MAT2377A - Probability and Statistic for Engineers

Prof. Mahmoud Zarepour   Transcribed by Ted Morin

Created January 7th, 2014, Updated January 16th, 2014

Contents

Sets

Set A set is a collection of items.

$\vee \leftrightarrow A = \{2, 3, 4\}$

$A = B$ implies that any $x \in A \to x \in B$

Operations on Sets

Let A and B be two sets.

$A \cap B = A$ intersects $B = A$ and $B = \{x : x \in A$ and $x \in B\}$

$A \cup B = A$ union $B = A$ or $B = \{x : x \in A$ or $x \in B\}$

(1) Example:

$ = {2,3,4}$ and $B = \{3, 4, 5\} \to A \cap B = \{3, 4\}, A \cup B = \{2, 3, 4, 5\}$

Sample Space

Sample Space All the possible outcomes in a random experiment.

$S =$ the set of outcomes in a random experiment.

(2) Example: Flip a coin

$S = \{H, T\}$

(3) Example: Roll a die

$S = \{1, 2, 3, 4, 5, 6\}$

(4) Example: Flip a coin until a heads appears.

$S = \{H, TH, TTH, \ldots\}$

(5) Example: For a circle of radius 1, any point within the circle is what set?

$S = \{(x \times y) : x^2 + y^2 \leq 1\}$

Countable Set A set in which you can list the entries, and an uncountable set is one which will not let you list the entries.

Event

Event Any subset of S (sample space) is called an event.

(6) Example:

Coin is heads: $E = \{H\} \subset \{H, T\}$

So E is an event.

(7) Example:

Die is on even number: $E = \{2, 4, 6\}$

(8) Another example:

$E = \{(x, y) : x^2 + y^2 \leq \frac{1}{4}\}$

Enumeration (Counting)

1. Multiplication Rule.

   If task 1 can be completed in n_1 different ways and task 2 can be completed in n_2 different ways, then both tasks can be completed by $n_1 \times n_2$ ways.

   (9) Example: How many 5 digit numbers are there?

   Task 1: 1,2,3,4,5,6,7,8,9. $n_1 = 9$.

   Task 2: 0-9. $n_2 = 10$.

   ...

   Task 5: 0-9. $n_5 = 10$.

   $9 \times 10^4 = 90,000$

   (10) Example: How many 5 digit numbers with no repeats do we have?

   $n_1 = 9, n_2 = 9, n_3 = 8, n_4 = 7, n_5 = 6$

   $= 9 * 9 * 8 * 7$

   (11) Example: In how many different ways can we write 1,2,3,4,5 in unique order?

   $5 * 4 * 3 * 2 * 1 = 5!$

(12) Example: How many different orders of these letters is possible, where repeat letters are to be treated as non-unique: INDEPENDENT?

$11!/(3! * 2! * 3!)$

2. Permutation

(13) Example: With letters a, b, c, d, write all possible permutations with 2 letters.

ab, ac, ad,
ba, bc, bd,
ca, cb, cd,
da, db, dc.

$P_2^4 = 4 \times 3 = 12$

$P_r^n = ?$

n letters, $\{a_r \dots a_n\}$, $n \leq r$

r letters.

n = number of ways for completing task 1.

n-1 = number of ways for completing task 2.

$\dots$ until task r:

$n - (r - 1)$ = number of ways for completing task r.

Rule of Permutations $P_r^n = n(n-1) \times \dots \times (n - r + 1)$

Permutations of r letters from n letters. ($r \leq n$)

$P_r^n = n(n-1) \dots (n - r + 1)$

(14) Example

$P_2^4 = 4 \times 3 = 12 \rightarrow \{a, b, c, d\}$

In general for Square Permutations

$P_n^n = n(n-1) \dots (3)(2)(1) = n!$

In general for Any Permutations

$P_r^n = \frac{n(n-1) \dots (n-r+1)(n-r)(n-r-1) \dots 1}{(n-r)(n-r-1) \dots 1} = \frac{n!}{(n-r)!}$

Combinations

$C^n_r = \binom{n}{r}$

Combination # of combination of r letters from n leters.

(15) Example: Let $n = 4, r = 2$

$a, b, c, d \to ab, ac, ad, bc, bd, cd$

$\binom{4}{2} = 6$

$P^4_2 = 2\binom{4}{2}$

(16) Example

$\binom{4}{3} =?$

$S = \{a, b, c, d\}$

$abc, abd, acd, bcd \to \binom{4}{3} = 4$

$P^4_3 = 3!\binom{4}{3}$

General Combination Formula

$\therefore P^n_r = r!\binom{n}{r} \to \binom{n}{r} = \frac{P^n_r}{r!} = \frac{\frac{n!}{(n-r)!}}{r!} = \frac{n!}{r!(n-r)!}$

$\binom{n}{r} = \frac{n!}{r!(n-r)!}$

$P^n_r = \frac{n!}{(n-r)!}$

(17) Example: How many hands of 5 cards do we have?

$\binom{52}{5} = \frac{52!}{5!47!} = \frac{52\times51\times50\times49\times48}{120}$

(18) Example: How many diagonals do we have in a polygone with n sides?

$\binom{5}{2} - 5 = 5$

In general, the solution is:

$\binom{n}{2} - n = \frac{n(n-1)}{2} - n$

$= \frac{n(n-1)-2n}{n} = \frac{n(n-1-2)}{2} = \frac{n(n-3)}{2}$

(19) Example: A city has m by n streets in a grid. How many ways can someone traverse from one corner of the city to the other?

$\frac{(m+n)!}{m!n!} = \binom{m+n}{m} = \binom{m+n}{n}$

(20) Example: How many ways can we write out $(a + b)^5$ expanded?

$(a + b)^5 = a^5 + \frac{5!}{4!1!}a^4b + \frac{5!}{3!2!}a^3b^2 + \frac{5!}{2!3!}a^2b^3 + \frac{5!}{1!4!}ab^4 + b^5$

A General Polynomial Combinational Formula

$$(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^i b^{n-i}$$

$$(a+b+c)^n = \sum \sum \tbinom{n}{i,j} a^i b^j c^{n-i-j}$$

Properties of Combinations

(i) $\binom{n}{r} = \binom{n}{n-r}$

$\frac{n!}{r!(n-r)!} = \frac{n!}{r!(n-r)!(n-(n-r))!}$

(ii) $\binom{n}{r-1} + \binom{n}{r} = \binom{n+1}{r}$

$\frac{n!}{(r-1)!(n-r+1)1} + \frac{n!}{r!(n-r)!}$

$= \frac{rn!+n!(n-r+1)}{r!(n-r+1)!}$

$= \frac{(n+1)!}{r!(n-r+1)!}$

$= \binom{n+1}{r}$

(21) Example: How many subsets do we have if the set has n elements?

$S = \{a, b, c\}$

$\{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}$

You are effectively doing, in this example, n choose 0, n choose 1, n choose etc until n.

$\binom{n}{0} + \binom{n}{1} + \ldots + \binom{n}{n} = 2^n$

To explain this, we can use the multiplication rule and say, "For task i, choose the i^{th} letter or do not choose it."

Defining Probability

(22) Example: Flip 2 coins.

$S = \{HH, HT, TH, TT\}$

Define E = having at least 1 head.

$E = \{TH, HT, HH\}$

$P(E) = \frac{3}{4}$

Define F = exactly one head = $\{TH, HT\}$

$P(F) = \frac{2}{4}$

(23) Example: Flip a coin until you see the first head.

$S = \{H, TH, TTH, \dots\}$

Event E; Stop before or at 4^{th} trial $= \{H, TH, TTH, TTTH\}$

$P(E) = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16}$

(24) Example: A point is picked at random from a circle with radius r

$S = \{(x, y) : x^2 + y^2 \leq R^2\}$

Let $E = \{(x, y) : x^2 + y^2 \leq \frac{R^2}{4}\}$

$\frac{R^2}{4} = (\frac{R}{2})^2$

$P(E) = \frac{Area(E)}{Area(S)} = \frac{\pi \times (\frac{R}{2})^2}{\pi \times R^2} = \frac{1}{4}$

Probability

Let S be the sample space of a random experiment. Let P be a set function with the following probabilities:

Assumptions:

i) $P(E) \geq 0$

ii) If E_1, E_2, ... are disjoint ($E_i \cap E_j = 0; i \neq j$) then $P(E_1 \cup E_2 \dots) = \sum_i P(E_i)$

iii) $P(S) = 1$

Properties of P

1. $P(\emptyset) = 0$
2. $P(A - B) = P(A) - P(A \cap B)$

   Note: $A - B = \{x | x \in A, x \notin B\}$

   $P((A - B) \cup (A \cap B)) = P(A)$

   $P(A - B) + P(A \cap B) = P(A)$

   $P(A - B) = P(A) - P(A \cap B)$
3. If $A \subset B \rightarrow P(A) \leq P(B)$

   $B = A \cup (B - A)$

   $P(B) = P(A) + P(B - A) \geq 0$

   $\therefore P(A) \leq P(B)$

4. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

   Proof:

   $A \cup B = A \cup (B - A)$

   $P(A \cup B) = P(A) + P(B - A) = P(A) + P(B) - P(A \cap B)$

5. $0 \leq P(E) \leq 1$

   If $\emptyset \subset E \subset S$

   $0 = P(\emptyset) \leq P(E) \leq P(S) = 1$

   $P(E') = 1 - P(E)$

(25) Example: Union of 3 sets

Let $B \cup C = D$

$P(A \cup B \cup C) = P(A \cup D) = P(A) + P(D) - P(A \cap D)$

$A \cap D = A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

$P(A \cap D) = P(A \cap B) + P(A \cap C) - P(A \cap B \cap C)$

$\therefore P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(B \cap C) - P(A \cap B) - P(A \cap C) + P(A \cap B \cap C)$

(26) Example: Union of 4 sets

$P(A \cup B \cup C \cup D)$

$= P(A) + P(B) + P(C) + P(D) - P(A \cap B) - P(A \cap C) - P(B \cap C) - P(C \cap D) - P(A \cap D) - P(B \cap D) + P(A \cap B \cap C) + P(A \cap B \cap D) + P(A \cap C \cap D) + P(B \cap C \cap D) - P(A \cap B \cap C \cap D)$

(27) Example: 100 different letters are sent to 100 different addresses at random. What is the probability that *at least* 1 letter reaches its designate address?

Hint: In questions, you can look for key phrases: *at least* suggests union, and *all* suggests intersection.

$E_i = i^{th}$ letter goes to the right address.

$P(E_1 \cup E_2 \cup E_3 \cup \ldots \cup E_100) = P(E_1) + P(E_2) + \ldots + P(E_100) - P(E_1 \cap E_2) - P(E_1 \cap E_3) - \ldots - P(E_99 \cap E_100) + P(E_1 \cap E_2 \cap E_3) + \ldots \ldots - P(E_1 \cap \ldots \cap E_100)$

$P(E_i) = \frac{1}{100}$

$i \neq j$, $P(E_i \cap E_j) = \frac{1}{100} \times \frac{1}{99}$

$P(E_i \cap E_j \cap E_k) = \frac{1}{100} \times \frac{1}{99} \times \frac{1}{98}$

$\rightarrow 100 * (\frac{1}{100}) - \binom{100}{2} * \frac{1}{100*99} + \binom{100}{3} * \frac{1}{100*99*98} - \ldots$

$= 1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \ldots - \frac{1}{100!}$

$= 1 - e^{-1}$ is the probability that 1 letter will reach the correct address.

Conditional Probability

Let S be the sample space for a random experiment.

Let $B \subset S$ (B is an event)

Such that $P(B) > 0$

For any event A we define $P(A|B) = \frac{P(A \cap B)}{P(B)}$

(28) Example: A die is rolled. Let A = result is more than 4, let B = outcome is even.

Find $P(A|B)$

Solution:

- $A = \{5, 6\}$
- $B = \{2, 4, 6\}$

Given that you have a result in B, what are the chances that it is a result in A? Since B has 2, 4, and 6, one third of the time, A will be met as well.

$A \cap B = \{6\}$

$P(A \cap B) = \frac{1}{6}$

$P(B) = \frac{3}{6}$

$P(A|B) = \frac{\frac{1}{6}}{\frac{3}{6}}$

$= \frac{1}{3}$

Total Probability Rule

Total Probability Rule Let S be a sample space in a random experiment.

Let A be any event ($A \subset S$)

Also we assume $S = E_1 \cup E_2 \cup \ldots \cup E_k$, such that E_i's are disjoint.

Then $P(A) = \displaystyle\sum_{i=1}^{k} P(A|E_i)P(E_i)$

Proof $A \subset S$

$A \cap S = A$

$A = A \cap (E_1 \cup E_2 \cup \ldots \cup E_k)$

$= (A \cap E_1) \cup (A \cap E_2) \cup \ldots \cup (A \cap E_k)$

$\therefore P(A) = P(A \cap E_1) + P(A \cap E_2) + \ldots + P(A \cap E_k)$

$= \frac{P(A \cap E_1)}{P(E_1)} P(E_1) + \ldots + \frac{P(A \cap E_k)}{P(E_k)} P(E_k)$

$= P(A|E_1)P(E_1) + \ldots + P(A|E_k)P(E_k)$

(29) Example (Polya): There are m white chips and n black chips in the box. You remove one chip from the box without looking at it, and it is put inside. A second chip is then drawn. What is the probability that the second chip is white?

Let $P(A)$ represent the probability that the 2^{nd} chip is white.

Solution:

- $E_1 = 1^{\text{st}}$ chip is white
- $E_2 = 1^{\text{st}}$ chip is black.

$P(A) = P(A|E_1)P(E_1) + P(A|E_2)P(E_2)$
$= \frac{m-1}{m+n-1} \times \frac{m}{m+n} + \frac{m}{m+n-1} \times \frac{n}{m+n}$
$= \dfrac{m}{m+n}$

Slightly more explanation: Probability doesn't measure the physical characteristics of the box, it measures based on information that you have. Not knowing the first chip, the second chip, or the k chip, will not change the chance of the k+1's chip.

(30) Example: You have two boxes, both with m white chips and n black chips, and you take one random chip from the first box and place it in the second box. Now you take a random chip from the second box. What is the probability that this chip is white?

$E_1 = 1^{\text{st}}$ chip is white

$E_2 = 2^{\text{nd}}$ chip is black

$P(\ 2^{\text{nd}}$ chip is white$) = P(A|E_1)P(E_1) + P(A|E_2)P(E_2)$
$= \frac{m+1}{m+n+1} \times \frac{m}{m+n} + \frac{m}{m+n+1} \times \frac{n}{m+n}$
$= \dfrac{m}{m+n}$

The total probability rule is useful for tangled experiments, it can help untangle them and find a simple solution.

(31) Example: You roll a die. You now flip a coin the number of times given by the die. What is the probability that no heads were flipped?

$P(\text{no heads}) =?$

- E_1 = Die is 1
- E_2 = Die is 2

- E_3 = Die is 3
- E_4 = Die is 4
- E_5 = Die is 5
- E_6 = Die is 6

A = no heads

$P(A) = P(A|E_1)P(E_1) + \ldots + P(A|E_6)P(E_6)$

$P(A|E_1) = \frac{1}{2}$

$P(A|E_2) = \frac{1}{4}$

$P(A|E_3) = \frac{1}{8}$

$P(A|E_4) = \frac{1}{16}$

$P(A|E_5) = \frac{1}{32}$

$P(A|E_6) = \frac{1}{64}$

$P(E_i) = \frac{1}{6}$ where $1 \leq i \leq 6$

$P(A) = \frac{1}{2} \times \frac{1}{6} + \ldots + \frac{1}{64} \times \frac{1}{6}$

Baye's Rule

Baye's Rule Let S be the sample space and $S = E_1 \cup \ldots \cup E_k; E_i$'s disjoint and $A \subset C$

$$P(E_1|A) = \frac{P(A|E_1)P(E_1)}{\sum_k P(A|E_k)P(E_k)}$$

Proof:

$P(E_1|A) = \frac{(P(A \cap E_1)/P(E_1))P(E_1)}{P(A)}$

$$P(E_1|A) = \frac{P(A|E_1)P(E_1)}{\sum_k P(A|E_k)P(E_k)}$$

(32) In a transmittion system, you send either 0 or 1. Let $P(\{0\}) = P(\{1\}) = 0.5$

Trans	Receive
0	0
1	1

$P[Rec0|Send0] = 0.99$

$P[Rec1|Send1] = 0.95$

$P[1 \text{ was sent } |Rec1]$

$E_1 = \text{send } 1$

$E_2 = \text{send } 0$

$A = \text{Receive } 1$

Applying the base rule:

$$P(E_1|A) = \frac{P(A|E_1)P(E_1)}{P(A|E_1)P(E_1) + P(A|E_2)P(E_2)}$$

$= \frac{.95 \times .5}{.95 \times .5 + .01 \times .5}$

$\therefore = \frac{95}{96}$

(33) Example: 0.01 of people in a population have a certain disease.

$P(+|sick) = 0.98$

$P(-|healthy) = 0.99$

$P(sick|+) = ?$

$E_1 = sick$

$E_2 = healthy$

$A = \text{test is } +$

$P(E_1|A) = \frac{P(A|E_1)P(E_1)}{P(A|E_1)P(E_1)+P(A|E_2)P(E_2)}$

$\frac{.98 \times .01}{.98 \times .01 + .01 \times .99} - \frac{.98}{.98 + .99}$