# MATH 1119B: Week 1, Lecture 2

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# Types of solutions to linear equations

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1. Unique solution (consistent)
2. No solution (inconsistent)
3. Infinite solutions (also called parametric solutions for reasons we'll see later) (consistent).

Given any system of linear equations (i.e., any number of equations and any number of variables), in this class we are going to learn to solve the system in each of the previous three scenarios.

# Size matters...

- Systems of equations can have any number of
  - equations;
  - or variables.

An easy example: The following is a "system" of equations:

$$x + y + z = 0.$$

Here, we have 1 equation in 3 unknowns (or variables). We will call this a $1 \times 3$ system (for obvious reasons, later).

# $x+y+z = 0$

A solution of $x + y + z = 0$ is:

$$x = 1$$
$$y = -1$$
$$z = 0$$

Check this for yourself!!

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An easy way of representing this solution is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

This is called a 3-vector, or a $3 \times 1$-vector or a 3-column vector.

# Coefficient matrices

- The essential information of a linear system can be recorded in an array called a matrix.

Consider the system of equations:

$$\begin{array}{rcrcrcr} 5x & - & y & + & 2z & = & 7 \\ -2x & + & 6y & + & 9z & = & 0 \\ -7x & + & 5y & - & 3z & = & -7 \end{array}$$

The matrix

$$\begin{bmatrix} 5 & -1 & 2 \\ -2 & 6 & 9 \\ -7 & 5 & -3 \end{bmatrix}$$

is called the coefficient matrix of the system.

# Augmented matrices

Remember the system of equations:

$$\begin{array}{rcrcrcr} 5x & - & y & + & 2z & = & 7 \\ -2x & + & 6y & + & 9z & = & 0 \\ -7x & + & 5y & - & 3z & = & -7 \end{array}$$

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$$\left[\begin{array}{ccc|c} 5 & -1 & 2 & 7 \\ -2 & 6 & 9 & 0 \\ -7 & 5 & -3 & -7 \end{array}\right]$$

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# The size of a matrix

The size of a matrix tells you how many rows and columns it contains. The augmented matrix

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has 3 rows and 4 columns, so we call it a $3 \times 4$ (read 3 by 4) matrix.

# Solving by elimination

Solve

$$\begin{array}{rcrcrcr} x & - & 2y & + & 2z & = & 5 \\ & + & 4y & + & 8z & = & 0 \\ -3x & + & 2y & - & 4z & = & -5 \end{array}$$

by elimination. At each step, write the corresponding augmented matrix.

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by elimination. At each step, write the corresponding augmented matrix.

Ans.:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}$$

# Taking it further...

We used elimination to reduce

$$\begin{array}{rcrcrcr} x & - & 2y & + & 2z & = & 5 \\ & + & 4y & + & 8z & = & 0 \\ -3x & + & 2y & - & 4z & = & -5 \end{array}$$

to

$$\begin{array}{rcrcrcr} x & - & 2y & + & 2z & = & 5 \\ & + & 4y & + & 8z & = & 0 \\ & & & & 10z & = & 10 \end{array}$$

We can continue eliminating (keep writing the corresponding matrices next door!) until the coefficient matrix looks like

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What is on the right-hand side of the augmented matrix?

# Elimination seems long...

The previous example showed how performing elimination on a system of equations was like performing operations on the rows of a matrix.

We have a name for this: row reduction. There are three basic row operations which allow us to reduce our matrices without changing the system of equations.

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Interchange (row-swap). Swap two rows of a matrix:

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$$\left[\begin{array}{ccc|c} 5 & -1 & 2 & 7 \\ -2 & 6 & 9 & 0 \\ -7 & 5 & -3 & -7 \end{array}\right] \sim R_1 \leftrightarrow R_3 \left[\begin{array}{ccc|c} -7 & 5 & -3 & -7 \\ -2 & 6 & 9 & 0 \\ 5 & -1 & 2 & 7 \end{array}\right].$$

(Why doesn't this change the system of equations?)

# More row operations

Scaling. Replace one row by a multiple of it.

$$\left[\begin{array}{ccc|c} 5 & -1 & 2 & 7 \\ -2 & 6 & 9 & 0 \\ -7 & 5 & -3 & -7 \end{array}\right] \sim R_1 \leftarrow 5R_1$$

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Replacement ("linear combination"). Replace a row by itself $\pm$ a multiple of another row. (e.g., $R_1 \leftarrow R_1 + cR_2$).

$$\left[\begin{array}{ccc|c} 5 & -1 & 2 & 7 \\ -2 & 6 & 9 & 0 \\ -7 & 5 & -3 & -7 \end{array}\right] \sim R_1 \leftarrow R_1 + 2R_2$$

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(Slightly less clear, but hopefully still understandable: why doesn't this change the system of equations?)

# Row operations are friendly

- Row operations can be applied to any matrix. We like them because they do not change the solution of systems of linear equations.
- Two matrices are row-equivalent if you can row-reduce one to the other.
- Row operations are reversible.
  - The reverse of $R_1 \leftrightarrow R_3$ is:

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  - The reverse of $R_3 \leftarrow R_3 + 3R_1$ is: $R_3 \leftarrow R_3 - 3R_1$.

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We can solve systems of linear equations by row-reducing matrices. Remember, when a solution exists, the system is called consistent. When no solution exists, the system is called inconsistent.

**Two important questions:**

1. Does a solution exist (i.e., is the system consistent?).
2. There are two possibilities for numbers of solutions in a consistent system
   - unique,
   - infinite solutions.

   Is the solution unique?

# Your first row-reduction example

Determine if the following system of equations is consistent.

$$\begin{array}{rcrcrcr} x_1 & & & - & 7x_3 & = & 8 \\ -4x_1 & + & 7x_2 & + & x_3 & = & -1 \\ -2x_1 & + & x_2 & + & 11x_3 & = & -9 \end{array}$$

Ans.: Put the system into a matrix and row-reduce to get:

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Ans.: Put the system into a matrix and row-reduce to get:

$$\begin{bmatrix} 1 & 0 & -7 & 8 \\ 0 & 1 & -3 & 7 \\ 0 & 0 & -6 & -18 \end{bmatrix}$$

Therefore:

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Therefore: Yes. It has a unique solution.

Homework: On Monday, tell me the solution by:
(i) back-substitution, (ii) continuing to row-reduce.

# Your second row-reduction example

This is the augmented matrix of an system of equations:

$$\begin{bmatrix} 1 & 3 & 5 & 7 \\ 2 & 7 & 12 & 17 \\ 0 & 2 & 4 & 18. \end{bmatrix}$$

Is the system consistent?

Ans.: The matrix row-reduces to

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What does the last line mean?

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No, the system is not consistent.

# A type-question

Find values for ♠ and ♣ such that the following system of equations is

i. consistent;

ii. inconsistent.

$$\begin{array}{rcrcl} x & - & 2y & = & 4 \\ 2x & + & \spadesuit y & = & \clubsuit \end{array}$$

Ans.:

# A type-question

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$$\begin{array}{rcrcr} x & - & 2y & = & 4 \\ 2x & + & \spadesuit y & = & \clubsuit \end{array}$$

Ans.: i. - Not ii. (see below)
ii. - $\spadesuit = -4, \clubsuit \neq 8$.

Homework. In [i.], determine when there is a unique solution and when there are infinite solutions.