end{bmatrix}$$

into row-echelon form.

Ans.

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Ans.

$$\begin{bmatrix} 1 & -1 & -3 & 8 & 1 \\ 0 & 1 & 1 & 4 & 3 \\ 0 & 0 & -1 & -2 & 2 \end{bmatrix}$$

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into row echelon form.

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into row echelon form.

Ans.

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & 1/2 \\ 0 & 0 & 0 \end{bmatrix}$$

Reduced row-echelon form

Definition. A matrix is in reduced echelon form (or reduced row-echelon form or RREF) if

1. It is is row-echelon form:
    1.1 All non-zero rows are above any zero rows

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   1.2 Each leading entry of a row is in a column to the right of the leading entry of the row above it.
   1.3 All entries in a column below a leading entry are zeroes.
2. The leading entry in each nonzero row is 1.
3. Each leading 1 is the only nonzero entry in its column.

Row-echelon	Reduced row-echelon
non-unique	unique
type of solutions	precise solutions (if any)

Back to the tall example

The matrix

$$\begin{bmatrix} 0 & 1 & 2 \\ 0 & 2 & 3 \\ 1 & 0 & 1 \\ 2 & 0 & 3 \end{bmatrix}$$

has row echelon form

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & 1/2 \\ 0 & 0 & 0 \end{bmatrix}.$$

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Finish reducing into RREF.

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has row-echelon form

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Finish reducing into RREF.

Ans.

$$\begin{bmatrix} 1 & 0 & 0 & 16 & 0 \\ 0 & 1 & 0 & 2 & 5 \\ 0 & 0 & 1 & 2 & -2 \end{bmatrix}$$

Giving the wide example a solution

Suppose the matrix

$$\begin{bmatrix} 1 & 0 & 0 & 16 & 0 \\ 0 & 1 & 0 & 2 & 5 \\ 0 & 0 & 1 & 2 & -2 \end{bmatrix}$$

in RREF is the augmented matrix of a system of linear equations. We can re-write the system of equations as:

$$\begin{array}{ccccccc} x_1 & & & + & 16x_4 & = & 0 \\ & x_2 & & + & 2x_4 & = & 5 \\ & & x_3 & + & 2x_4 & = & -2 \end{array}$$

All of x_1, x_2, x_3 depend on x_4, and there is no constraint on x_4.

The variable x_4 is called a free variable. There is an infinite number of solutions because x_4 can take any value.

Expressing parametric solutions

We have the system

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Thus, we have

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Thus, we have

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -16x_4 \\ 5 - 2x_4 \\ -2 - 2x_4 \\ x_4 \end{bmatrix} =$$

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Thus, we have

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -16x_4 \\ 5-2x_4 \\ -2-2x_4 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \\ -2 \\ 0 \end{bmatrix} + \begin{bmatrix} -16x_4 \\ -2x_4 \\ -2x_4 \\ x_4 \end{bmatrix}.$$

Finishing off the example

Since we have

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \\ -2 \\ 0 \end{bmatrix} + \begin{bmatrix} -16x_4 \\ -2x_4 \\ -2x_4 \\ x_4 \end{bmatrix},$$

we know that x_4 is free, so let $x_4 = t \in \mathbb{R}$ and factor it out to get the solution

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \\ -2 \\ 0 \end{bmatrix} + t \cdot \begin{bmatrix} -16 \\ -2 \\ -2 \\ 1 \end{bmatrix}.$$

A twel.. er, five-step process

We have a five step process to solve any linear system of equations now

1. Write the augmented matrix of the system.

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   2.1 If the system if inconsistent, break.

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   2.1 If the system if inconsistent, break.
   2.2 If the system is consistent, continue.

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We have a five step process to solve any linear system of equations now

1. Write the augmented matrix of the system.
2. Row-reduce the matrix to obtain an equivalent augmented matrix in echelon form.
   2.1 If the system if inconsistent, break.
   2.2 If the system is consistent, continue.
3. Continue row reduction to obtain the reduced echelon form.
4. Write the system of equations corresponding to the matrix in Step 3.
5. Rewrite each nonzero equation so one basic variable is expressed in terms of free variables in the equation.

Pivot!!

Giving a formal name to leading ones

- Once a matrix is in echelon form (there may be many echelon forms!), any further row operations do not change the position of the leading entries.
- Since the reduced echelon form is unique, the leading entries are always in the same position in any echelon form.
- These leading entries correspond to the leading 1s in the reduced echelon form.

Definition. A *pivot* position in a matrix A is a location in A that corresponds to a leading 1 in the reduced echelon form of A. A *pivot column* is a column of A that contains a pivot position (similarly for a pivot row).

An beautiful example explaining pivots

Row reduce the matrix A below to echelon form, and find the pivot positions.

$$A = \begin{bmatrix} 0 & -6 & -12 & 8 & 18 \\ -1 & -2 & -1 & 3 & 1 \\ -3 & -5 & -1 & 6 & 0 \\ 0 & 2 & 4 & -6 & -6 \end{bmatrix}$$

Ans.

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Ans.

$$\begin{bmatrix} \color{red}{1} & 4 & 5 & -9 & -7 \\ 0 & \color{red}{2} & 4 & -6 & -6 \\ 0 & 0 & 0 & \color{red}{-5} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Why do we care about pivots?

1. Because it always reminds me of that Friends episode.

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1. Because it always reminds me of that Friends episode.
2. A pivot allows us to further reduce the matrix into RREF (i.e., we can remove all entries in the same column in the rows above the pivot).
3. The variables corresponding to pivot columns are the basic variables.
4. The variables in non-pivot column are the free variables.
5. If there is a pivot in the final column of an augmented matrix, the corresponding system of equations has no solution.