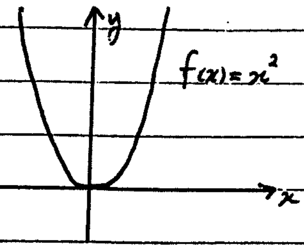
Chapter 1 : Function and Models

§1.1 : Four Ways to Represent a Function



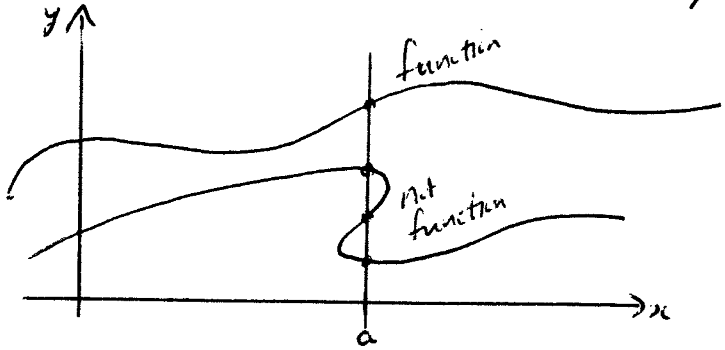
$f(x) = x^2$ (graph)

we'll begin with some definitions :
consider the function $f(x) = x^2$ (p11) 12
what is a function ? a *function* is a *rule* that takes certain input numbers (called the *independent variable*) and produces unique output numbers for each input
(the values of) the output is called the *dependent variable* (because it depends on the independent variable value)
so, for $f(x) = x^2$, x is the independent variable, $f(x)$ is the dependent and $f(x) = x^2$ produces a unique output for each input (this means that $f(x)$ is unique for any x, it does *not* mean the $f(x)$ values are all distinct)
the set of all possible inputs is called the *domain* of the function and the set of all outputs is called the *range*
so, for $f(x) = x^2$, the domain is all real x ($x \in \mathbb{R}$, $-\infty < x < \infty$)
and the range is nonnegative real numbers $f(x) \geq 0$ ($[0,\infty)$)
in this case, we'd call x a *continuous* variable since it can have any value in a given interval (all $\mathbb{R}$ in this case)

for other functions, say we're keeping track of the temperature each day, the variable can only have a certain number of fixed isolated (often integer) values - in this case, the variable is *discrete* (p2)

we must be careful not to confuse the function with its formula
when we say we have the function $f(x) = x^2$, the function is really "take the square of the input" and not x^2
so $f(x) = x^2$ and $g(t) = t^2$ are really the same function

and sometimes, we may not have a formula for the function - ie the function may be given to us in the form of a data table (see the examples in the book) we can test if a curve in the xy plane is a function using the Vertical Line Test (p 17)

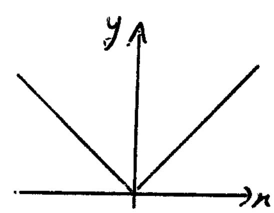


functions that have different formulas in different parts of their domains are called piecewise defined functions (p 18)

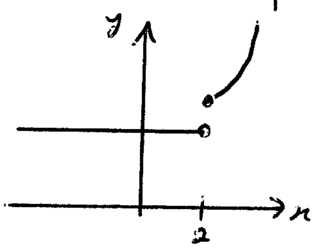
examples :

i, the absolute value function (p 19)

$$f(x) = |x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$



ii, $f(x) = \begin{cases} 3 & x < 2 \\ x^2 & x \geq 2 \end{cases}$



some functions have symmetry properties (p 20) 19

$f(x)$ is called even if $f(-x) = f(x)$ for all x

examples : x^2, x^4, x^{2n}, $\cos(x)$, x^2+2, etc...

even functions are symmetric about the y axis

$f(x)$ is called odd if $f(-x) = -f(x)$ for all x

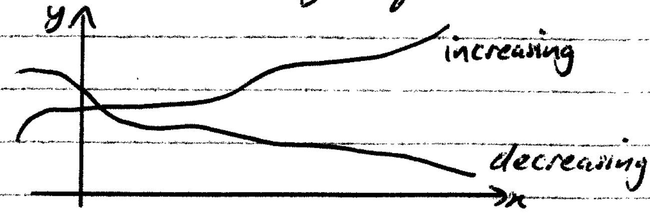
examples : x, x^3, x^{2k+1}, $\sin x$, $2x^3+7x$, etc...

odd functions are symmetric about the origin

note : most functions are neither even nor odd

example : $f(x) = x^2 + x - 1$ contains both even and odd powers

a function f is called <u>increasing</u> if $f(x_1) < f(x_2)$ for $x_1 < x_2$
and <u>decreasing</u> if $f(x_1) > f(x_2)$ (p 21)



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§1.2: Mathematical Models: A Catalogue of Essential Functions

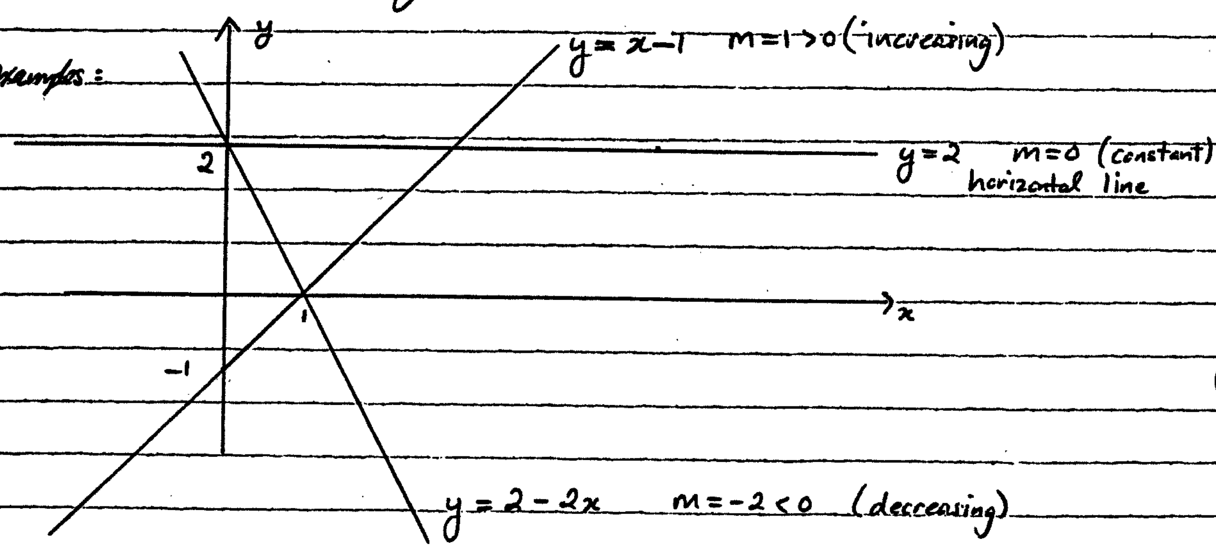
one of the simplest types of functions is a <u>linear function</u> (p 26)
linear functions are characterized by a constant rate of increase/decrease
ie they have constant slope (or rate of change)
they can be represented by the formula $y = mx + b$

where $m = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{f(x_2) - f(x_1)}{x_2 - x_1} = \dfrac{\text{rise}}{\text{run}} = \dfrac{\Delta y}{\Delta x}$ (recall the Δ notation for change) (p 5)

(difference quotient)

is the <u>slope</u> of the line (p 5)
and b is the <u>y-intercept</u>

examples:



we can recognize a linear function from a data table by looking for y differences that are constant for equal differences in x
(ie $\Delta y / \Delta x$ = constant)

example:

x	y	Δy
1	-5	2
2	-3	2
3	-1	2
4	1	2
5	3	

this is the linear function
$y = 2x - 7$
(can you get this ?)

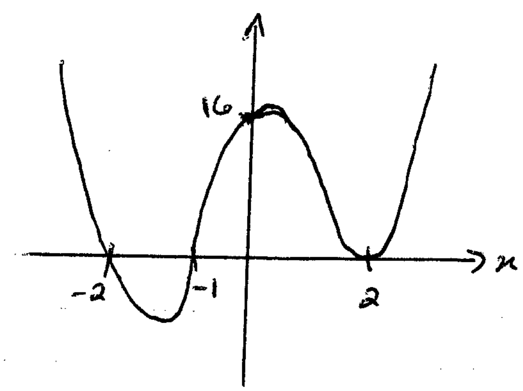
a <u>polynomial of degree n</u> has the form (p29)

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$$

where $a_n \neq 0$ is called the <u>leading coefficient</u>

a polynomial of degree n has at most n roots or zeros and $n-1$ "turning points" (see figures on p 29-30)

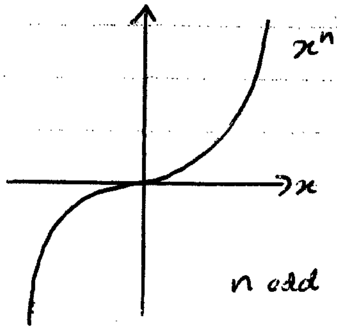
example:

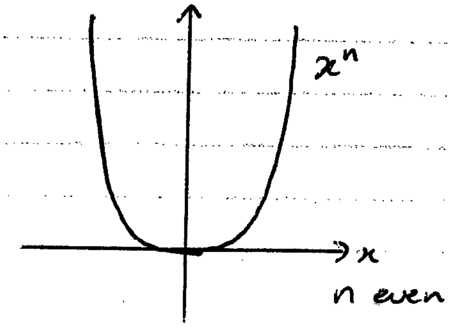


this function has 3 turnings, so it appears to be a quartic
it has roots at $x = -2, -1$ and 2, with $x=2$ being a double root
so the function has the form $p(x) = k(x+2)(x+1)(x-2)^2$
since $p(0) = 16$ $\quad p(0) = k(2)(1)(-2)^2 = 8k \Rightarrow k = 2$
and so $p(x) = 2(x+2)(x+1)(x-2)^2$

a <u>power function</u> has the form $f(x) = x^a$ (p 31 30)
for constant a

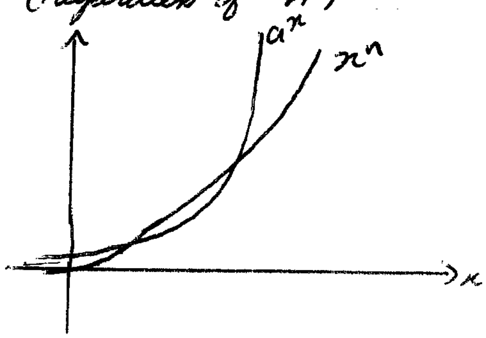
for positive integer powers, say $f(x) = x^n$, there are 2 basic shapes, depending on whether n is even or odd





the larger n is the steeper the curves are as $x \to \pm\infty$ and the flatter near $x = 0$

interesting fact: any exponential function a^x $(a>1)$ will dominate any power function eventually
ie for some value of x a^x becomes larger than x^n (regardless of n)



if $a = 1/n$, we'll have a <u>root function</u> like $f(x) = x^{1/2} = \sqrt{x}$ (p31)

and if $a = -1$, we have the <u>reciprocal function</u> $f(x) = x^{-1} = 1/x$ (p32)

a <u>rational function</u> has the form $f(x) = \dfrac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomials (p32)

domain is $\{x \mid q(x) \neq 0\}$

the graph of a rational function may have features that the graph of a polynomial won't:

vertical asymptotes occur when the denominator is zero (and the numerator is not)

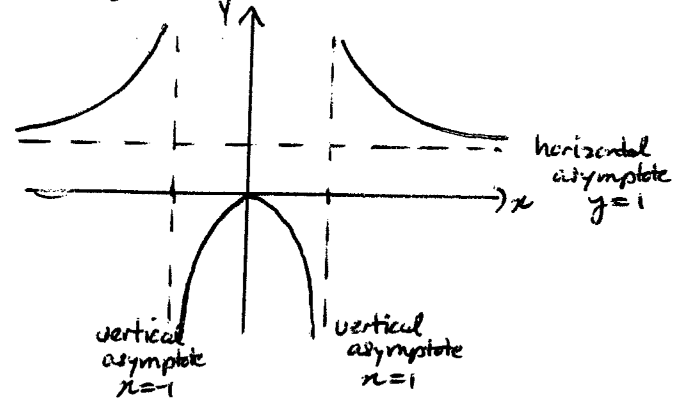
example: $y = f(x) = \dfrac{x^2}{x^2-1}$ this function has vertical asymptotes at $x = \pm 1$

⑥ the end behavior, as $x \to \pm\infty$, $\frac{x^2}{x^2-1} \approx \frac{x^2}{x^2} = 1$

so $y \to 1$ as $x \to \pm\infty$ and hence $y=1$ is a horizontal asymptote

$y(0)=0$ and $y=0$ only if $x=0$, so $(0,0)$ is the only intercept

the graph looks like



a function f is called an algebraic function (p 32) if it is constructed using algebraic operations $(+,-,\times,\div,\sqrt{\ })$

examples: polynomials, powers, rational functions

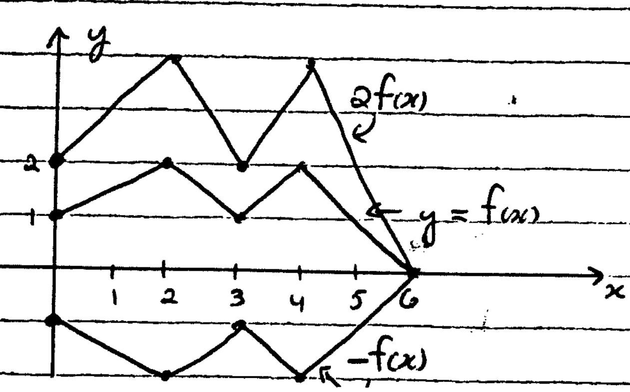
$\sqrt{x^2+17}$, $\frac{x+3\sqrt{x}}{x^2+4}$, $\frac{(x-7)^4}{\sqrt{x+3}}$, etc...

other functions, such as the trigonometric, inverse trig, exponential and logarithmic functions are not algebraic, rather they are transcendental (p 32) - we'll study them later

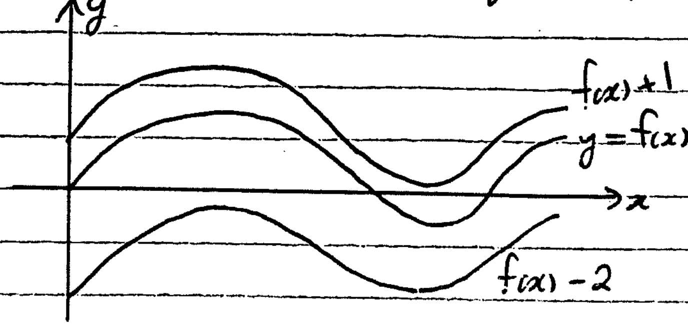
§1.3: New Functions from Old Functions

given a function $y=f(x)$, we can do things to it to produce new functions (p 38-39)

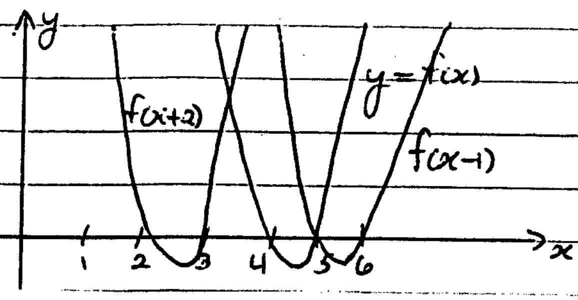
- if we multiply by a constant c, $cf(x)$, the graph will stretch if $|c|>1$, shrink if $|c|<1$ and be flipped upside down if $c<0$



- if we add k, we move the graph up or down) ⑦

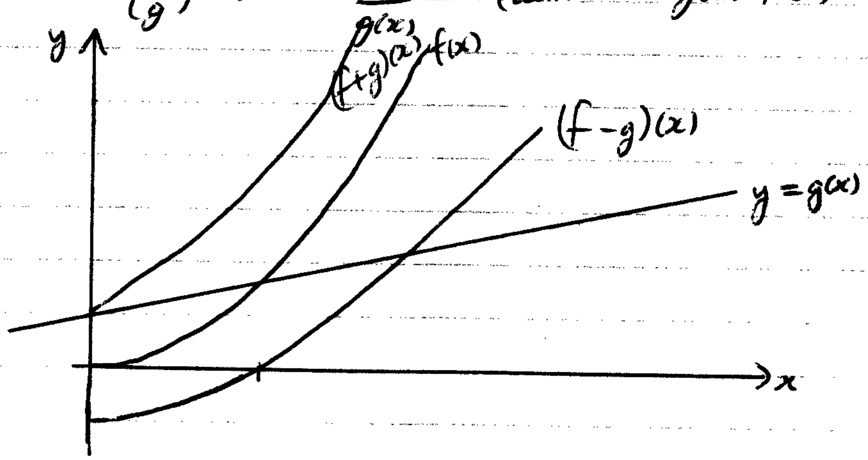


- if we replace x by $x-h$, the graph moves to the right by h if $h>0$ or to the left by h if $h<0$



we can also add, subtract, multiply and divide (p 42)

$$(f+g)(x) = f(x) + g(x)$$
$$(f-g)(x) = f(x) - g(x)$$
$$(fg)(x) = f(x)\,g(x)$$
$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} \quad (\text{domain}: g(x) \neq 0)$$



another way to make new functions from old is to compose them ie use the output of one function as the input of another (p44)

$(f \circ g)(x) = f(g(x))$ — outer, inner

the domain is all x in domain of g such that $g(x)$ is in domain of f

(8) example: blowing up a spherical balloon

the volume is $V = \frac{4}{3}\pi r^3 = f(r)$

but the radius is a function of time, say $r = g(t) = 2t^2$

so V is really a function of t

ie $V(t) = \underset{\text{outer function}}{f}(\underset{\text{inner function}}{g(t)}) = \frac{4}{3}\pi(2t^2)^3 = \frac{4}{3}\pi\, 8t^6 = \frac{32}{3}\pi t^6$

example: let $f(x) = x^2+2$, $g(x) = \cos(2x)$

then $h(x) = f(g(x)) = f(\cos(2x)) = (\cos(2x))^2 + 2 = \cos^2(2x) + 2$

but $p(x) = g(f(x)) = \cos(2(x^2+2)) = \cos(2x^2+4)$

ie $f(g(x)) \neq g(f(x))$ (in general)

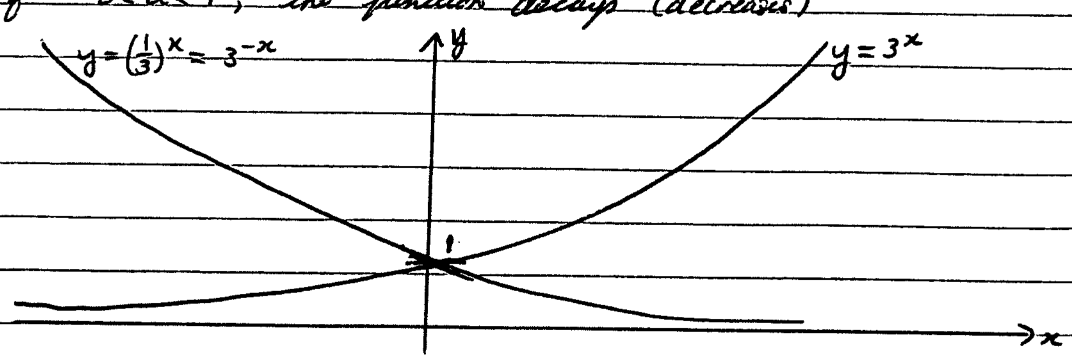
§1.5: Exponential Functions

another important type of function is an exponential function (p55)
exponential functions are characterized by growth (or decay) by a constant factor for equal intervals of the independent variable

example: $y = f(x) = 2^x$ doubles everytime x increases by 1

x	2^x
0	1
1	2
2	4
3	8 etc...

exponential functions can be represented by the formula $f(x) = a^x$
where a is any positive real number (and is the growth/decay factor)
if $a>1$, the function grows (increases)
if $0<a<1$, the function decays (decreases)



notice that whether we have an increasing or decreasing exponential, the curve bends upwards (as we move to the right)

a graph that bends upwards is called concave up

one that bends downwards is called concave down

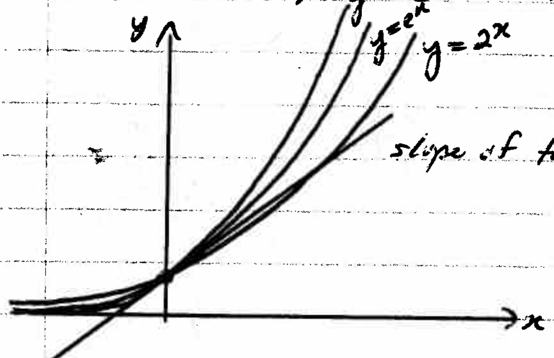
(a line does not bend and so is neither)

note that concave up/down is independent of increasing/decreasing

basic laws of exponents: (p57) 54

$a^{x+y} = a^x a^y$, $a^{x-y} = a^x/a^y$, $(a^x)^y = a^{xy}$, $(ab)^x = a^x b^x$

recall that there is the special base of exponentials, the number called e, $e \approx 2.7182818\ldots$ (p57)



slope of tangent to $y=e^x$ at $x=0$ is 1

examples:

i, the population of a city is 600,000 and growing at a rate of 2.1 % per year

what is the function that describes the population and what will the population be in 3 years?

$$P(t) = P_0 a^t = P_0(1+r)^t = 600\,000\,(1.021)^t$$

so $P(3) = 600\,000\,(1.021)^3 \simeq 638\,600$

ii, what is the population of the city using base e?

then $P(t) = P_0 e^{kt} = P_0(1+r)^t \Rightarrow e^k = 1+r$

so $e^k = 1.021 \Rightarrow k = \ln(1.021) \simeq 0.0208$

so $P(t) = 600\,000\, e^{0.0208t}$

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Appendix C: Trigonometry

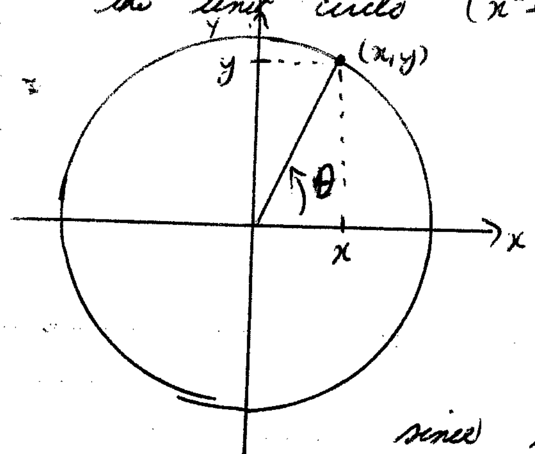
in calculus, we measure angles in radians, not degrees (there are good reasons for this that we'll see later)

an angle of 1 radian is defined to be the angle at the centre of the unit circle that cuts off an arclength of 1

since the circumference of the unit circle is 2π, the total angle contained is 2π radians (p A18)17

ie $360^\circ = 2\pi$ radians or $180^\circ = \pi$ radians

we can define the trigonometric functions with reference to the unit circle $(x^2+y^2=1)$



at an angle of θ radians (measured counter-clockwise from the x axis), we have

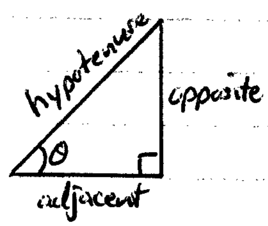
$\cos\theta = x$ (p 30)

$\sin\theta = y$

since $x^2+y^2=1$, $\cos^2\theta + \sin^2\theta = 1$

notation convention: $\cos^2\theta = (\cos\theta)^2$, $\sin^2\theta = (\sin\theta)^2$

or we can define the functions with a right triangle (p A20)



$\cos\theta = \frac{adj}{hyp}$, $\sin\theta = \frac{opp}{hyp}$, $\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{opp}{adj}$

$\sec\theta = \frac{1}{\cos\theta} = \frac{hyp}{adj}$, $\csc\theta = \frac{1}{\sin\theta} = \frac{hyp}{opp}$

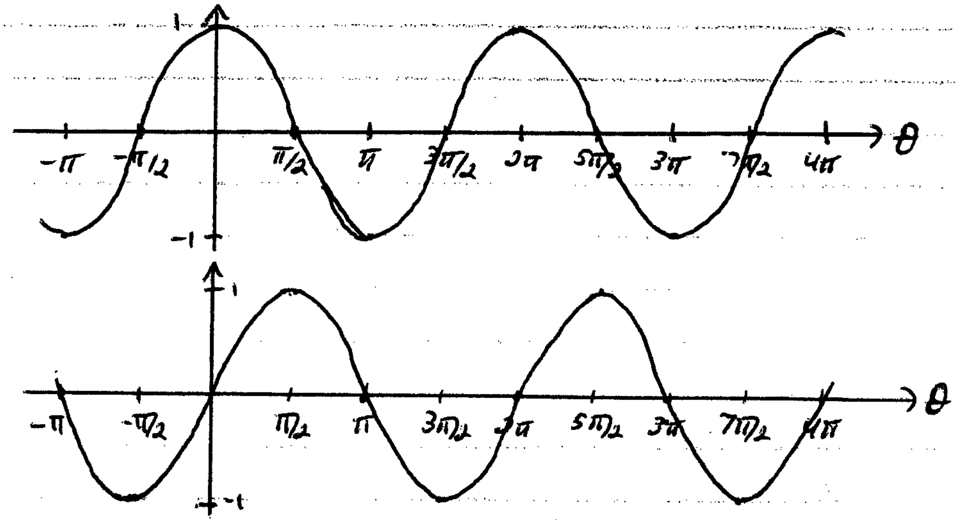
and $\cot\theta = \frac{1}{\tan\theta} = \frac{\cos\theta}{\sin\theta} = \frac{adj}{opp}$

at the special angles, we get the following values: (p A21) ...

θ	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$
$\cos\theta$	1	$\sqrt{3}/2$	$1/\sqrt{2}$	$1/2$	0
$\sin\theta$	0	$1/2$	$1/\sqrt{2}$	$\sqrt{3}/2$	1

and so on ...

if we follow the values as we go around the circle (which we can do several times), we'll get the following graphs



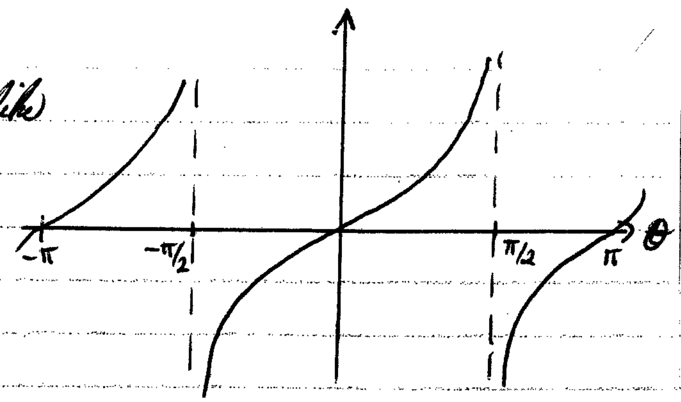
$\cos\theta$

(2π periodic)

$\sin\theta$

the graph of $\tan\theta$ looks like

($\tan\theta$ is π periodic)

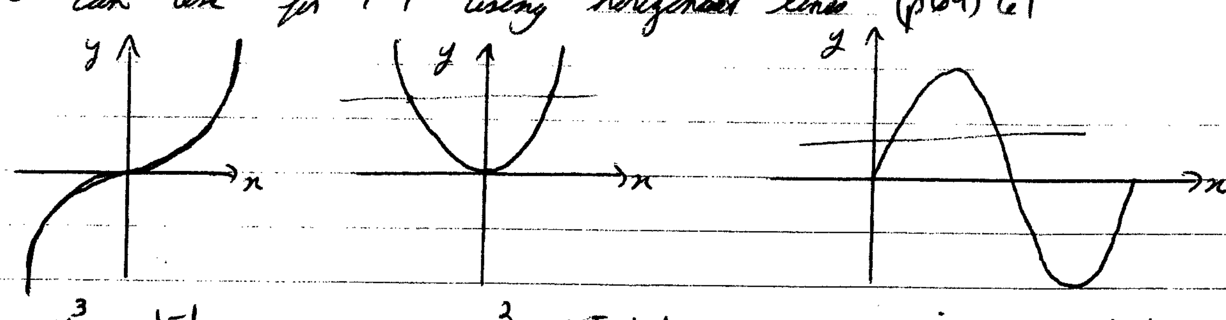


§ 1.6 : Inverse Functions and Logarithms

definition : a function f is called <u>1-1</u> if it never takes the same value twice

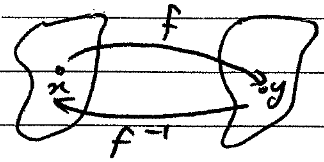
ie if $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$ (p64 / 61)

we can test for 1-1 using horizontal lines (p64) 61



$y = x^3$ 1-1 $\qquad$ $y = x^2$ NOT 1-1 $\qquad$ $y = \sin x$ NOT 1-1 if a function f is 1-1, there is another function, called the <u>inverse of f</u> or <u>inverse function</u>, written f^{-1}, that undoes what f does (p64) 62

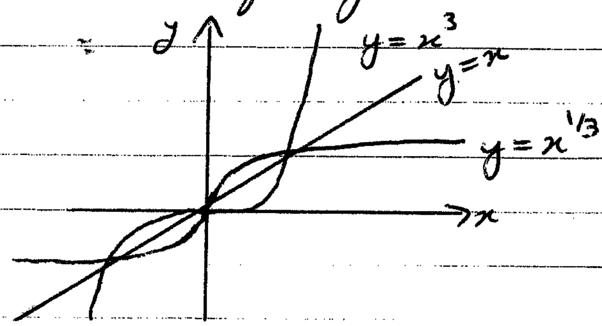
ie if $f(x) = y$, then $f^{-1}(y) = x$



NOTE: $f^{-1} \neq \frac{1}{f}$

and the domain of f^{-1} is the range of f (p65) 62
" range " f^{-1} " " domain " f

the inverse of $y = x^3$ is $y = x^{1/3}$



<u>graphically</u>
f and f^{-1} are reflections of each other in line $y = x$ (p67) 64

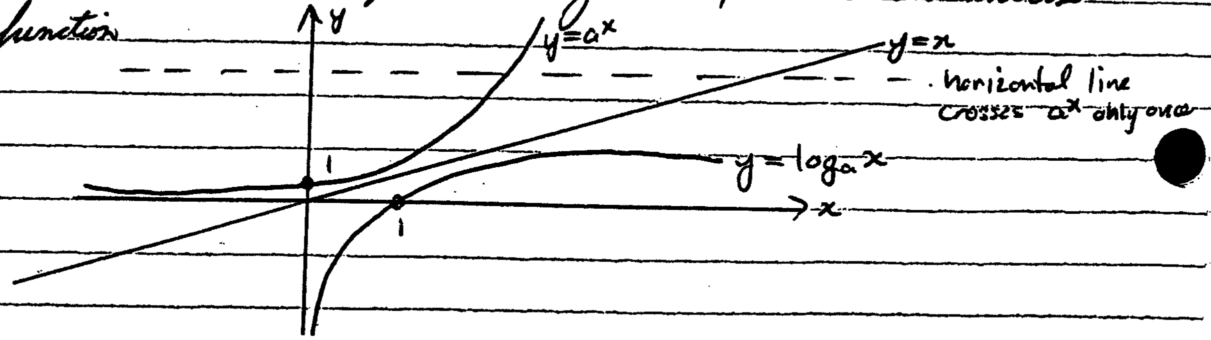
given $y = f(x)$, how do we find f^{-1}?
solve for x, swap variables

$$y = x^3 = f(x)$$
$$\text{so} \quad x = y^{1/3}$$
$$\therefore y = f^{-1}(x) = x^{1/3}$$

also $f^{-1}(f(x)) = x$ (special compositions)
$f(f^{-1}(x)) = x$

ie $(x^3)^{1/3} = x = (x^{1/3})^3$

consider the exponential function $y = a^x$, it has an inverse function



as we saw ~~in the previous section~~ above, the logarithmic function $y = \log_a x$ $(x>0)$ is defined as the inverse of the exponential $y = a^x$

ie $y = \log_a x$ means $x = a^y$

the most useful bases are 10 and e (p67) 65

the common logarithm $y = \log_{10} x$ is usually written $y = \log x$

the natural logarithm $y = \log_e x$ is written $y = \ln x$ (p68) 66

the properties of logs (see page ~~68~~ 65) follow from the properties of exponentials :

i, $\log_a(AB) = \log_a A + \log_a B$

ii, $\log_a(A/B) = \log_a A - \log_a B$

iii, $\log_a(A^p) = p \log_a A$

iv, $\log_a(a^x) = x$

v, $a^{\log_a x} = x$

vi, $\log_a 1 = 0$

iii, $f(x) = y = \ln(x+2)$

$e^y = x+2$

$x = e^y - 2$

$\therefore y = f^{-1}(x) = e^x - 2$

examples :

i, solve $200 = 20e^{3x} \Rightarrow e^{3x} = 10 \Rightarrow 3x = \ln(10) \Rightarrow x \simeq 0.7675$

ii, p 28 #46 : a picture supposedly painted by Vermeer (1632-1675) contains 99.5% of its carbon 14 is it a fake ?

the half-life of carbon 14 is 5730 years

the amount of carbon 14 present in an object can be modelled by $Q(t) = Q_0 e^{-kt}$

when $t = 5730$ years, $Q(t) = \frac{1}{2} Q_0$, so $e^{-5730k} = 0.5$

then $-5730k = \ln(0.5) \Rightarrow k = \simeq 1.21 \times 10^{-4}$

so $Q(t) = Q_0 e^{-(1.21\times10^{-4})t}$

the amount present now is $0.995 Q_0$, so $e^{-(1.21\times10^{-4})t} = 0.995 \Rightarrow t$

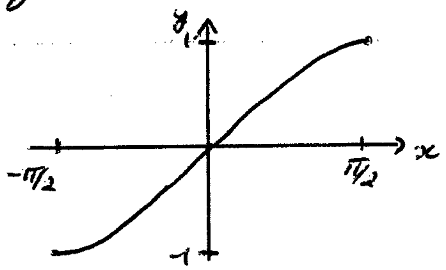
the painting is clearly a fake

§ 3.6: Inv. Trig. Functions and their Derivatives

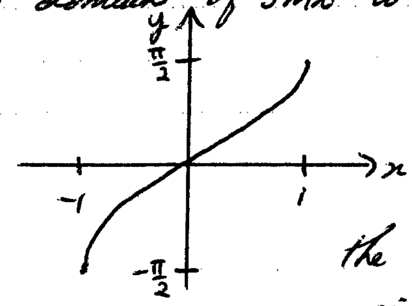
~~Back to Appendix~~ C:

to define the inverse function of $y = \sin x$, we need to remember that the function must pass the horizontal line test
for this reason, we restrict the domain of $\sin x$ to $[-\pi/2, \pi/2]$

(p ~~A25~~ 216)



$y = \sin x \quad [-\pi/2, \pi/2]$

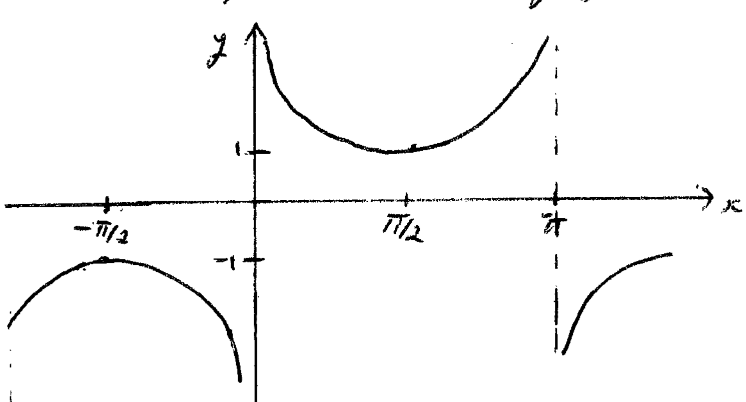


the inverse function
$y = \sin^{-1} x = \arcsin x$

(we'll use the notation $\arcsin x$ to avoid notational confusion)

so for $\underset{\text{domain of arcsin}x}{\overset{\text{range of sin}y}{-1 \le x \le 1}}$, $y = \arcsin x$ means $x = \sin y$ for $\underset{\text{range of arcsin}x}{\overset{\text{domain of sin}y}{-\pi/2 \le y \le \pi/2}}$

it is important to recognize that $\arcsin x$ is <u>not</u> $\csc x = \dfrac{1}{\sin x}$



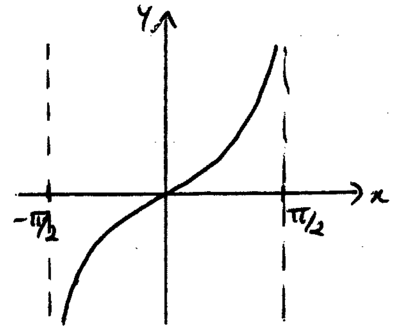
graph of $y = \csc x$

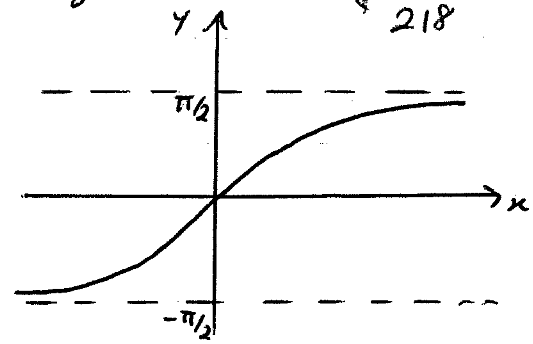
example: solve $\quad 3\sin(2x-3) = 2$
then $\quad \sin(2x-3) = 2/3$
so $\quad 2x - 3 = \arcsin(2/3)$

so $\quad 2x = \arcsin(2/3) + 3$ and hence
$x = \frac{1}{2}(\arcsin(2/3) + 3) \simeq 1.8649$ (radians)

See also Ex 1 p 217
2 p 218

to define the inverse of the tangent function, we must also restrict the domain of $\tan x$ (p ~~A26~~ 218)

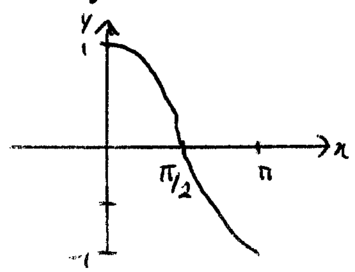


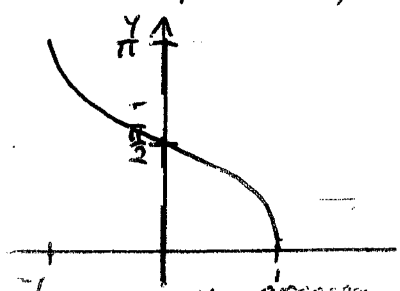


$y = \tan x \quad (-\pi/2, \pi/2)$ $\qquad\qquad$ $y = \tan^{-1} x = \arctan x$

for any $\underset{\text{domain of arctan } x}{\overset{\text{range of tan } y}{x}}$, $y = \arctan x$ means $x = \tan y$ for $\underset{\text{range of arctan } x}{\overset{\text{domain of tan } y}{-\pi/2 < y < \pi/2}}$

similarly, we must restrict the domain of $\cos x$ in order to define $\arccos x$: (see # ~~46 p A28~~) p 218





$y = \cos x \quad [0, \pi]$ $\qquad\qquad$ $y = \arccos x$

for $\underset{\text{domain of arccos } x}{\overset{\text{range of cos } y}{-1 \le x \le 1}}$, $y = \arccos x$ means $x = \cos y$ for $\underset{\text{range of arccos } x}{\overset{\text{domain of cos } y}{0 \le y \le \pi}}$

what is $\sin(\arctan x)$?

let $\theta = \arctan x$ then $x = \tan\theta = \dfrac{\sin\theta}{\cos\theta}$

want $\sin\theta$, know that $\sin^2\theta + \cos^2\theta = 1$

$\cos\theta = \sqrt{1 - \sin^2\theta}$

$$x^2 = \frac{\sin^2\theta}{\left(\sqrt{1-\sin^2\theta}\right)^2} = \frac{\sin^2\theta}{1-\sin^2\theta}$$

$$1 + \cot^2\theta = \csc^2\theta$$

$$1 + \frac{1}{x^2} = \frac{1}{\sin^2\theta}$$

$$x^2 - x^2\sin^2\theta = \sin^2\theta$$

$$\sin^2\theta = \frac{x^2}{1+x^2}$$

$$\sin^2\theta = \frac{1}{1+\frac{1}{x^2}} = \frac{1}{\frac{x^2+1}{x^2}} = \frac{x^2}{1+x^2}$$

$$\sin\theta = \frac{x}{\sqrt{1+x^2}} = \sin(\arctan x)$$

$\sin\theta = \dfrac{x}{\sqrt{1+x^2}}$

(signs okay)

Chapter 2 : Limits and Derivatives

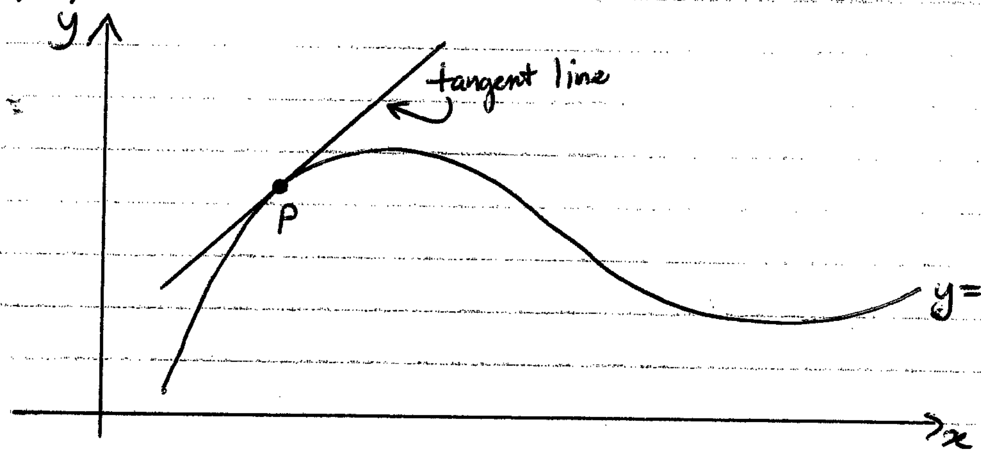
§ 2.1 : the Tangent and Velocity Problems

§ 2.6 : ~~Tangents, Velocities and Other Rates of Change~~ Derivatives and Rates of Change

say we have the graph of some curve $y = f(x)$ and we want to draw a tangent line at some point P

what is it?

the tangent line to the curve $f(x)$ at the point P is a straight line that passes through the point P and best approximates the curve near P



why bother?

because the slope of the tangent line will tell us the slope of the curve at point P

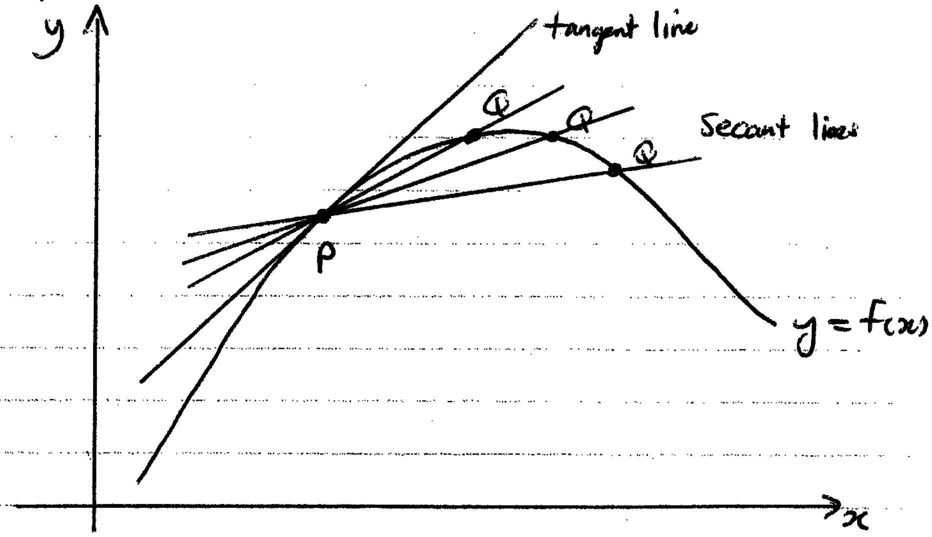
so this is what we'd like to do:

given the graph of the function $y = f(x)$, or just the function itself, we'd like to be able to find the slope of the tangent line at a point P, or more often, the slope of the tangent line at any point on the curve

well, we could always draw the curve and the tangent line and try to calculate its slope

yes, but this would only be an approximation because we can't draw a "perfect" representation

we get around this by drawing secant lines
a secant line of the curve is a line that joins two points, P and Q, on the curve



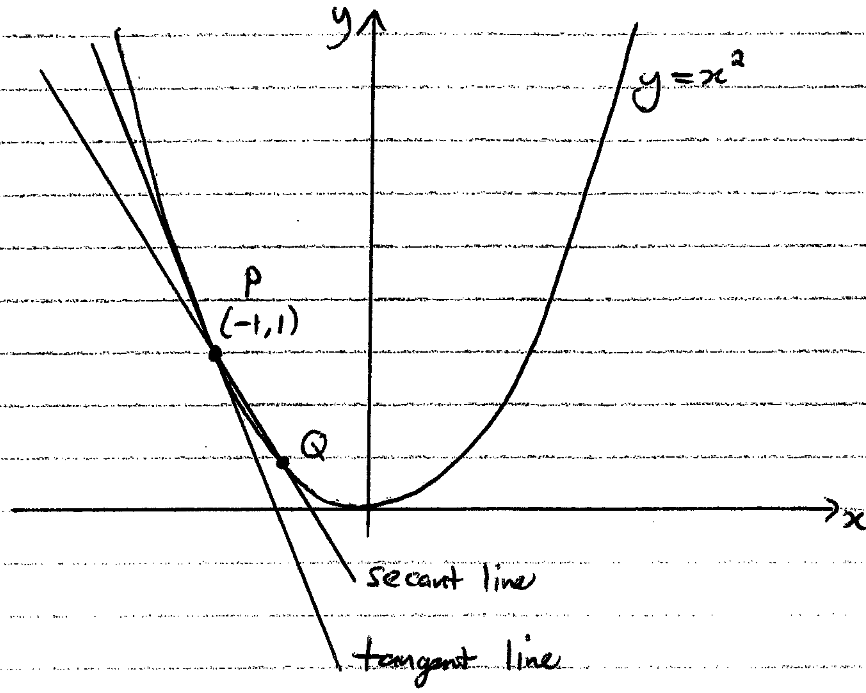
the advantage - because we know P and Q, we can calculate the slope of the secant line exactly
also, notice that if we take Q closer to P, the better the secant line approximates the tangent line

this is how we do it:
the slope of the tangent line to a curve at a point P is the limiting slope of the secant line PQ as the point Q slides along the curve towards P

we'll illustrate the method by example:
let's find the slope of the tangent line to the parabola $y = x^2$ at the point $(-1, 1)$

so we have $P = (-1,1)$ and we want Q to be a nearby point on the curve so that we can calculate the slope of the secant line PQ
take $Q = (x, y)$, where $x = -1 + h$ for some small

(nonzero) number h, then $y = (-1+h)^2 = h^2 - 2h + 1$
so $Q = (-1+h,\ 1+h^2-2h)$



then the slope of the secant line is

$$\frac{\Delta y}{\Delta x} = \frac{(1+h^2-2h)-1}{(-1+h)-(-1)} = \frac{h^2-2h}{h} = h-2$$

now, we'd like to "slide" the point Q closer to P;
this corresponds to taking smaller values for h
ie we want to let h get closer and closer to 0
the notation we'll use for "h approaches 0" is $h \to 0$

the slope of the tangent line is then the limiting value
of the slope of the secant line $\Delta y/\Delta x$ as $h \to 0$
we write this as

$$m = \lim_{h\to 0} \frac{\Delta y}{\Delta x}$$

$$= \lim_{h\to 0} (h-2)$$

well, as $h \to 0$, $h-2$ approaches -2, so

$$m = \lim_{h\to 0} (h-2) = -2$$

and the slope of the tangent line is -2

now, let's generalize this method for any function $f(x)$ and point $P = (a, f(a))$

take $Q = (x, y) = (a+h, f(a+h))$ (also on curve)

then the slope of the secant line PQ is

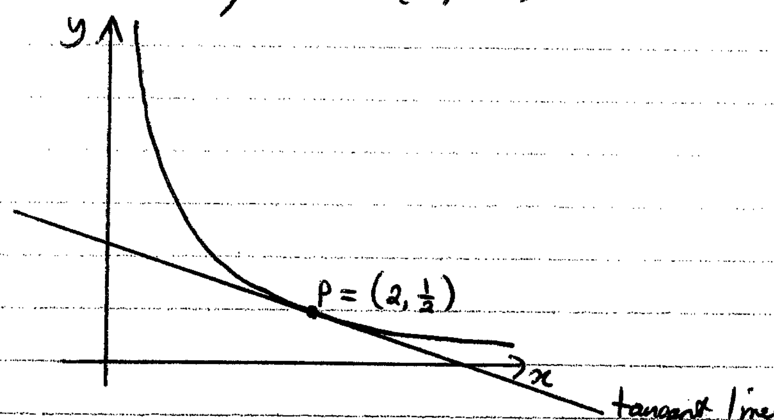
$$\frac{\Delta y}{\Delta x} = \frac{f(a+h) - f(a)}{(a+h) - a} = \frac{f(a+h) - f(a)}{h}$$

which is called <u>the difference quotient</u>

the <u>slope of the tangent line</u> to the graph of $y = f(x)$ at the point $P = (a, f(a))$ is

$$\boxed{m = \lim_{h \to 0} \frac{\Delta y}{\Delta x} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}}$$ (p. ~~141~~ 136)

example: find the slope of the tangent line to $y = 1/x$ at the point $(2, 1/2)$



$$m = \lim_{h \to 0} \frac{\Delta y}{\Delta x} = \lim_{h \to 0} \frac{f(2+h) - f(2)}{h}$$

$$= \lim_{h \to 0} \frac{\frac{1}{2+h} - \frac{1}{2}}{h}$$

$$= \lim_{h \to 0} \frac{\frac{2 - (2+h)}{2(2+h)}}{h}$$

$$= \lim_{h \to 0} \frac{-h}{h\,2(2+h)}$$

$$= \lim_{h \to 0} \frac{-1}{2(2+h)} = \lim_{h \to 0} \frac{-1}{4+2h} = \frac{-1}{4}$$

when we talk about the function $y = f(x)$, we call x the <u>independent variable</u> and y the <u>dependent variable</u> because the value of y depends on the value of x through the function $f(x)$

often we are interested in knowing how rapidly y changes when there is a change in x
(changes in x will produce a change in y, unless $f(x)$ is constant)
this concept is called "rate of change"
and we can ask about the rate of change of the price of something as time passes
or the rate of change of temperature as elevation changes (say you're climbing a mountain)
and so on...

the best place to start is by discussing a very familiar rate of change - velocity

let's say that we drove 500 km in 6 hours, then our <u>average velocity</u> was $\frac{500\text{ km}}{6\text{ hr}} = 83.\bar{3}$ km/hr

but this doesn't tell us what our speed was at any given instant - it is highly unlikely that we were able to drive at the same speed the whole time
so how could we find our <u>instantaneous velocity</u> - ie our speed at any particular time?

we do this by recognizing that velocity is the rate of change of position with respect to time

also, we usually drop the "instantaneous" and speak of "velocity"

again, it's best to illustrate by example:
if an object is dropped from rest, its position, after t seconds, will be $s(t) = \frac{1}{2} g t^2$, where g is the acceleration of gravity

io after t seconds, the object will have fallen
$\frac{1}{2}gt^2$ metres

$g = 9.8\ m/s^2$ (often just use 10)
so $s(t) = 4.9t^2$

let's say we have a sky-diver who jumps from a plane at a height of 10 000 m
what is her average velocity between $t = 1s$ and $t = 5s$

$$\text{average velocity} = \frac{\text{distance fallen}}{\text{time}} = \frac{\Delta s}{\Delta t}$$

$$= \frac{s(5) - s(1)}{5-1}$$

$$= \frac{4.9(5)^2 - 4.9(1)^2}{4} \quad (m/s)$$

$$= 29.4\ m/s$$

what is her instantaneous velocity at time $t = 2s$?
how could we find this?
by taking a short time interval between $t = 2s$ and $t = 2 + \Delta t\ s$ (ie time interval $= \Delta t$)
and finding her average velocity over this interval

$$\text{average velocity} = \frac{\Delta s}{\Delta t} = \frac{s(2+\Delta t) - s(2)}{\Delta t}$$

$$= \frac{4.9(2+\Delta t)^2 - 4.9(2)^2}{\Delta t} \quad (m/s)$$

$$= \frac{4.9(4 + 4\Delta t + (\Delta t)^2 - 4)}{\Delta t}$$

$$= \frac{4.9(4\Delta t + (\Delta t)^2)}{\Delta t}$$

$$= 4.9(4 + \Delta t) = 19.6 + (4.9)\Delta t \quad m/s$$

but this is still an average velocity over time interval Δt
how do we get instantaneous velocity at time $t = 2$?
by finding the limiting value of this average as $\Delta t \to 0$

ie $v = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \lim_{\Delta t \to 0} 19.6 + 4.9\Delta t = 19.6\ m/s$

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so, at time $t = 2s$, the sky-diver is falling with a velocity of 19.6 m/s

we can generalize this to the case of any position function $s(t)$ (see page ~~142~~ 137)

the velocity of an object with position function $s(t)$ at time t_0 is $v(t_0) = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \lim_{\Delta t \to 0} \frac{s(t_0+\Delta t) - s(t_0)}{\Delta t}$ $\left(\begin{matrix} t_0 = a \\ \Delta t = h \end{matrix}\right)$

looks familiar?

it should because it's just the slope of the tangent line to the curve $s(t)$ at $t = t_0$

let's generalize even more to consider the rate of change of function $y = f(x)$

if x changes from a to $a + \Delta x$, then y changes from $f(a)$ to $f(a+\Delta x)$

so a change of Δx in x produces a change of $\Delta y = f(a+\Delta x) - f(a)$ in y

then the difference quotient $\frac{\Delta y}{\Delta x} = \frac{f(a+\Delta x) - f(a)}{\Delta x}$ $(\Delta x = h)$ would (p ~~143~~ 139–140) represent the average rate of change of f (or y) over the interval from a to $a + \Delta x$

we can get the instantaneous rate of change by taking $\Delta x \to 0$

so the rate of change of $y = f(x)$ with respect to x when $x = a$ is $\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(a+\Delta x) - f(a)}{\Delta x}$

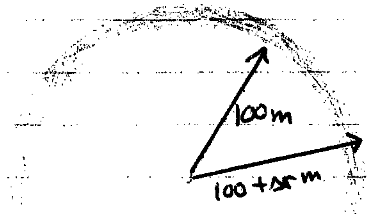
so the rate of change of $f(x)$ at $x = a$ is equal to the slope of the tangent line at the point $(a, f(a))$

(another reason why we are interested in these slopes)

~~Example of p 143~~

a circular oil slick on the ocean is spreading outwards

find the rate of change of the area of the oil slick when its radius is 100 m (with respect to radius



the area of a circle is
$A = \pi r^2 \quad (m^2)$

the change in area when the radius changes from 100 to $100+\Delta r$
is $\Delta A = \pi\left((100+\Delta r)^2 - (100)^2\right)$
$= \pi\left(10000 + 200\,\Delta r + (\Delta r)^2 - 10000\right)$
$= \pi\left(200\,\Delta r + (\Delta r)^2\right)$

so the average rate of change of area with respect to radius is $\dfrac{\Delta A}{\Delta r} = \dfrac{\pi\left(200\,\Delta r + (\Delta r)^2\right)}{\Delta r} = 200\pi + \Delta r$

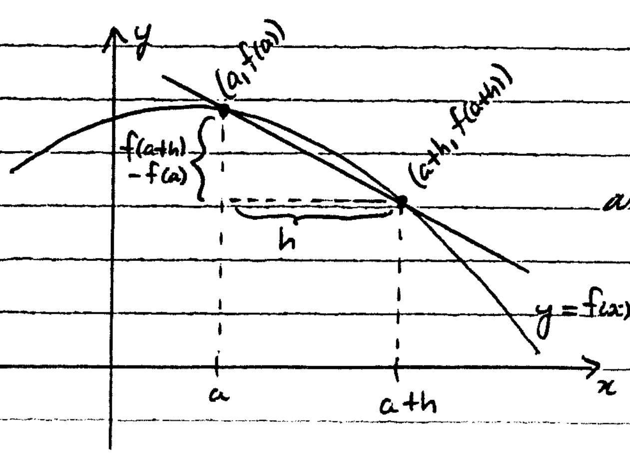
so the instantaneous rate of change is $\displaystyle\lim_{\Delta r\to 0} \frac{\Delta A}{\Delta r}$

$\displaystyle = \lim_{\Delta r\to 0} 200\pi + \Delta r = 200\pi$

so, when the radius is 100 m, the rate of change of area with respect to radius is 200π m²/m
(note that this rate of change is equal to the circumference of the circle $C = 2\pi r$)

§2.7: Derivatives

the average rate of change of f over the interval from a to $a+h$
is $\dfrac{f(a+h) - f(a)}{h}$ ← this is called the difference quotient

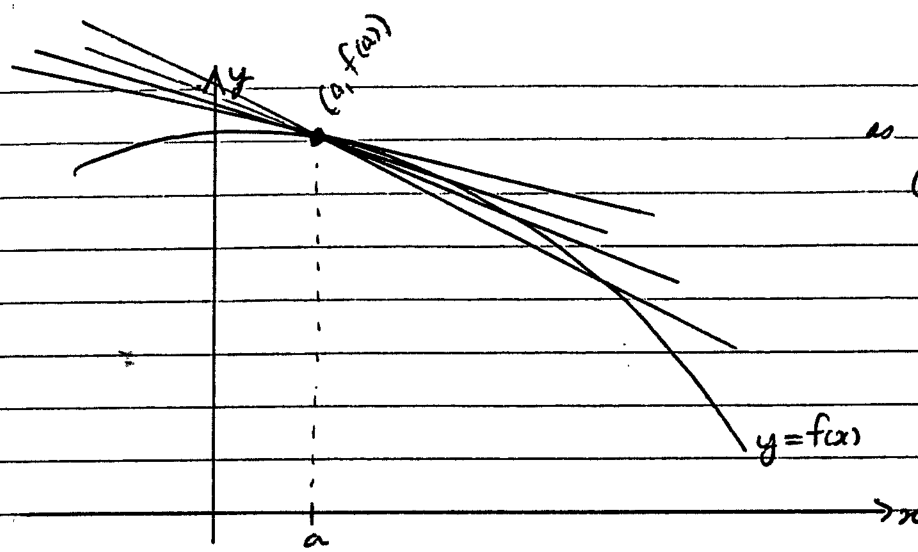


average rate of change is the slope of this secant line the instantaneous rate of change of $f(x)$ at $x=a$ is also called the derivative of f at a, denoted $f'(a)$

$$f'(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$ (p. 148) 140

if the limit exists, we say f is differentiable at a

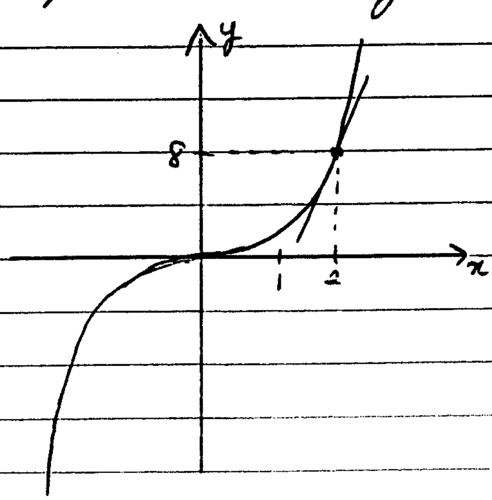
when the function is $y=s(t)$, a position function, the derivative at a is $s'(a)=v(a)$, the velocity at time a



as we slide the point $(a+h, f(a+h))$ towards the point $(a, f(a))$ (ie as $h\to 0$), the secant line becomes the tangent line to the curve at $x=a$

so the derivative of f at $x=a$ is
i) the slope of the curve at $(a, f(a))$
ii) the rate of change of f at a
iii) the slope of the tangent line to the curve at $(a, f(a))$

example: consider $y=f(x)=x^3$



what is $f'(2)$?
since the tangent line would have a positive slope, we know that $f'(2)>0$

$$f'(2)=\lim_{h\to 0}\frac{f(2+h)-f(2)}{h}$$
$$=\lim_{h\to 0}\frac{(2+h)^3-(2)^3}{h}$$
$$=\lim_{h\to 0}\frac{(2+h)(4+4h+h^2)-8}{h}$$

what is the equation of the tangent line to $y=f(x)=x^3$ at $x=2$?

$$= \lim_{h\to 0} \frac{8+12h+6h^2+h^3-8}{h}$$

$$= \lim_{h\to 0} \frac{12h+6h^2+h^3}{h}$$

$$= \lim_{h\to 0} 12+6h+h^2 = 12$$

slope is $m=12$, point $(2,8)$

$y-y_0 = m(x-x_0)$

$y-8 = 12(x-2)$

$y = 12x-16$

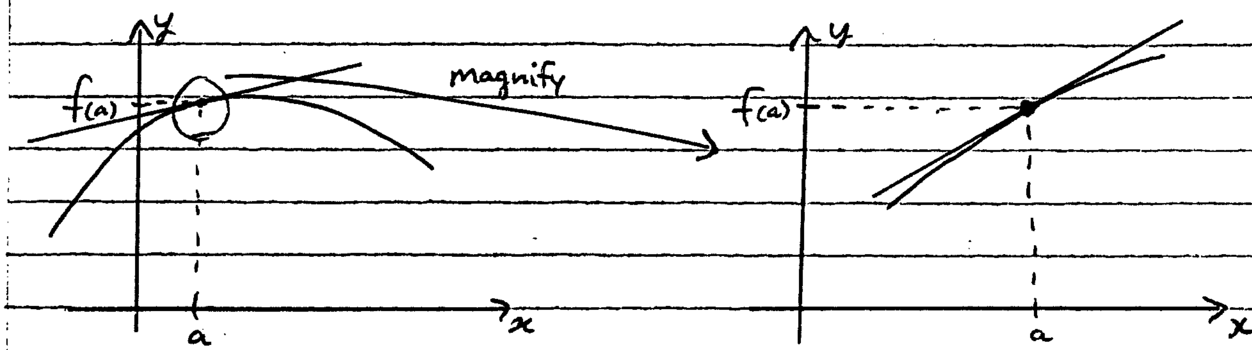
§ ~~2.8~~ 2.7 the Derivative as a Function

we have seen the derivative of a function at a point $f'(a)$
obviously, we can talk about $f'(a)$ for any a where the limit

$$\lim_{h\to 0} \frac{f(a+h)-f(a)}{h} \text{ exists}$$

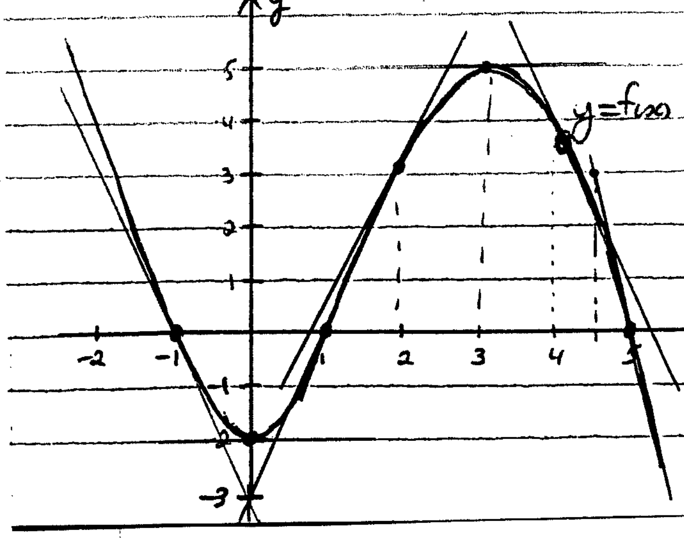
and we know that $f'(a)$ is the slope of the curve at that point, the rate of change, or the slope of the tangent line to the curve at that point

and, we can also see the fact that if we zoom in on the graph of $y=f(x)$ closely enough, the curve will appear linear



ie if we zoom in close enough, the curve and the tangent line to the curve at the point $(a, f(a))$ become indistinguishable near $(a, f(a))$
(and so the tangent line can be used to approximate the curve)

consider the following graph of $y = f(x)$



if we try to estimate the value of the derivative at $x = -1, 0, 1, 2, 3, 4, 5$ we'll see the following (rough estimates only)

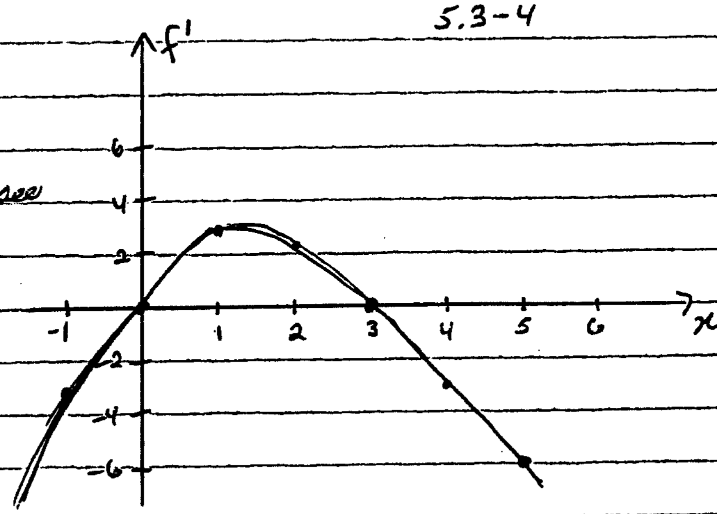
(can't make good estimates from hand-drawn graph)

$f'(-1) \approx \dfrac{-3-0}{0--1} = -3$ $\quad$ $f'(0) = 0$, $\quad$ $f'(1) \approx \dfrac{0--3}{1-0} = 3$,

$f'(2) \approx \dfrac{3-0}{2-0.8} = 2.5$ $\quad$ $f'(3) = 0$, $\quad$ $f'(4) \approx \dfrac{0-3.5}{5.3-4} \approx -2.7$,

$f'(5) \approx \dfrac{0-3}{5-4.5} = -6$

so if we graph f' we see



so the derivative of $f(x)$ is also a function of x

the derivative function f', which is the rate of change of f at x, is
$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$ (p155) 146

if $f'(x)$ exists, we say that f is differentiable at that point x
if f is differentiable for all x in its domain, we say that f is differentiable everywhere (p160) 150

examples:
i, if $y = f(x) = b$, then $f'(x) = 0$ because horizontal line has slope 0

ii, if $y = f(x) = mx + b$, then $f'(x) = m$, the slope of the line, iii, if $y = f(x) = x^2$, then $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$

$$= \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h\to 0} \frac{x^2+2xh+h^2-x^2}{h}$$

$$= \lim_{h\to 0} \frac{2xh+h^2}{h} = \lim_{h\to 0} (2x+h) = 2x$$

we have used the notation $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ for the derivative
(Newton')

there is another notation, from Leibniz, that follows from (p159) 150

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\Delta f}{h} = \lim_{\Delta x\to 0} \frac{\Delta f}{\Delta x} = \lim_{\Delta x\to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx}$$

so if $y = f(x)$, we have $f'(x) = \frac{dy}{dx} = \frac{df}{dx}$

if $s(t)$ is the position function of a moving object, then its velocity is $v(t) = s'(t) = \frac{ds}{dt}$

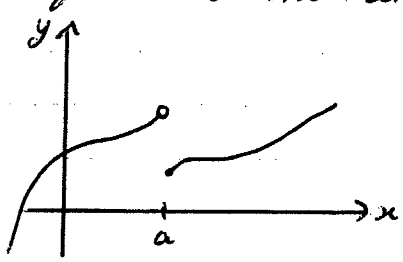
we can also think of $\frac{d}{dx}$ as an operator that "takes the derivative with respect to x"

ie $\frac{dy}{dx} = \frac{d}{dx}(y)$

when we evaluate the derivative at a specific x, the notation is $f'(2) = \left.\frac{dy}{dx}\right|_{x=2} = \left.\frac{df}{dx}\right|_{x=2}$

so how does a function fail to be differentiable at a point? (p162) 152

i, if the function is not continuous at the point



$f'(a)$ does not exist

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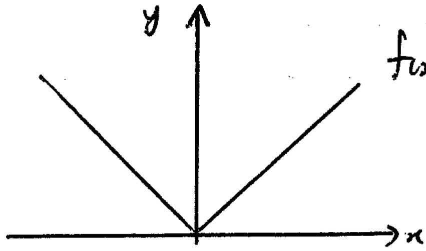
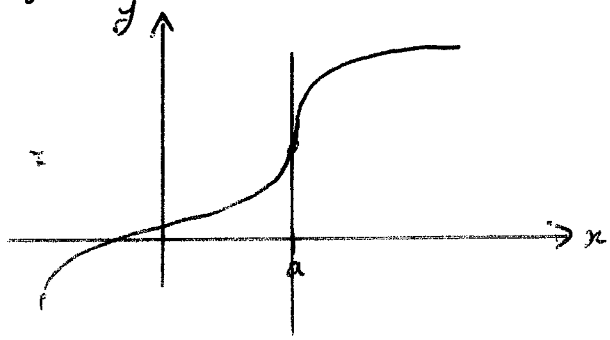
ii, the graph has a sharp corner (cusp) at the point



$$f'(0) = \lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0} \frac{|h|}{h} \quad \text{does not exist}$$

(since the one-sided limits do not agree

iii, if the function has a vertical tangent line at the point



$f'(a)$ does not exist
(ie slope is infinite)

notice that examples ii, and iii, are continuous at the point
so $f(x)$ continuous at $x=a$ $\not\Rightarrow$ $f(x)$ differentiable at $x=a$
But $f(x)$ differentiable at $x=a$ $\Rightarrow$ $f(x)$ continuous at $x=a$ (p161)

we've seen that $f'(x)$ is a function of x, so we can look at its derivative $(f'(x))'$, which is the derivative of the derivative of $f(x)$ or the <u>second derivative</u> of $f(x)$, which we denote by $f''(x)$ (p163) 153

if $y=f(x)$, we have $\frac{dy}{dx} = f'(x)$

and now $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(f'(x)) = f''(x)$

the third derivative is $f'''(x) = \frac{d^3y}{dx^3}$

and so on...

we've seen that velocity $v(t)$ is the rate of change of position $s(t)$

ie $v(t) = s'(t) = \lim_{h \to 0} \frac{s(t+h) - s(t)}{h}$

acceleration is the rate of change of velocity (w.r.t. time) (p 163) 153

and so $a(t) = v'(t) = \lim_{h \to 0} \frac{v(t+h) - v(t)}{h} = s''(t)$

§ ~~2.9~~ 2.8: What Does f' Say About f?

if $f'(x) > 0$ on an interval, then $f(x)$ is increasing on that interval
if $f'(x) < 0$ on an interval, then $f(x)$ is decreasing on that interval (p 164) 153

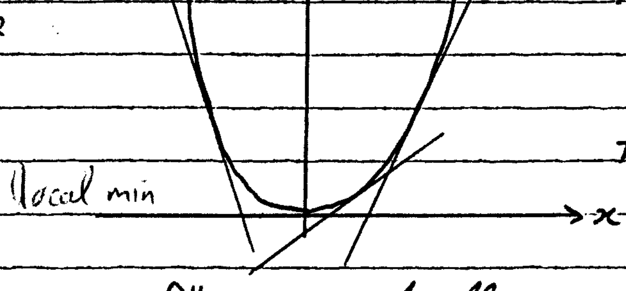
so what does $f''(x) > 0$ on some interval mean?
it would mean that $f'(x)$ is increasing on that interval
since f' is the rate of change of f, f'' would be the rate of change of the rate of change
so if $f'' > 0$, we'd have that the rate of change of the rate of change of f is positive, this corresponds to the graph bending upwards
so if $f'' > 0$ on some interval, the graph is concave up on that interval (p170) 159

if $f''(x) < 0$, then $f'(x)$ is decreasing and the rate of change of the rate of change of f is negative, the graph would bend down
so if $f'' < 0$ on some interval, the graph is concave down on that interval

examples:

i) $y = f(x) = x^2$

$f'(x) = 2x$
$f''(x) = 2$

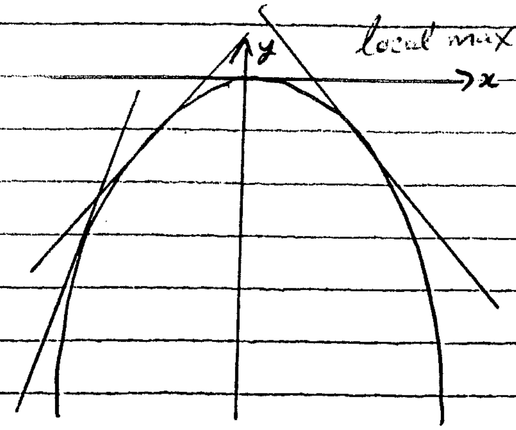


$f'(x) < 0$ if $x < 0$
decreasing

$f'(x) > 0$ if $x > 0$
increasing

$f''(x) > 0$ for all $x \Rightarrow$ always concave up

notice that this means tangent lines lie below the curve ii, $y = f(x) = -x^2$



$f'(x) = -2x$
$f''(x) = -2$

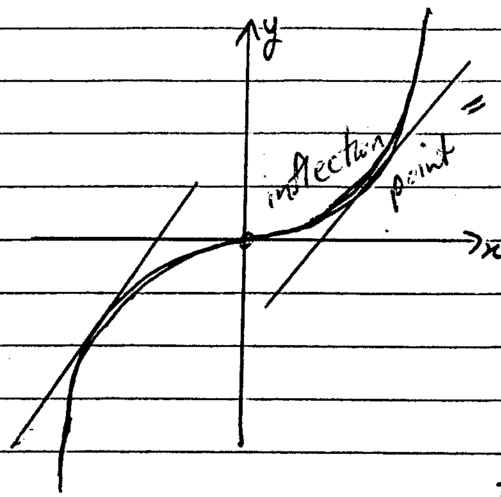
$f'(x) > 0$ if $x < 0$
increasing

$f'(x) < 0$ if $x > 0$
decreasing

$f''(x) < 0$ always concave down

notice that tangent lines lie above the curve

iii, $y = f(x) = x^3$



$f'(x) = 3x^2$
$f''(x) = 6x$

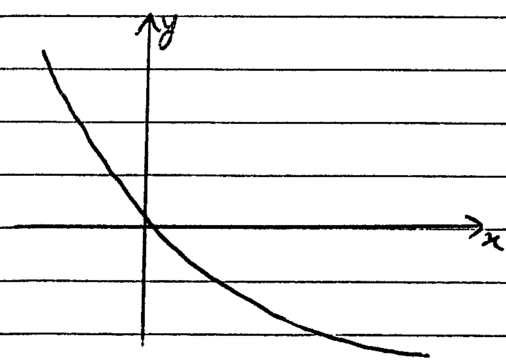
$f'(x) > 0$ for all $x \neq 0$
always increasing
(except $x=0$)

$f''(x) < 0$ if $x < 0$
concave down

$f''(x) > 0$ if $x > 0$
concave up

note : increasing/decreasing and concave up/down are independent

iv,



this function is decreasing, so $f'(x) < 0$
but it is concave up, so $f''(x) > 0$

ie the rate at which the function is decreasing is "becoming more positive" (increasing)

($|f'(x)|$ is decreasing)

Chapter 3: Differentiation Rules

§3.1: Derivatives of Polynomials and Exponential Functions

derivative of a constant $\frac{d}{dx}(c) = 0$ (p ~~183~~ 174)

Constant Multiple: $\frac{d}{dx}(c f(x)) = c f'(x)$ (p 180)

Sums and Differences: $\frac{d}{dx}[f(x) \pm g(x)] = f'(x) \pm g'(x)$ (p ~~187~~ 177-8)

Power Rule: $\frac{d}{dx}(x^n) = n x^{n-1}$ (any real n) (p ~~185~~ 175)

examples:

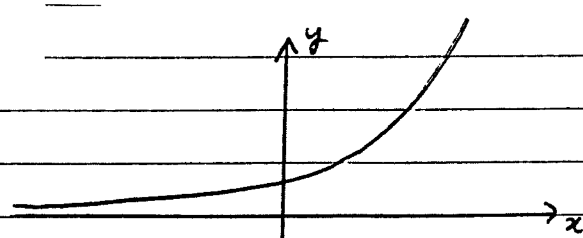
i) $\frac{d}{dx}(3x^2) = 6x$

ii) $\frac{d}{dx}(4x^{5/7}) = \frac{20}{7} x^{-2/7}$

iii) $\frac{d}{dx}\left(7x^2 - \frac{5}{x} + 2\right) = 14x + \frac{5}{x^2}$

iv) $\frac{d}{dx}\left(5x^4 + 2x^{\pi} + \sqrt{2}\, x^{1/3}\right) = 20x^3 + 2\pi x^{\pi-1} + \frac{\sqrt{2}}{3} x^{-2/3}$

see also Ex 1-7 p ~~184-8~~ 175-8



consider $y = f(x) = a^x$ $(a > 1)$

we know that the function is always increasing, so $f'(x) > 0$
and it's always concave up, so $f''(x) > 0$, too
so $f'(x)$ starts small and gets bigger (much like $f(x)$)
ie we can guess that a graph of $f'(x)$ would look like a graph of $f(x)$

if $y = f(x) = a^x$, then $f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$

$$= \lim_{h\to 0} \frac{a^{x+h} - a^x}{h}$$

(p ~~188~~) 179

$$= \lim_{h\to 0} \frac{a^x a^h - a^x}{h} = \lim_{h\to 0} a^x \frac{a^h - 1}{h} = a^x \lim_{h\to 0} \frac{a^h-1}{h}$$

now, $\lim_{h\to 0} \frac{a^h-1}{h}$ will be a constant, in fact $f'(0) = \lim_{h\to 0} \frac{a^h-1}{h}$

(depends on a)

so $f'(x) = f'(0)\, a^x$ ie the derivative is a constant times the function, so they have the same behaviour

but we need to figure out what $f'(0)$ is going to be

we do this in a neat way: is there a base a such that $f'(0) = 1$?

ie we want to find the a such that $\lim_{h\to 0} \frac{a^h-1}{h} = 1$

then if h is very small $\frac{a^h-1}{h} \approx 1$ or $a \approx (1+h)^{1/h}$

if we take the limit as $h\to 0$ $a = \lim_{h\to 0} (1+h)^{1/h} = e$

or $\lim_{h\to 0} \frac{e^h - 1}{h} = 1$ (p ~~189~~) 180

numerically, $e \simeq 2.718281\ldots$

so we have shown that if $y = f(x) = e^x$, then $f'(x) = e^x$

or $\frac{d}{dx}(e^x) = e^x$ (p ~~190~~) 180

example: $\frac{d}{dx}\left(2e^x + x^2 - 3x\right) = 2e^x + 2x - 3$

see also Ex 8 29 p ~~190~~ 180

§ 3.2: the Product and Quotient Rules

how do we find the derivative of a product?
it's tempting to think $\frac{d}{dx}(f(x)g(x)) = f'(x)g'(x)$, but that would not be right

example: if $f(x) = 2x^3$ and $g(x) = x^2$
then $f'(x) = 6x^2$ and $g'(x) = 2x$
so $f'(x)g'(x) = (6x^2)(2x) = 12x^3$
but $f(x)g(x) = (2x^3)(x^2) = 2x^5$ so $\frac{d}{dx}(f(x)g(x)) = 10x^4$

if $p(x) = f(x)g(x)$, then

$$p'(x) = \frac{d}{dx}(f(x)g(x)) = \lim_{h\to 0} \frac{p(x+h)-p(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x) + f(x)g(x+h) - f(x)g(x+h)}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x+h) + f(x)g(x+h) - f(x)g(x)}{h}$$

$$= \lim_{h\to 0} \left[\frac{f(x+h)g(x+h) - f(x)g(x+h)}{h} + \frac{f(x)g(x+h) - f(x)g(x)}{h}\right]$$

$$= \lim_{h\to 0} \left[\frac{f(x+h)g(x+h) - f(x)g(x+h)}{h}\right] + \lim_{h\to 0} \left[\frac{f(x)g(x+h) - f(x)g(x)}{h}\right]$$

$$= \lim_{h\to 0} \left[\left(\frac{f(x+h)-f(x)}{h}\right)\left(g(x+h)\right)\right] + \lim_{h\to 0}\left[f(x)\left(\frac{g(x+h)-g(x)}{h}\right)\right]$$

$$= \left(\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}\right)\left(\lim_{h\to 0} g(x+h)\right) + \left(f(x)\right)\left(\lim_{h\to 0} \frac{g(x+h)-g(x)}{h}\right)$$

$$= f'(x)g(x) + f(x)g'(x)$$

ie if $u = f(x)$ and $v = g(x)$, the Product Rule says

$$\frac{d}{dx}(f(x)g(x)) = (f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$$

or $\frac{d}{dx}(uv) = \frac{du}{dx}\cdot v + u\cdot\frac{dv}{dx}$ (p 194)

examples :

i) $\frac{d}{dx}(x^3 e^x) = \left(\frac{d}{dx}(x^3)\right)e^x + (x^3)\left(\frac{d}{dx}(e^x)\right) = 3x^2e^x + x^3e^x = (3x^2+x^3)e^x$

ii) $\frac{d}{dx}(2\sqrt{x}\,e^x) = \frac{2}{2\sqrt{x}}\,e^x + 2\sqrt{x}\,e^x = \left(\frac{1}{\sqrt{x}} + 2\sqrt{x}\right)e^x$

see also Ex 1-4 p ~~194-6~~ 185-6

how do we find $Q'(x)$ if $Q(x) = \frac{f(x)}{g(x)}$?

easy, rewrite as $f(x) = Q(x)g(x)$, then $f'(x) = Q'(x)g(x) + Q(x)g'(x)$

so $Q'(x) = \frac{f'(x) - Q(x)g'(x)}{g(x)} = \frac{f'(x) - \frac{f(x)}{g(x)}g'(x)}{g(x)}$

or $Q'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$ the Quotient Rule (p ~~197~~) 187

or $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{\frac{du}{dx}\cdot v - u\cdot\frac{dv}{dx}}{v^2}$

examples :

i) $\frac{d}{dx}\left(\frac{e^x}{x}\right) = \frac{d}{dx}(x^{-1}e^x) = -x^{-2}e^x + x^{-1}e^x = (x^{-1} - x^{-2})e^x$

$= \frac{\left(\frac{d}{dx}(e^x)\right)(x) - (e^x)\left(\frac{d}{dx}(x)\right)}{(x)^2} = \frac{xe^x - e^x}{x^2}$ (same)

ii) $\frac{d}{dx}\left(\frac{3x^2+1}{7x+2}\right) = \frac{\left(\frac{d}{dx}(3x^2+1)\right)(7x+2) - (3x^2+1)\left(\frac{d}{dx}(7x+2)\right)}{(7x+2)^2}$

$= \frac{(6x)(7x+2) - (3x^2+1)(7)}{(7x+2)^2}$

$= \frac{42x^2 + 12x - 21x^2 - 7}{(7x+2)^2} = \frac{21x^2 + 12x - 7}{(7x+2)^2}$

see also Ex 5 & 6 p ~~197~~ 187-8

②

the derivative of a constant is $\frac{d}{dx}(c) = 0$

the Power Rule $\frac{d}{dx}(x^n) = nx^{n-1}$ for any real number n

the Constant Multiple Rule $\frac{d}{dx}(c\,f(x)) = c\,f'(x)$

the Sum and Difference Rules $\frac{d}{dx}\left(f(x) \pm g(x)\right) = f'(x) \pm g'(x)$

the derivative of the exponential function $\frac{d}{dx}(e^x) = e^x$

the Product Rule $\frac{d}{dx}\left(f(x)g(x)\right) = f'(x)g(x) + f(x)g'(x)$

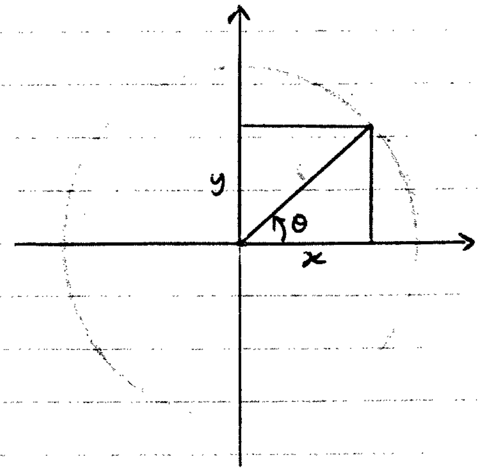
the Quotient Rule $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$

Chapter 3: Differentiation Rules

§ ~~3.4~~ 3.3 Derivatives of Trigonometric Functions

the trig functions are reviewed in Appendix C (p A18) 17

recall that we always use radians in calculus (the derivative formulas that we are about to find are only true for radians)



$$\sin\theta = \frac{y}{\sqrt{x^2+y^2}} \qquad \cos\theta = \frac{x}{\sqrt{x^2+y^2}}$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = y/x$$

$$\csc\theta = \frac{1}{\sin\theta}, \quad \sec\theta = \frac{1}{\cos\theta}, \quad \cot\theta = \frac{1}{\tan\theta}$$

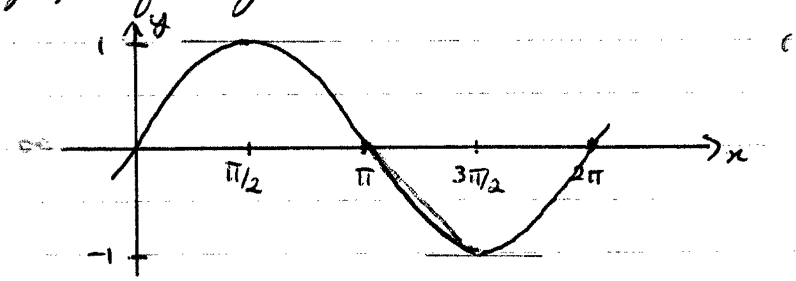
$$\cos^2\theta + \sin^2\theta = 1$$

sin and cos are 2π-periodic ie $\sin(\theta + 2\pi n) = \sin\theta$
$\cos(\theta + 2\pi m) = \cos\theta$

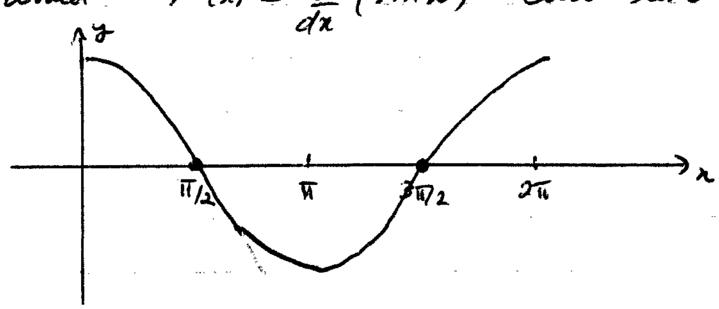
sin is an odd function $\sin(-\theta) = -\sin\theta$
cos is an even function $\cos(-\theta) = \cos\theta$

and $\sin(\theta \pm \phi) = \sin\theta\cos\phi \pm \cos\theta\sin\phi$
$\cos(\theta \pm \phi) = \cos\theta\cos\phi \mp \sin\theta\sin\phi$, etc...

the graph of $y = f(x) = \sin x$ looks like



what would $f'(x) = \frac{d}{dx}(\sin x)$ look like?



(p ~~213~~) 191

it looks like $\frac{d}{dx}(\sin x) = \cos x$, let's verify that if $f(x) = \sin x$, then

$$\frac{d}{dx}(\sin x) = f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h}$$

$$= \lim_{h \to 0} \frac{\sin x \cosh + \cos x \sinh - \sin x}{h} \qquad \text{(trig identity)}$$

$$= \lim_{h \to 0} \frac{\sin x (\cosh - 1) + \cos x \sinh}{h}$$

$$= \lim_{h \to 0} \sin x \left(\frac{\cosh - 1}{h}\right) + \lim_{h \to 0} \cos x \left(\frac{\sinh}{h}\right)$$

$$= \sin x \left(\lim_{h \to 0} \frac{\cosh - 1}{h}\right) + \cos x \left(\lim_{h \to 0} \frac{\sinh}{h}\right)$$

what are these limits?
the textbook gives a geometric argument on pages ~~214-5~~ 192

another way to get them:
if h is very small (and in radians), then
$\cosh \approx 1 - \frac{1}{2}h^2$ and $\sinh \approx h - \frac{1}{6}h^3$

and these approximations get better and better as $h \to 0$

so $\lim_{h \to 0} \frac{\cosh - 1}{h} = \lim_{h \to 0} \frac{1 - \frac{1}{2}h^2 - 1}{h} = \lim_{h \to 0} \frac{-\frac{1}{2}h^2}{h} = \lim_{h \to 0} -\frac{1}{2}h = 0$

and $\lim_{h \to 0} \frac{\sinh}{h} = \lim_{h \to 0} \frac{h - \frac{1}{6}h^3}{h} = \lim_{h \to 0} 1 - \frac{1}{6}h^2 = 1$

$\therefore \frac{d}{dx}(\sin x) = \sin x(0) + \cos x(1) = \cos x$ (p ~~215~~) 193

and in a very similar way, we could show that
$\frac{d}{dx}(\cos x) = -\sin x$ (p ~~216~~) 193

since $\tan x = \dfrac{\sin x}{\cos x}$, we can use Quotient Rule to show

$$\frac{d}{dx}(\tan x) = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) = \frac{\left(\frac{d}{dx}(\sin x)\right)(\cos x) - (\sin x)\left(\frac{d}{dx}(\cos x)\right)}{(\cos x)^2}$$

$$= \frac{(\cos x)(\cos x) - (\sin x)(-\sin x)}{(\cos x)^2}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

$$= \frac{1}{\cos^2 x} = \sec^2 x \qquad (p\ \cancel{210})\ 194$$

we can use QR to find

$$\frac{d}{dx}(\sec x) = \frac{d}{dx}\left(\frac{1}{\cos x}\right) = \frac{(0)(\cos x) - (1)(-\sin x)}{\cos^2 x} = \frac{\sin x}{\cos^2 x} = \sec x \tan$$

~~(§3.4 p 223 #14)~~

$$\frac{d}{dx}(\csc x) = -\csc x \cot x \qquad \cancel{(\#13)}$$

$$\frac{d}{dx}(\cot x) = -\csc^2 x \qquad \cancel{(\#15)}$$

examples:

i, if $g(t) = t^2 \cos t$, $g'(t) = \left(\frac{d}{dt}(t^2)\right)(\cos t) + (t^2)\left(\frac{d}{dt}(\cos t)\right)$ Product Rule

$$= 2t\cos t - t^2 \sin t$$

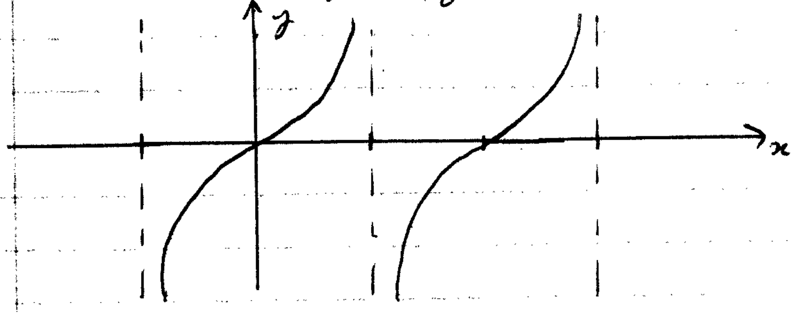
ii, $f(x) = \dfrac{e^x}{\sin x + \cos x}$,
$$f'(x) = \frac{e^x(\sin x + \cos x) - e^x(\cos x - \sin x)}{(\sin x + \cos x)^2}$$

$$= \frac{2e^x \sin x}{\sin^2 x + 2\sin x\cos x + \cos^2 x}$$

$$= \frac{2e^x \sin x}{1 + 2\sin x \cos x} = \frac{2e^x \sin x}{1 + \sin(2x)}$$

⑥

iii, a) where does the graph of $\tan x$ have a horizontal tangent?



it looks like $\tan x$ is always increasing

since $\frac{d}{dx}(\tan x) = \sec^2 x$ and $|\sec x| \geq 1$, we have

that $\frac{d}{dx}(\tan x) \geq 1$ for all x

so the graph of $\tan x$ is always increasing and there are no horizontal tangents

b) what is the equation of the tangent line to the graph of $y = \tan x$ at $x = \pi$? (pt is $(\pi, 0)$)

$\frac{dy}{dx} = \sec^2 x$ so the slope of the tangent is $m = \sec^2(\pi) = 1$

so the equation of the line is $y - 0 = (1)(x - \pi)$
or $y = x - \pi$

see also Ex 1-4 p ~~216-8~~ 193-5

§ ~~3.5~~ 3.4 the Chain Rule

recall that if we have two functions $f(x)$ and $g(x)$ we can <u>compose</u> them to form the <u>composite</u> functions

$F(x) = (f \circ g)(x) = f(g(x))$ (f is outer, g is inner)

and $G(x) = (g \circ f)(x) = g(f(x))$ (g is outer, f is inner)

but these are not the same function! the order matters

eg if $f(x) = x^2$ and $g(x) = \sqrt{x+1}$

then $(f \circ g)(x) = f(g(x)) = f(\sqrt{x+1}) = (\sqrt{x+1})^2 = x+1$

and $(g \circ f)(x) = g(f(x)) = g(x^2) = \sqrt{x^2+1}$

the Chain Rule will tell us what $\frac{d}{dx}(f(g(x)))$ is (and hence will allow us to differentiate many more functions)

the <u>Chain Rule</u> says that if f and g are both differentiable then their composition is differentiable and

(p ~~220~~ 198) $\frac{d}{dx}(f(g(x))) = f'(g(x))\,g'(x)$ (prime notation)

or if $y = f(u)$ where $u = g(x)$ (so $y = f(g(x))$)

then $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$ (Leibniz notation)

are these saying the same thing?

yes, both say that $\frac{d}{dx}(f(g(x))$ = derivative of outer function (f) (evaluated at inner) × derivative of inner function (g)

how? if $y = f(u)$ and $u = g(x)$, then obviously $y = f(g(x))$, so $\frac{dy}{dx} = \frac{d}{dx}(f(g(x)))$

but $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \frac{df}{du}\frac{dg}{dx} = f'(u)\,g'(x) = f'(g(x))\,g'(x)$

we can demonstrate easily that the Chain Rule should be true:

if $y = f(u)$ where $u = g(x)$

then a change of Δx in x produces a change

$\Delta u = g(x+\Delta x) - g(x)$ in u

so then $\frac{dy}{dx} = \lim_{\Delta x \to 0}\frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0}\frac{\Delta y}{\Delta x}\frac{\Delta u}{\Delta u} = \lim_{\Delta x \to 0}\frac{\Delta y}{\Delta u}\frac{\Delta u}{\Delta x}$

(p ~~221~~ 198) $= \left(\lim_{\Delta x \to 0}\frac{\Delta y}{\Delta u}\right)\left(\lim_{\Delta x \to 0}\frac{\Delta u}{\Delta x}\right) = \left(\lim_{\Delta u \to 0}\frac{\Delta y}{\Delta u}\right)\left(\lim_{\Delta x \to 0}\frac{\Delta u}{\Delta x}\right)$

(proper proof on p ~~227~~ 204)

$= \frac{dy}{du}\frac{du}{dx}$

examples :

i, $\frac{d}{dx}\left(\cos(x^2)\right) = \left(-\sin(x^2)\right)\left(\frac{d}{dx}(x^2)\right) = -2x\sin(x^2)$

ii, $\frac{d}{dx}\left(\cos^2 x\right) = \frac{d}{dx}\left((\cos x)^2\right) = \left(2\cos x\right)\left(\frac{d}{dx}(\cos x)\right) = -2\cos x \sin x$

iii, $\frac{d}{dt}\left(\sqrt{t^3+1}\right) = \frac{d}{dt}\left((t^3+1)^{1/2}\right) = \frac{1}{2}(t^3+1)^{-1/2}\,(3t^2) = \frac{3t^2}{2\sqrt{t^3+1}}$

see also Ex 1 & 2 p ~~221-2~~ 199 - 200

when the outer function is simply a power, we have a special case of the Chain Rule sometimes called the Power of a Function Rule (p ~~223~~ 200)

Can call FTC Chain Rule question

$\frac{d}{dx}\left(\int_1^{x^2} \sqrt{t^3+1}\,dt\right)$

ie let $y = f(u)$ where $u = g(x)$, but $f(u) = u^n$
so then $y = (g(x))^n$

the Chain Rule says $\frac{dy}{dx} = \frac{d}{dx}(u^n) = n u^{n-1}\frac{du}{dx}$

or $\frac{dy}{dx} = n\,(g(x))^{n-1}\,g'(x)$

examples :

i, $y = \left(\frac{x+1}{x^2+2}\right)^4$

$$\frac{dy}{dx} = 4\left(\frac{x+1}{x^2+2}\right)^3 \frac{d}{dx}\left(\frac{x+1}{x^2+2}\right)$$
$$= 4\left(\frac{x+1}{x^2+2}\right)^3\left(\frac{(1)(x^2+2)-(x+1)(2x)}{(x^2+2)^2}\right)$$
$$= 4\left(\frac{x+1}{x^2+2}\right)^3\left(\frac{x^2+2-2x^2-2x}{(x^2+2)^2}\right)$$
$$= \frac{4(x+1)^3(2-2x-x^2)}{(x^2+2)^5}$$

ii, $f(x) = (x^2+1)^4(2x+1)^3$
$f'(x) = 4(x^2+1)^3(2x)(2x+1)^3 + (x^2+1)^4(3)(2x+1)^2(2)$

$= 2(x^2+1)^3(2x+1)^2\left[4x(2x+1) + 3(x^2+1)\right]$

$= 2(x^2+1)^3(2x+1)^2(11x^2+4x+3)$

see also Ex 3-6 p ~~223-4~~ 200-1

what is the derivative of $e^{g(x)}$?

$$\frac{d}{dx}\left(e^{g(x)}\right) = e^{g(x)}\,g'(x)$$

so $\frac{d}{dx}\left(e^{x^2}\right) = 2x e^{x^2}$, etc...

but also $a^x = (e^{\ln a})^x = e^{(\ln a)x}$

so $\frac{d}{dx}(a^x) = (\ln a)\,e^{(\ln a)x} = a^x \ln a$ the derivative of any exponential function

(p ~~224~~ 202)

eg $\frac{d}{dx}(2^x) = 2^x \ln 2$, etc...

more examples:

i) $\frac{d}{dx}\left(e^{\cos x}\right) = e^{\cos x}(-\sin x) = (-\sin x)e^{\cos x}$

ii) $\frac{d}{dt}\left(\cos(e^t)\right) = -\sin(e^t)(e^t) = -e^t\sin(e^t)$

iii) if $s(\omega) = 2^{\omega^2}$, then $s'(\omega) = 2^{\omega^2}(\ln 2)(2\omega)$

see also Ex 7-9 p ~~225~~ 201-2

what if we have $y = f(g(h(x)))$?

how does the Chain Rule extend?

then we have $y = f(u)$, where $u = g(v)$, $v = h(x)$

so $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \frac{dy}{du}\left(\frac{du}{dv}\frac{dv}{dx}\right) = \frac{dy}{du}\frac{du}{dv}\frac{dv}{dx}$

or $\frac{dy}{dx} = \frac{df}{du}\frac{dg}{dv}\frac{dh}{dx} = f'(u)\,g'(v)\,h'(x)$

$= f'(g(v))\,g'(h(x))\,h'(x)$

$= f'(g(h(x)))\,g'(h(x))\,h'(x)$

example: $\frac{d}{dx}\left(\sin(\tan(e^x))\right) = \cos(\tan(e^x))\left(\frac{d}{dx}(\tan(e^x))\right)$

$$= \cos(\tan(e^x))\sec^2(e^x)\left(\frac{d}{dx}(e^x)\right)$$

$$= e^x\cos(\tan(e^x))\sec^2(e^x)$$

if we know that $w(x) = f(g(x))$ and we want $w'(a)$,
what do we need to know?
since $w(x) = f(g(x))$, $w'(x) = f'(g(x))\,g'(x)$
then $w'(a) = f'(g(a))\,g'(a)$
ie we need to know $g'(a)$, $g(a)$ and $f'(g(a))$

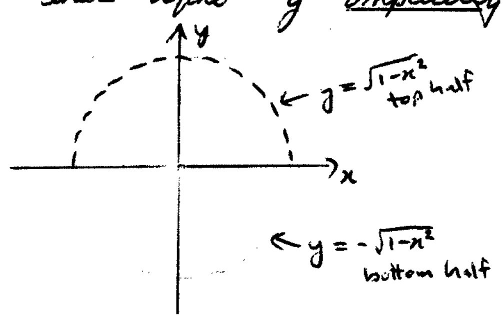
example: $w(x) = f(g(x))$ $\quad f(2) = 3$, $f'(2) = 4$, $g(1) = 2$, $g'(1) = -$
$f(1) = 7$, $f'(1) = 0$, $g(2) = -5$, $g'(2) =$

then $w'(1) = f'(g(1))\,g'(1)$
$= f'(2)\,(-1)$
$= (4)(-1)$
$= -4$

§ ~~3.6~~ 3.8 : Implicit Differentiation

the functions that we are most familiar with, like $f(x) = mx+b$, $f(x) = x^2$, $f(x) = e^x$, $f(x) = \cos x$, etc...
are called explicit functions because $y = f(x)$ gives y explicitly as a function of x

but there are other relations like $x^2+y^2=1$ (the unit circle) that define y implicitly as a function of x



$x^2 + y^2 = 1$
$y^2 = 1 - x^2$

so $y = \pm\sqrt{1-x^2}$ 2 functions

other relations, like $xy^3 + y^4 = 6x$ are much more difficult to solve for y (and really don't want to)

but only certain y values will satisfy the equation for given x values and hence y is still a function (implicit) of x and the expression still represents a curve in the xy plane

what is the equation of the tangent line to the unit circle at the point $\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$?

this point lies on the top half of the circle, so $y = \sqrt{1-x^2}$ + then the slope of the tangent line is $\frac{dy}{dx} = \frac{1}{2}\frac{1}{\sqrt{1-x^2}}(-2x) = \frac{-x}{\sqrt{1-x^2}}$

$$\text{so } m = \left.\frac{dy}{dx}\right|_{x=-\frac{\sqrt{3}}{2}} = \frac{-(-\sqrt{3}/2)}{1/2} = \sqrt{3}$$

and the tangent line is $y - \frac{1}{2} = \sqrt{3}\left(x + \frac{\sqrt{3}}{2}\right)$

$$y = \sqrt{3}x + 2$$

but we could have found the slope of the tangent line at any point on the circle by differentiating implicitly:

(like Ex 1 p ~~237~~ 210) if $x^2 + y^2 = 1$, then $\frac{d}{dx}(x^2+y^2) = \frac{d}{dx}(1)$

$$\text{or } \frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = 0 \quad \text{or} \quad 2x + \frac{d}{dx}(y^2) = 0$$

but what's $\frac{d}{dx}(y^2)$?

since y is an implicit function of x, we can (think of y as $f(x)$ and write $y = f(x)$)

then $\frac{d}{dx}(y^2) = \frac{d}{dx}\left((f(x))^2\right) = 2f(x)\,f'(x) = 2y\frac{dy}{dx}$ (Chain Rule)

so we have $2x + 2y\frac{dy}{dx} = 0$ or $\frac{dy}{dx} = \frac{-x}{y}$

(and this formula is valid for all points on the circle except where $y=0$) (top or bottom)

(12)

so at $\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$, $\frac{dy}{dx} = \frac{-(-\sqrt{3}/2)}{1/2} = \sqrt{3}$

at $\left(1/\sqrt{2}, -1/\sqrt{2}\right)$, $\frac{dy}{dx} = \frac{-1/\sqrt{2}}{-1/\sqrt{2}} = 1$, etc...

another way to see $\frac{d}{dx}(y^2) = 2y\frac{dy}{dx}$ is to use the Chain Rule in the following way: $\frac{d}{dx}\left(g(y(x))\right) = \frac{dg}{dy}\frac{dy}{dx}$

(p234)

so if $g(y) = y^2$, $\frac{dg}{dy} = 2y$, so $\frac{d}{dx}(y^2) = 2y\frac{dy}{dx}$

A107

examples:

i, $x^2y^2 + e^xy = x$

then $\frac{d}{dx}(x^2y^2 + e^xy) = \frac{d}{dx}(x)$

or $2xy^2 + 2x^2y\frac{dy}{dx} + e^xy + e^x\frac{dy}{dx} = 1$

or $\frac{dy}{dx} = \frac{1 - 2xy^2 - e^xy}{2x^2y + e^x}$

ii, $\sin(x+y) = x^2 + y^2$

$\cos(x+y)\left(1 + \frac{dy}{dx}\right) = 2x + 2y\frac{dy}{dx}$

$\frac{dy}{dx} = \frac{2x - \cos(x+y)}{\cos(x+y) - 2y}$

iii, $(x^2-y^2)^2 = 4x^2$

$2(x^2-y^2)(2x - 2yy') = \cancel{8}x$ $\;{}_{4}$

$2x(x^2-y^2) - \underbrace{2y(x^2-y^2)y'}_{=4x}$

$\frac{dy}{dx} = y' = \frac{2x(x^2-y^2) - 4x}{2y(x^2-y^2)}$

$= \frac{x(x^2-y^2) - 2x}{y(x^2-y^2)}$

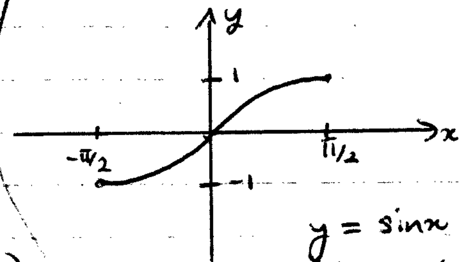
See also Ex 3 p ~~236~~ 212-3

§ 3.6 = Inv. Trig. Functions and their Derivatives

* Go to Book I p 14 *

(review of inverse functions § 1.6 (p 64))

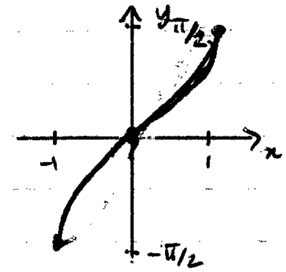
the inverse trig functions (Appendix C p A25-27)



$y = \sin x$

restrict domain $-\pi/2 \le x \le \pi/2$

$\longrightarrow$



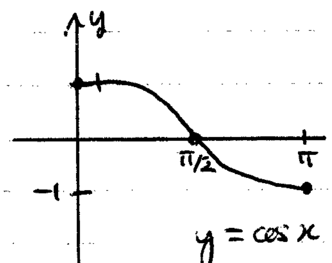
$y = \arcsin x$
$= \sin^{-1}(x)$
$\neq \csc x$
$= (\sin x)^{-1}$
$= \frac{1}{\sin x}$

book uses $\sin^{-1} x$, etc... BAD NOTATION

$y = \arcsin x$ is the angle $-\pi/2 \le y \le \pi/2$ whose sin is x

ie $y = \sin^{-1} x \iff x = \sin y$

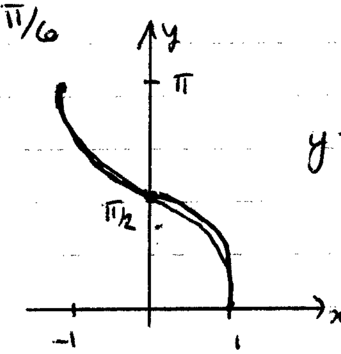
eg. $\arcsin(1/2) = \sin^{-1}(1/2) = \pi/6$



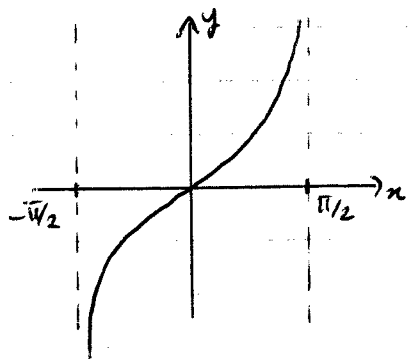
$y = \cos x$

restrict domain $0 \le x \le \pi$

$\longrightarrow$

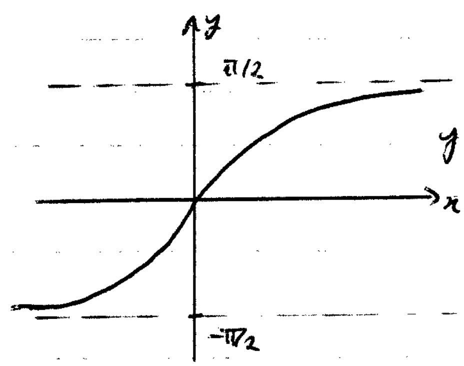


$y = \arccos x$
$= \cos^{-1} x$
$\neq \sec x$



$y = \tan x$

$-\pi/2 < x < \pi/2$



$y = \arctan x$
$= \tan^{-1} x$
$\neq \cot x$

we can use implicit differentiation to find the derivatives:

$y = \sin^{-1} x = \arcsin x \iff x = \sin y$

$1 = \cos y \frac{dy}{dx}$

so $$\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-\sin^2 y}} = \frac{1}{\sqrt{1-x^2}}$$

(since $-\pi/2 \le y \le \pi/2$, $\cos y \ge 0$)

(14)

ie $\frac{d}{dx}(\arcsin x) = \frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$ (p ~~237~~ 217)

$\frac{d}{dx}(\arccos x) = \frac{-1}{\sqrt{1-x^2}}$ p 218

and $y = \arctan x \iff x = \tan y$

$1 = \sec^2 y \frac{dy}{dx}$

so $\frac{dy}{dx} = \frac{1}{\sec^2 y} = \frac{1}{1+\tan^2 y} = \frac{1}{1+x^2}$

ie $\frac{d}{dx}(\arctan x) = \frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$ (p ~~237~~ 218)

examples:

i) $y = e^x \sin^{-1}(x^2)$

$\frac{dy}{dx} = e^x \sin^{-1}(x^2) + e^x \frac{1}{\sqrt{1-x^4}}(2x)$

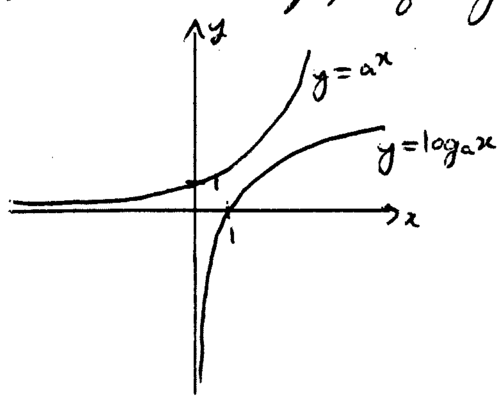
ii) $f(x) = \arctan(1+x)$

$f'(x) = \frac{1}{1+(1+x)^2} = \frac{1}{2+2x+x^2}$

See also Ex 5 p ~~238~~ 220

§ 3.7: Derivatives of Logarithmic Functions

recall that the graph of $y = a^x$ $(a>1)$ looks like



$y = \log_a x$ is the inverse of the function

ie $y = \log_a x \iff x = a^y$

we already know that $\frac{d}{dx}(a^x) = a^x(\ln a)$ where $\ln a = \log_e a$

so if $x = a^y$, then $1 = a^y(\ln a)\frac{dy}{dx}$

or $\frac{dy}{dx} = \frac{1}{a^y(\ln a)} = \frac{1}{x\ln a}$ ie $\frac{d}{dx}(\log_a x) = \frac{1}{x(\ln a)}$ (p ~~240~~ 221)

so, in particular, if we have here $a = e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$

(p ~~245~~ 225-6) $= \lim_{x\to 0}(1+x)^{1/x}$

we have $\frac{d}{dx}(e^x) = e^x(\ln e) = e^x$ (we knew)

and $\frac{d}{dx}(\log_e x) = \frac{d}{dx}(\ln x) = \frac{1}{x}$ (p ~~241~~ 221)

what if we have $f(x) = \ln(g(x))$?, then Chain Rule says
$f'(x) = \frac{g'(x)}{g(x)}$ (p ~~241~~ 222)

eg $\frac{d}{dx}\left(\ln(x^2+3x)\right) = \frac{2x+3}{x^2+3x}$

or in the Leibniz notation: $\frac{d}{dx}(\ln u) = \frac{1}{u}\frac{du}{dx}$

Example 6 (p ~~242~~ 223) $f(x) = \ln|x|$ $\quad f(x) = \begin{cases} \ln x & x>0 \\ \ln(-x) & x<0 \end{cases}$

note that $\frac{d}{dx}(\ln|x|) = \frac{1}{x}$ for all $x \neq 0$ (p ~~243~~ 223)

eg $\frac{d}{dx}\left(\ln|x+1|\right) = \frac{1}{x+1}$

example : $g(x) = \ln\left(\dfrac{\sqrt{x+1}}{x^2+2}\right)$

then $$g'(x) = \frac{x^2+2}{\sqrt{x+1}} \frac{d}{dx}\left(\frac{\sqrt{x+1}}{x^2+2}\right)$$

$$= \frac{x^2+2}{\sqrt{x+1}} \frac{\frac{1}{2}(x+1)^{-1/2}(x^2+2) - (x+1)^{1/2}(2x)}{(x^2+2)^2}$$

$$= \frac{x^2+2}{(x^2+2)^2} \frac{\frac{1}{2}(x+1)^{-1/2}(x^2+2) - (x+1)^{1/2}(2x)}{(x+1)^{1/2}}$$

$$= \frac{1}{x^2+2} \frac{\frac{1}{2}(x^2+2) - (x+1)(2x)}{(x+1)}$$

$$= \frac{-\frac{3}{2}x^2 - 2x + 1}{(x+1)(x^2+2)}$$

* See also Ex 1-5 p ~~241-2~~ 222-3

but the work would've been easier if we remembered the properties of logarithmic functions

ie $$g(x) = \ln\left(\frac{\sqrt{x+1}}{x^2+2}\right) = \ln(x+1)^{1/2} - \ln(x^2+2)$$
$$= \frac{1}{2}\ln(x+1) - \ln(x^2+2)$$

then $$g'(x) = \frac{1}{2}\frac{1}{x+1} - \frac{2x}{x^2+2} \left(= \frac{\frac{1}{2}(x^2+2) - 2x(x+1)}{(x+1)(x^2+2)}\right)$$

these properties of the log functions give us the technique of <u>Logarithmic Differentiation</u> (p ~~243~~ 223)

what if we wanted to differentiate $y = \dfrac{(2x+1)^3(\sqrt{x}+1)^4}{(x^2+1)^6}$?

if we use Product Rule and Quotient Rule, we'll get a big mess

but if we take the natural log of both sides

$$\ln y = \ln\left(\frac{(2x+1)^3(\sqrt{x}+1)^4}{(x^2+1)^6}\right) = 3\ln(2x+1) + 4\ln(\sqrt{x}+1) - 6\ln(x^2+1)$$

and then differentiate implicitly:

$$\frac{1}{y}\frac{dy}{dx} = \frac{6}{2x+1} + \frac{4\left(\tfrac{1}{2}x^{-1/2}\right)}{\sqrt{x}+1} - \frac{12x}{x^2+1}$$

or $$\frac{dy}{dx} = y\left(\frac{6}{2x+1} + \frac{2}{x+\sqrt{x}} - \frac{12x}{x^2+1}\right)$$

$$= \frac{(2x+1)^3(\sqrt{x}+1)^4}{(x^2+1)^6}\left(\frac{6}{2x+1} + \frac{2}{x+\sqrt{x}} - \frac{12x}{x^2+1}\right)$$

ie we can find y' much more "efficiently"

logarithmic differentiation allows us to differentiate new types of functions:
consider $f(x) = x^x$, what's $f'(x)$?

x^x is not a power of x like x^n
nor is it an exponential function like a^x
so the Rules $\frac{d}{dx}(x^n) = nx^{n-1}$ and $\frac{d}{dx}(a^x) = a^x(\ln a)$

will not help us

if we tried the definition: $f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$

$$= \lim_{h\to 0} \frac{(x+h)^{x+h} - x^x}{h} \quad \text{yikes!}$$

but watch this: if $f(x) = x^x$, then $\ln f(x) = \ln(x^x) = x\ln x$

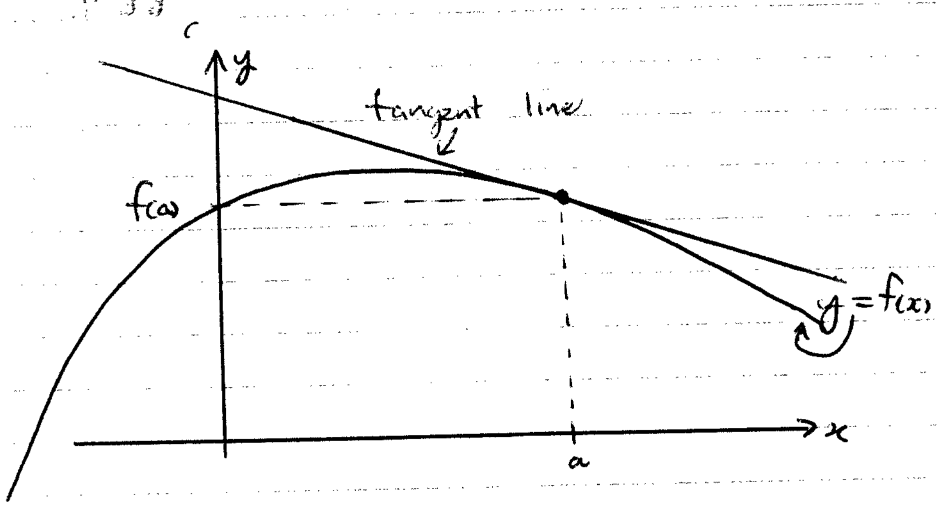
so $$\frac{d}{dx}\left(\ln f(x)\right) = \frac{d}{dx}(x\ln x)$$

or $$\frac{1}{f(x)} f'(x) = \ln x + 1 \quad \text{or} \quad f'(x) = x^x(\ln x + 1)$$

(§ 3.7 # 31 p 251)

see also Ex 7 & 8 p ~~243-4~~ 223-225

3.9
§ ~~3.8~~: Linear Approximations and Differentials



- recall that the tangent line to the curve $y = f(x)$ at $x = a$ has 2 important properties: i) it passes through the point $(a, f(a))$ on the curve and ii) it is the best straight line approximation to the curve near $x = a$

the slope of the tangent line is $m = f'(a)$
so the equation of the tangent line is

$$y - f(a) = f'(a)(x - a)$$

or

$$y = f(a) + f'(a)(x - a)$$

so for x near a, we have the __linear approximation__ or __tangent line approximation__

$$\underset{\substack{\uparrow \\ \text{value of function } f(x) \\ \text{(on curve)}}}{f(x)} \approx f(a) + f'(a)\underset{\substack{\uparrow \\ \text{corresponding value} \\ \text{on tangent line}}}{(x - a)}$$

(p ~~247~~) 241

the function $L(x) = f(a) + f'(a)(x - a)$ is called the __(local) linearization__ of $f(x)$ near a (p ~~248~~ 241)

(these kinds of approximations are used all the time in Science and Engineering) example: let's find the linear approximation of $f(x) = \sqrt{1+x^2}$ at $a = 1$

$$f(x) = (1+x^2)^{1/2}, \quad \text{so} \quad f'(x) = \tfrac{1}{2}(1+x^2)^{-1/2}(2x) = \frac{x}{\sqrt{1+x^2}}$$

so then $f(a) = f(1) = \sqrt{2}$ and $f'(a) = \frac{1}{\sqrt{2}} = f'(1)$

so near $x = 1$, $f(x) = \sqrt{1+x^2}$ is approximated by

$$L(x) = f(a) + f'(a)(x-a)$$

$$\text{or} \quad L(x) = \sqrt{2} + \tfrac{1}{\sqrt{2}}(x-1)$$

$$\text{or} \quad \sqrt{1+x^2} \approx \sqrt{2} + \tfrac{1}{\sqrt{2}}(x-1)$$

let's approximate $\sqrt{2.21}$ $(x = 1.1)$

$$\sqrt{2.21} \approx \sqrt{2} + \tfrac{1}{\sqrt{2}}(1.1-1) = 1.4849$$

(true value $\sqrt{2.21} \approx 1.4866$)

note how different these tangent lines are

what is the linear approximation at $a = 0$?

$f(a) = \sqrt{1} = 1$ $\quad f'(a) = f'(0) = 0$

then $L(x) = 1 + 0(x-0) = 1 + x$

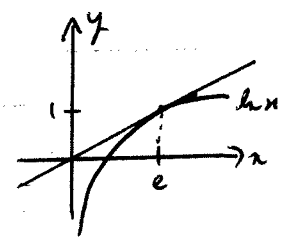
ie for x near 0, $\sqrt{1+x^2} \approx 1+x$

but how useful would this be for approximating $\sqrt{1.25}$?

example: find the linearization of $f(x) = \ln x$ at $a = e$

$$f(x) = \ln x, \quad f'(x) = \frac{1}{x}$$

so $f(a) = f(e) = \ln e = 1$, $\quad f'(a) = f'(e) = \frac{1}{e}$



then $L(x) = 1 + \frac{1}{e}(x - e) = \frac{1}{e}x$

then $\ln 3 \approx \frac{3}{e} = 1.1036$ (true value 1.0986)

See Ex 1-3 p 241-3

skip

Differentials

if $y = f(x)$ is differentiable, then $\frac{dy}{dx} = f'(x)$

we can think of $\frac{dy}{dx}$ as the single symbol representing the derivative of y wrt x

or we can think of it as being the ratio of the differentials dy and dx

and then we could write $dy = f'(x)\,dx$

and it represents that if we make an infinitesimal change in x of dx at value x, the corresponding infinitesimal change in y is dy

and since $\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$, we can use dy and dx

as approximations of Δy and Δx

example: let $f(x) = \sqrt{1+x^2}$, then $f'(x) = \frac{x}{\sqrt{1+x^2}}$

if we change x by 0.1 from 1 to 1.1

ie let $\Delta x = 0.1 = dx$, at $x = 1$

then $dy = f'(x)dx = f'(1)(0.1) = \frac{1}{\sqrt{2}}(0.1) = 0.0707$

so y changes from $f(1) = \sqrt{2}$ to $\underset{\text{approx}}{\wedge}$ $\sqrt{2} + dy = 1.4849$

ie $y = f(a) + dy = f(a) + f'(a)\,dx$

(note the connection with tangent line approximation)

true $\Delta y = f(1.1) - f(1) = 0.0724$

(and y changes to 1.4866)

example: the Canadian Space Agency has built a spherical satellite with radius $r = 50\,cm$
before it can be put into orbit, it needs to be coated with a special paint (heat resistant) in a layer 2 mm thick
approximately how much paint is required?

when we coat the satellite, we are changing the radius by $\Delta r = dr = 2\,mm = 0.2\,cm$ (ie $50 \to 50.2$)
this will change the volume of the object by $\Delta V \approx dV$
(and this change in volume is the amount of paint required)

$$V = \frac{4}{3}\pi r^3, \quad \text{so} \quad \frac{dV}{dr} = 4\pi r^2$$

$$\begin{aligned} \text{or} \quad dV &= 4\pi r^2 dr \\ &= 4\pi (50\,cm)^2 (0.2\,cm) \\ &= 4\pi (500)\,cm^3 \\ &= 2000\pi\,cm^3 \\ &\approx 6283.19\,cm^3 \\ &\simeq 6.3\,L \end{aligned}$$

(true $\Delta V = 6308.35\,cm^3$)

Chapter 4: Applications of Differentiation

§ 4.1 : Related Rates

if the variables or quantities x and y are related, ie $y = f(x)$, then their rates of change (w.r.t time) are related as well
the Chain Rule tells us that $\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$

the text offers a strategy for solving related rate problems on page ~~265~~ 258

example : the radius of a circle increases at a rate of $5\,cm/s$
at what rate is the area increasing if the radius is 20 cm ?

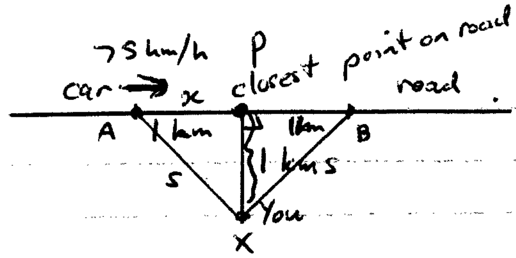
let r be the radius of the circle and A be the area
we're told $\frac{dr}{dt} = 5\,cm/s$

we want $\frac{dA}{dt}$ when $r = 20$ cm

how are A and r related? $A = \pi r^2$
so then $\frac{dA}{dt} = \frac{dA}{dr}\frac{dr}{dt} = 2\pi r \frac{dr}{dt}$

so, when $r = 20$ cm, $\frac{dA}{dt} = 2\pi(20cm)(5cm/s) = 200\pi\ cm^2/s$
$\simeq 628.3\ cm^2/s$

example : you are standing at a point 1 km away from a straight road that is perpendicular to your path
a car is travelling along the road at 75 km/h
how fast is the car moving towards you when it is 1 km away from the closest point on the road?
how fast is the car moving away from you when it is 1 km past the closest point on the road to you?



A) when the car is 1 km before closest point
let x be the distance between the car and the closest point P
and let s be the distance between you and the car
we're told $\frac{dx}{dt} = -75$ km/h (negative because x decreases)

we want $\frac{ds}{dt}$ when $x = 1$ km

by Pythagoras, $x^2 + (1)^2 = s^2$
then $\frac{d}{dt}(x^2+1) = \frac{d}{dt}(s^2)$

or $2x\frac{dx}{dt} = 2s\frac{ds}{dt}$

or $\frac{ds}{dt} = \frac{x}{s}\frac{ds}{dt}$

when $x=1$, $s=\sqrt{2}$, so $\frac{ds}{dt} = \frac{1}{\sqrt{2}}(-75\text{km/h}) \approx -53$ km/h

the negative sign indicates that the car is moving towards you (ie s is decreasing)

B) when the car is 1 km past P
then $\frac{dx}{dt} = 75$ km/h   positive because car is moving away

and hence $\frac{ds}{dt} = 53$ km/h   (x and s increasing)

at what velocity is the car travelling with respect to you when $x=0$, ie when the car is at P?
if $x=0$, $\frac{ds}{dt} = 0$

ie the car is not moving towards or away from you
(the distance s does not change in that instant)

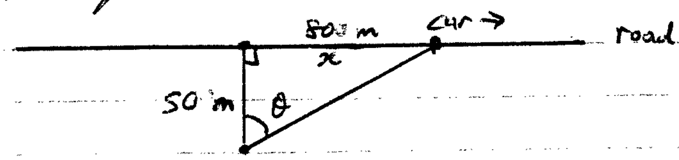
example: a surveillance camera is tracking a suspicious car

the camera is located 50 m from the road

the car is travelling at 40 km/h along the road

when the car is 80 m down the road (past the camera) (ie moving away)

how fast must the camera rotate to track it?



let x be the distance down the road the car is

let θ be the angle that the camera makes, with $\theta = 0$ corresponding to the closest point on the road

we're told $\frac{dx}{dt} = 40\,\text{km/h} = 40 \frac{\text{km}}{\text{h}} \times \frac{1\,\text{h}}{3600\,\text{s}} \times 1000 \frac{\text{m}}{\text{km}}$

$\simeq 11.11$ m/s

we want $\frac{d\theta}{dt}$ when $x = 80\,\text{m}$

$$\tan\theta = \frac{x}{50}$$

so $\theta = \arctan(x/50)$

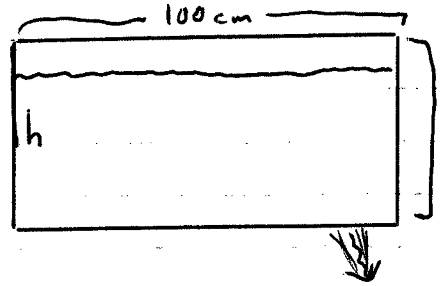
then $$\frac{d\theta}{dt} = \frac{d\theta}{dx}\frac{dx}{dt} = \frac{1}{1+(x/50)^2}\left(\frac{1}{50}\right)\frac{dx}{dt}$$

so, if $x = 80\,\text{m}$, $$\frac{d\theta}{dt} = \frac{1}{50\,\text{m}}\,\frac{1}{1+\left(\frac{8}{5}\right)^2}\,(11.11\,\text{m/s}) \simeq 0.0624\ \text{rad/s}$$ $(\simeq 3.58°/\text{s})$

what about when the car is only 10 m past the camera?

$$\frac{d\theta}{dt} = \frac{1}{50\,\text{m}}\,\frac{1}{1+\left(\frac{1}{5}\right)^2}\,(11.11\,\text{m/s}) \simeq 0.214\ \text{rad/s}\ (\simeq 12.2°/\text{s})$$

example : a rectangular aquarium of length 100 cm, width 60 cm and height 50 cm is leaking at a rate of 1 L/min through a hole at the bottom
at what rate is the height of the water changing when the water in the ~~tank~~ tank is 40 cm deep?



let h be the height of the water in the tank
let V be the volume of the water in the tank

∴ we're told $\frac{dV}{dt} = -1$ L/min $= -1000$ cm^3/min

we want $\frac{dh}{dt}$ when $h = 40$ cm

$$V = lwh = (100\text{ cm})(60\text{ cm})\,h = 6000h \quad (\text{cm}^3)$$

so $h = \frac{V}{6000}$

then $\frac{dh}{dt} = \frac{1}{6000}\frac{dV}{dt} = \frac{-1}{6}$ cm/min $\simeq -0.17$ cm/min

see also examples 1-5 on pages ~~263-6~~ 256-9

§4.2 : Maximum and Minimum Values

we want to find the maximum (largest) and minimum (smallest) values of a function

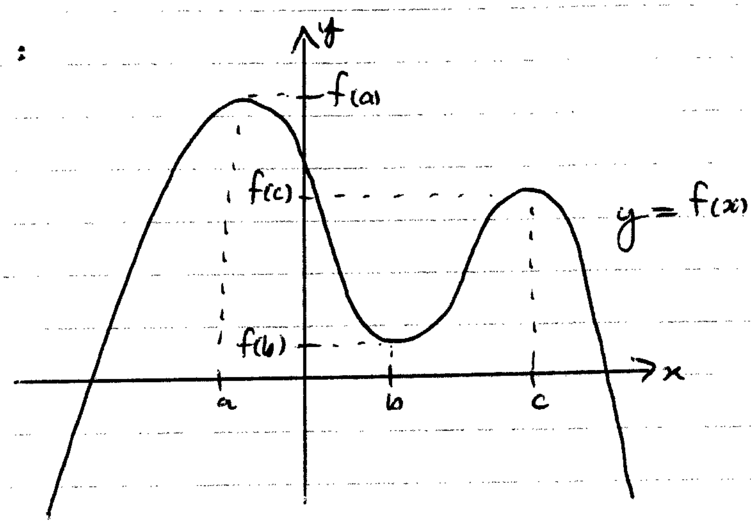
the function f has an absolute or global maximum at $x = c$ if
(p ~~269~~ 262) $f(c) \geq f(x)$ for all x in the domain of f
then $f(c)$ is called the maximum value of f on the domain
(ie $f(c)$ is the largest value the function can take on)

function f has an absolute or global minimum at $x = c$ if

$f(c) \le f(x)$ for all x in the domain
and $f(c)$ is called the <u>minimum value</u> of f on the domain
(ie $f(c)$ is the smallest value $f(x)$ can take)

together, the max and min values are called <u>extreme values</u>

example:



this function does not have an absolute minimum
but it does have an absolute max at $x = a$

but what happens at $x = b$ and $x = c$?
these are not global extrema, but they are <u>local</u> ones

a function f has a <u>local</u> or <u>relative maximum</u> at
$x = c$ if $f(c) \ge f(x)$ for all x near c (for all x in
(p 270) 263 some open interval around c)

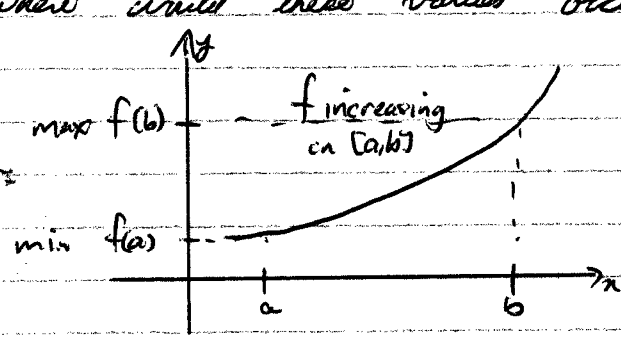
f has a <u>local</u> or <u>relative minimum</u> at $x = c$ if
$f(c) \le f(x)$ for all x near c

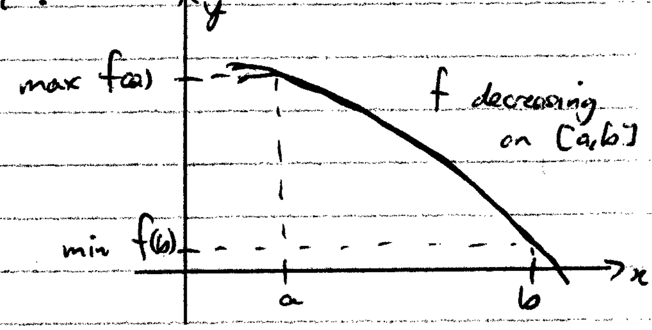
so, in our example above: no absolute min, an absolute and
local max at $x = a$, a local min at $x = b$ and
a local max at $x = c$

have a look at the examples on p 269–74 262–7 we're going to be specifically interested in finding the max and min values of continuous functions (and typically differentiable) defined on closed intervals [a,b]

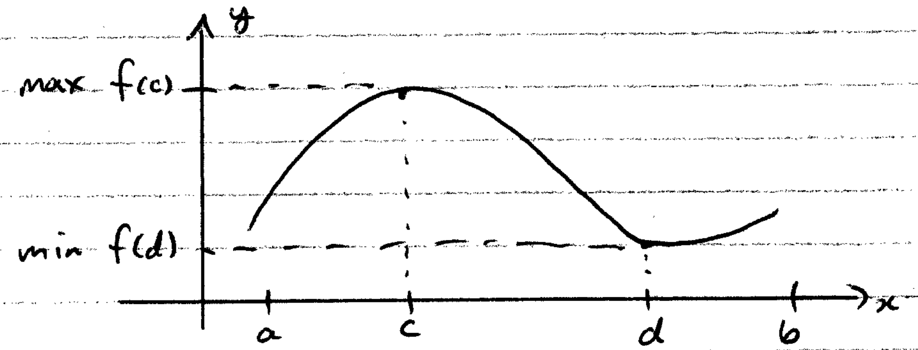
<u>the Extreme Value Theorem</u>: (p ~~271~~ 264) if f is continuous on a closed interval $[a,b]$, then f attains an absolute max $f(c)$ and an absolute min $f(d)$ at $x=c$ and $x=d$, where $a \le c \le b$, $a \le d \le b$

where would these values occur?





to have a min or max between a and b?



what happens at c and d?

if the max or min occurs inside the interval, it happens at a <u>local</u> max or min
and we expect that $f'(c)=0$ and $f'(d)=0$

<u>Fermat's theorem</u>: if f has a local max or min at $x=c$ (p ~~272~~ 265) and if $f'(c)$ exists (ie f differentiable at $x=c$), then $f'(c)=$

be careful - the theorem says if there is a local extremum at c, then $f'(c)=0$, it does NOT say that if

$f'(c)=0$, there is a local extremum there

example : $f(x)=x^3$, $f'(x)=3x^2$, $f'(0)=0$, but no local extremum at $x=0$

we must also be aware that there can be a local extremum at places where f' does not exist

example: $f(x)=|x|$ has local (and absolute) min at $x=0$ but $f'(0)$ does not exist

a critical number (or point) of function f is a number c in the domain of f such that $f'(c)=0$ or $f'(c)$ does not exist (p ~~272~~) 266

example: $f(x) = x - x^{1/2}$
$f'(x) = 1 - \frac{1}{2}x^{-1/2}$
$f'(x)=0$ if $1-\frac{1}{2}x^{-1/2}=0$ $\quad 1 = \frac{1}{2}\frac{1}{\sqrt{x}}$, $\sqrt{x}=\frac{1}{2}$, $x=\frac{1}{4}$
and $f'(x)$ does not exist at $x=0$
$\therefore$ 0 and $\frac{1}{4}$ are the critical numbers of $f(x)$

so we have that if f has a local max or min at $x=c$, c is a critical number of f (p ~~273~~) 266

so we put together everything we've seen to get the Closed Interval Method (p ~~273~~) 266
to find the absolute max and min values of a continuous function f on a closed interval $[a,b]$

i, evaluate f at the critical numbers inside the interval $((a,b)$,
ii, evaluate f at the endpoints (a + b)
iii, compare the values

examples : find the max and min values of
i, $f(x) = 2x^3-9x^2+12x+7$ on the interval $[0,3]$

$f'(x) = 6x^2 - 18x + 12 = 6(x^2-3x+2) = 6(x-1)(x-2)$

$f'(x) = 0$ if $x=1, x=2$ (both inside the interval)

($f'(x)$ is a polynomial, so defined for all x)

$f(1) = 2(1)^3 - 9(1)^2 + 12(1) + 7 = 2 - 9 + 12 + 7 = 12$ (local max)

$f(2) = 2(2)^3 - 9(2)^2 + 12(2) + 7 = 16 - 36 + 24 + 7 = 11$ (local min)

the endpoints are 0 & 3

$f(0) = 7$ (abs min)

$f(3) = 2(3)^3 - 9(3)^2 + 12(3) + 7 = 54 - 81 + 36 + 7 = 16$ (abs max)

so the min value is 7 (at $x=0$) and the max value is 16 (at $x=3$)

ii) $f(x) = x - x^{1/2}$ on $[0,2]$

we already know that $f'(x) = 1 - \frac{1}{2\sqrt{x}} = 0$ if $x = 1/4$ and is undefined at $x=0$

$f(0) = 0$

$f(1/4) = 1/4 - \sqrt{1/4} = 1/4 - 1/2 = -1/4$

$f(2) = 2 - \sqrt{2} \simeq 0.5859$

so the min value is $-1/4$ (at $x = 1/4$) and the max value is $2-\sqrt{2}$ at ($x=2$)

A10

§4.3: Derivatives and the Shapes of Curves

in this section, we'll review what the first and second derivatives can tell us about the shape of a curve

<u>the Mean Value Theorem</u>: if f is a differentiable function on the interval $[a,b]$, then there exists c, $a \le c \le b$, such that

skip? (p ~~279~~ 272) $$f'(c) = \frac{f(b)-f(a)}{b-a}$$

ie there is some point between a and b where the slope of the tangent line to the curve is equal to the slope of the line joining $(a, f(a))$ and $(b, f(b))$

(30)

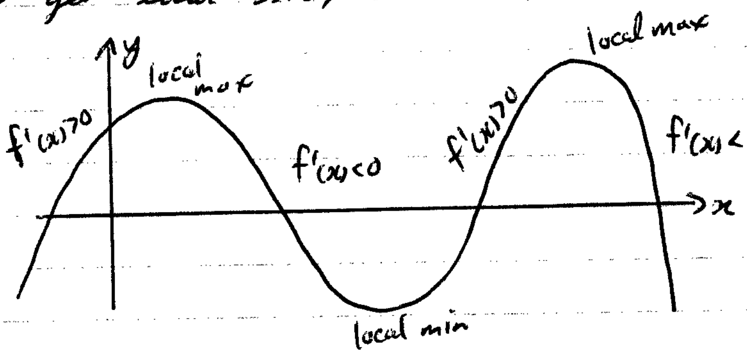
the sign of $f'(x)$ tells us whether the function is increasing or decreasing:

(~~p230~~ 273) if $f'(x) > 0$ on some interval, then f is increasing on that interval

" $f'(x) < 0$ " " " , " " " decreasing " " "

what happens when $f'(x)$ changes sign?

we get local max's and min's



but remember, $f'(c) = 0$ does not mean there has to be a local max or min (eg $f(x) = x^3$)

(~~p231~~ 274)

<u>The First Derivative Test</u>: suppose that c is a critical number of continuous function f

a) if f' changes from + to − at c, then f has a local max at c

b) if f' " " − to + " " , " " " " " min " "

c) if f' does not change sign at c, no local max or min

see the examples on p ~~231~~ 274

example: find the local extrema of $f(x) = x^3 - 3x^2 - 9x + 10$

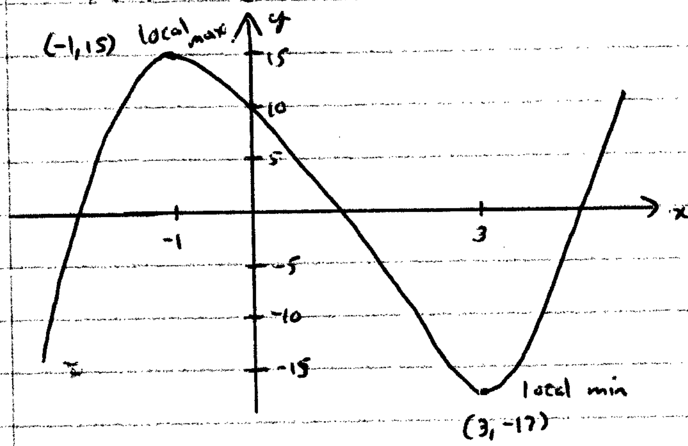
$f'(x) = 3x^2 - 6x - 9 = 3(x^2 - 2x - 3) = 3(x-3)(x+1)$

so $f'(x) = 0$ if $x = -1$ or $x = 3$

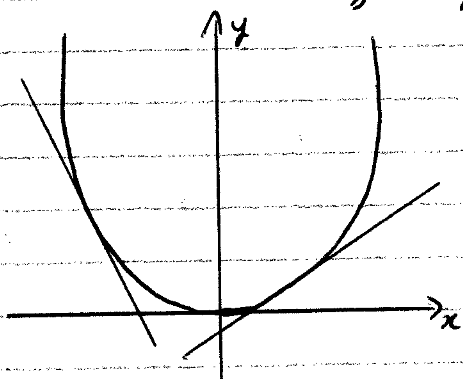
interval	$x+1$	$x-3$	$f'(x)$	$f(x)$
$x < -1$	−	−	+	increasing
$-1 < x < 3$	+	−	−	decreasing
$x > 3$	+	+	+	increasing

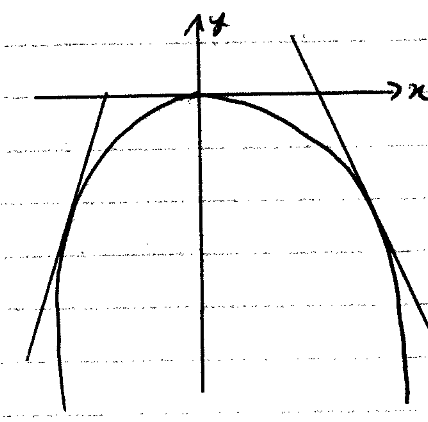
at $x = -1$, $f'(x)$ changes sign from $+$ to $-$, so local max at $x=$
at $x = 3$, " " " " $-$ to $+$, " " min " $x=$

$f(-1) = (-1)^3 - 3(-1)^2 - 9(-1) + 10 = -1 - 3 + 9 + 10 = 15$
$f(3) = (3)^3 - 3(3)^2 - 9(3) + 10 = 27 - 27 - 27 + 10 = -17$



consider now the following functions:





$y = f(x) = x^2$
$f'(x) = 2x$
increasing if $x>0$
decreasing if $x<0$
local min at $x=0$
(& abs)

tangent lines always lie below the curve
$f''(x) = 2 \Rightarrow f'(x)$ increasing

$y = f(x) = -x^2$
$f'(x) = -2x$
increasing if $x<0$
decreasing if $x>0$
local (& abs) max at $x=0$

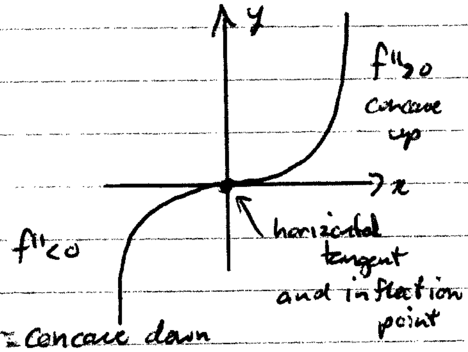
tangent lines always lie above the curve
$f''(x) = -2 \Rightarrow f'(x)$ decreasing

f is called <u>concave up</u> on interval I if f' is increasing on I
" " " <u>concave down</u> " " " " " " decreasing " "

(p ~~281~~) 279

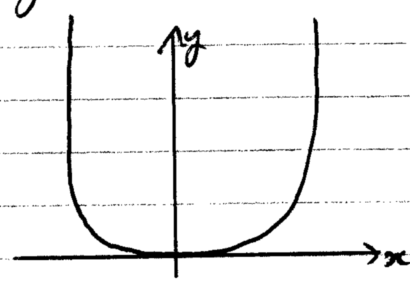
a if $f''(x) > 0$ on I, f is concave up on I (p ~~282~~ 275)
" $f''(x) < 0$ " " , " " " down " "

what happens when $f''(x)$ changes sign?
we have inflection points



$y = f(x) = x^3$
$f'(x) = 3x^2 \geq 0$ for all x
so no local extrema
$f''(x) = 6x$ < 0 if $x < 0$
> 0 if $x > 0$
so inflection point at $x = 0$

CAUTION: $f''(x) = 0$ does not mean there has to be an inflection pt
eg $f(x) = x^4$



$f'(x) = 4x^3$
$f''(x) = 12x^2$
$f''(0) = 0$, but $f''(x) \geq 0$ for all x
no sign change
there is a local (and abs) min at $x = 0$

we can use concavity to identify local max and min's:
(look back at x^2 & $-x^2$)

<u>The Second Derivative Test</u>: suppose f'' is continuous near $x = c$
(p ~~282~~ 275) a) if $f'(c) = 0$ and $f''(c) > 0$, then f has a local min at c
b) " " " " $f''(c) < 0$, " " " " " max " "

example: $f(x) = x^3 - 3x^2 - 9x + 10$
$f'(x) = 3x^2 - 6x - 9$
$f''(x) = 6x - 6 = 6(x - 1)$
$f''(-1) < 0 \Rightarrow$ local max at $x = -1$
$f''(3) > 0 \Rightarrow$ local min at $x = 3$

Example : $y = f(x) = x + \frac{1}{x}$

$f(x)$ not defined at $x=0$

$\lim_{x \to 0^-} f(x) = -\infty \qquad \lim_{x \to 0^+} f(x) = \infty$ $\therefore$ vertical asymptote at $x=0$

$\lim_{x \to \infty} f(x) = \infty \qquad \lim_{x \to -\infty} f(x) = -\infty$ no horizontal asymptote

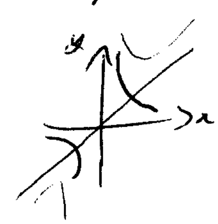
$f(x) = 0 \Rightarrow x + \frac{1}{x} = 0 \quad x = \frac{-1}{x} \Rightarrow x^2 = -1$ no intercepts

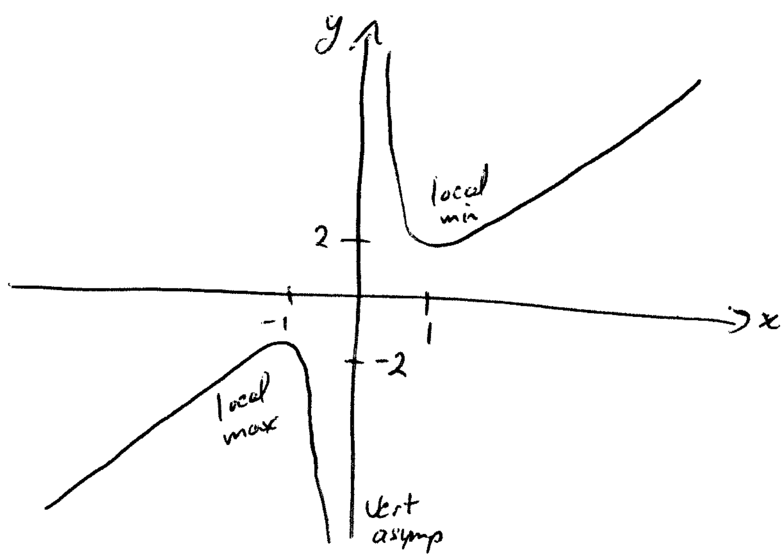
$f'(x) = 1 - \frac{1}{x^2} \qquad f'(x) = 0$ if $x = \pm 1$

$x < -1$	$f'(x) > 0$	$f(x)$ increasing	$\therefore$ local max at $(-1,-2)$
$-1 < x < 0$	$f'(x) < 0$	$f(x)$ decreasing	
$0 < x < 1$	$f'(x) < 0$	$f(x)$ decreasing	$\therefore$ local min at $(1,2)$
$x > 1$	$f'(x) > 0$	$f(x)$ increasing	

$f''(x) = \frac{2}{x^3} \qquad f''(x) \neq 0$ for ~~all~~ any x

$x < 0$	$f''(x) < 0$	concave down	no inflection pts
$x > 0$	$f''(x) > 0$	concave up	





Example : $y = f(x) = x + \sin x$ interval $[0, 2\pi]$

defined for all $x \Rightarrow$ no vertical ~~asympt~~ asymptotes

$f(0) = 0$, $f(2\pi) = 2\pi$

($f(x) = 0 \Rightarrow x = -\sin x \Rightarrow$ need Newton)
no solution

$f'(x) = 1 + \cos x$ $f'(x) = 0$ if $\cos x = -1 \Rightarrow x = \pi$

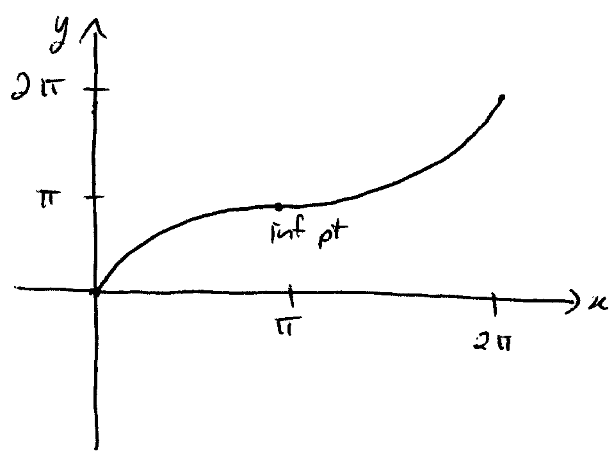
$0 < x < \pi$ $f'(x) > 0$ $f(x)$ increasing

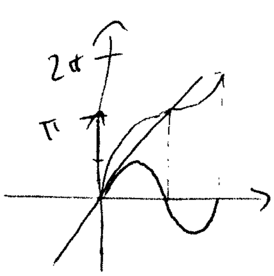
$\pi < x < 2\pi$ $f'(x) > 0$ $f(x)$ increasing $\therefore$ no local extremum

$f''(x) = -\sin x$ $f''(x) = 0$ if $\sin x = 0 \Rightarrow x = 0, \pi, 2\pi$

$0 < x < \pi$ $f''(x) < 0$ concave down

$\pi < x < 2\pi$ $f''(x) > 0$ concave up $\therefore$ inflection pt at (π, π)





<u>Example</u> : $y = f(x) = e^{-x^2}$

defined for all $x \Rightarrow$ no vertical asymptotes

$\lim_{x\to\infty} e^{-x^2} = 0 \qquad \lim_{x\to-\infty} e^{-x^2} = 0$ $\qquad y=0$ is horizontal asymptote (from above since $f(x)>0$ for all x)

$f(0) = 1 \quad , \quad f(x) \neq 0$

$f'(x) = -2x e^{-x^2} \qquad f'(x) = 0$ if $x = 0$

$x<0 \quad f'(x)>0 \quad f(x)$ increasing
$x>0 \quad f'(x)<0 \quad f(x)$ decreasing $\quad \therefore$ local max at $(0,1)$

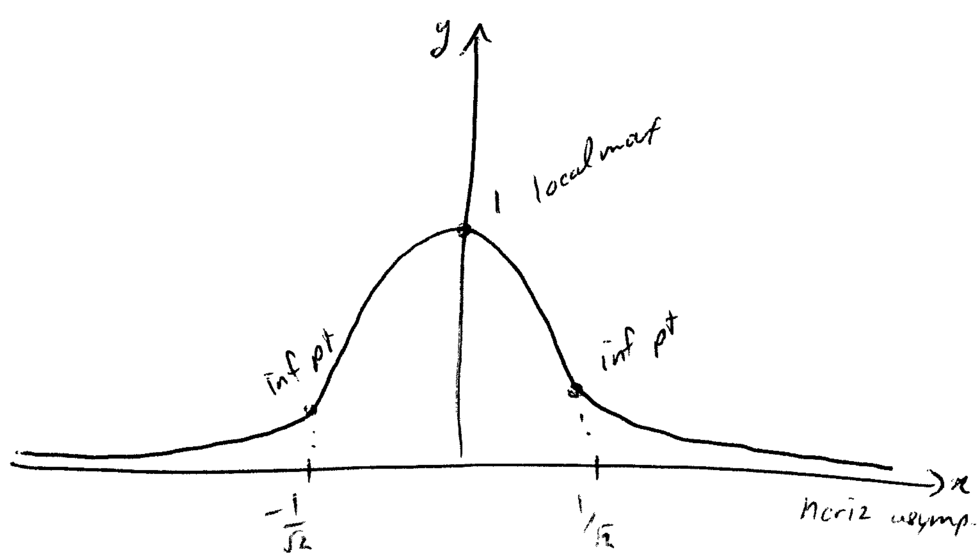
$f''(x) = -2e^{-x^2} + 4x^2 e^{-x^2} = 2e^{-x^2}(2x^2-1)$

$f''(x) = 0$ if $x = \pm \frac{1}{\sqrt{2}}$

$x < -\frac{1}{\sqrt{2}} \quad f''(x) > 0 \quad f(x)$ concave up

$-\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}} \quad f''(x) < 0 \quad f(x)$ concave down $\quad \therefore$ inflection pts at $\left(\pm\frac{1}{\sqrt{2}}, e^{-1/2}\right)$

$x > 1/\sqrt{2} \quad f''(x) > 0 \quad f(x)$ concave up



Should not do in 401

do in A08

Example : $y = f(x) = \frac{\ln x}{x}$

only defined for $x > 0$ $\quad$ $x = 0$ is vertical asymptote

$$\lim_{x \to 0^+} \frac{\ln x}{x} = -\infty$$

$f(x) = 0$ if $\ln x = 0 \Rightarrow x = 1$

$\lim_{x \to \infty} f(x) = 0$ $\quad$ $y = 0$ is horizontal asymptote (from above (since $f(x) > 0$ if $x > 1$))

$$f'(x) = \frac{1}{x^2} - \frac{\ln x}{x^2} = \frac{1}{x^2}(1 - \ln x)$$

$f'(x) = 0$ if $x = e$ $\quad$ $0 < x < e$ $\quad f'(x) > 0$ $\quad f(x)$ increasing

$x > e$ $\quad f'(x) < 0$ $\quad f(x)$ decreasing

$\therefore$ $(e, \frac{1}{e})$ is local max

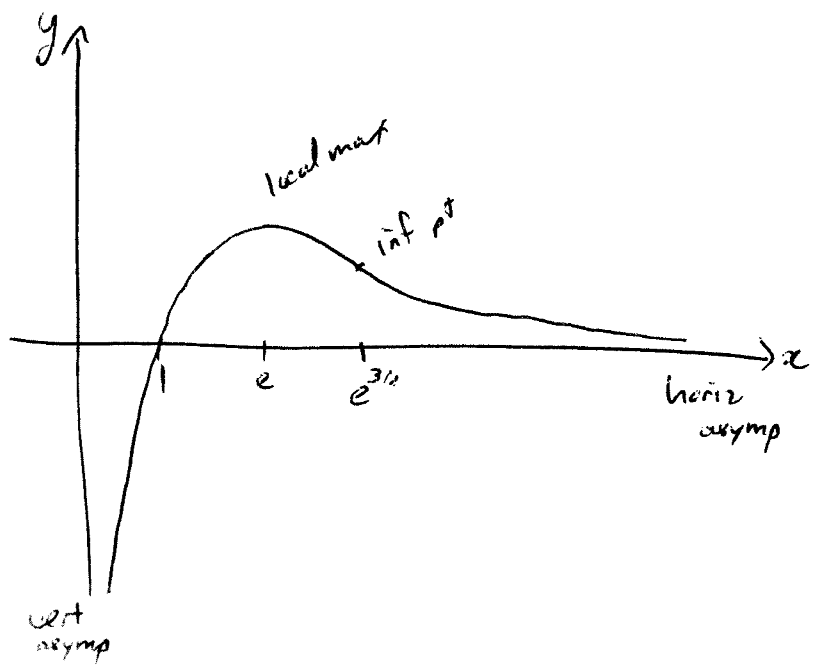
$$f''(x) = \frac{-2}{x^3} - \frac{1}{x^3} + 2\frac{\ln x}{x^3} = \frac{1}{x^3}(2\ln x - 3)$$

$f''(x) = 0$ if $\ln x = \frac{3}{2} \Rightarrow x = e^{3/2}$

$0 < x < e^{3/2}$ $\quad f''(x) < 0$ $\quad f(x)$ concave down

$x > e^{3/2}$ $\quad f''(x) > 0$ $\quad f(x)$ concave up

$\therefore$ inflection pt at $\left(e^{3/2}, \frac{3/2}{e^{3/2}}\right)$



NOTE: the SDT cannot be used if $f''(c) = 0$ or if $f''(c)$ does not exist; in that case, use the FDT (see x^4)

see also Ex #5 on page ~~233-4~~ 226-7

example: (not in 3rd) $f(x) = \dfrac{x}{x^2+9}$

a) $f(x)$ is defined for all $x \Rightarrow$ no vertical asymptotes

$\lim_{x\to\infty} \dfrac{x}{x^2+9} = 0$ (from above) $\quad \lim_{x\to-\infty} \dfrac{x}{x^2+9} = 0$ (from below)

so $y=0$ is a horizontal asymptote

b) $f'(x) = \dfrac{(1)(x^2+9) - (x)(2x)}{(x^2+9)^2} = \dfrac{9-x^2}{(x^2+9)^2}$

$f'(x)$ defined for all x, $f'(x) = 0$ if $x = \pm 3$

if $x < -3$, $f'(x) < 0$ $\quad f(x)$ decreasing

if $-3 < x < 3$, $f'(x) > 0$ $\quad f(x)$ increasing

if $x > 3$, $f'(x) < 0$ $\quad f(x)$ decreasing

c) FDT tells us local min at $(-3, -1/6)$ and local max at $(3, 1/6)$

d) $f''(x) = \dfrac{-2x(x^2+9)^2 - (9-x^2)\,2(x^2+9)(2x)}{((x^2+9)^2)^2}$

$= \dfrac{-2x(x^2+9) - 4x(9-x^2)}{(x^2+9)^3} = \dfrac{2x^3 - 54x}{(x^2+9)^3} = \dfrac{2x(x^2-27)}{(x^2+9)^3}$

$f''(x)$ defined for all x, $f''(x) = 0$ if $x = 0, \pm 3\sqrt{3}$

if $x < -3\sqrt{3}$, $f''(x) < 0$ $\quad f(x)$ concave down

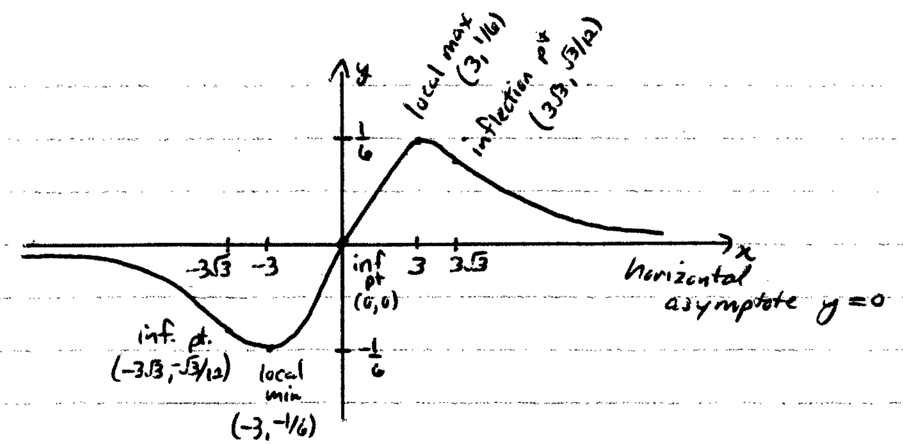
if $-3\sqrt{3} < x < 0$, $f''(x) > 0$ $\quad f(x)$ concave up

if $0 < x < 3\sqrt{3}$, $f''(x) < 0$ $\quad f(x)$ concave down

if $x > 3\sqrt{3}$, $f''(x) > 0$ $\quad f(x)$ concave up

so there are inflection points at $(-3\sqrt{3}, -\sqrt{3}/12)$, $(0,0)$ and $(3\sqrt{3}, \sqrt{3}/12)$

e)



H03 A08 A10

§4.5: Indeterminate Forms and L'Hôpital's Rule

when we were trying to find $\frac{d}{dx}(\sin x)$, we ran into the limit $\lim\limits_{h\to 0}\frac{\sin h}{h}$ and we were stuck as to what it should be

why? because both the numerator and denominator $\to 0$ but the limit does exist (we saw that $\lim\limits_{h\to 0}\frac{\sin h}{h}=1$)

what about $\lim\limits_{x\to\infty}\frac{e^x}{x}$? here $e^x\to\infty$ and $x\to\infty$, so what happens to the ratio as $x\to\infty$?

it could go to 0, a positive number or ∞, depending on which of e^x or x "dominates"

L'Hôpital's Rule will allow us to find limits like these: (p ~~248~~) 291

suppose that f and g are differentiable and $g'(x)\neq 0$ near a and suppose that $\lim\limits_{x\to a} f(x)=0$ and $\lim\limits_{x\to a} g(x)=0$

or $\lim\limits_{x\to a} f(x)=\pm\infty$ and $\lim\limits_{x\to a} g(x)=\pm\infty$

(so the limit $\lim\limits_{x\to a}\frac{f(x)}{g(x)}$ is an indeterminate form of type $\frac{0}{0}$ or $\frac{\infty}{\infty}$) (p ~~248~~) 290

then $\lim\limits_{x\to a}\frac{f(x)}{g(x)} = \lim\limits_{x\to a}\frac{f'(x)}{g'(x)}$ (provided RHS exists or is $\pm\infty$)

L'Hôpital's Rule is also valid if $a = \pm\infty$ or the limit is one-sided

examples:

i) $\lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0} \frac{\cos x}{1} = 1$

ii) $\lim_{x\to\infty} \frac{e^x}{x} = \lim_{x\to\infty} \frac{e^x}{1} = \infty$

iii) $\lim_{x\to\infty} \frac{e^x}{x^2} = \lim_{x\to\infty} \frac{e^x}{2x} = \lim_{x\to\infty} \frac{e^x}{2} = \infty$

in fact $\lim_{x\to\infty} \frac{e^x}{x^n} = \infty$ for all n (ie $e^x \to \infty$ faster than <u>any</u> power of x)

iv) $\lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{1/x}{1} = 0$ see also Ex 1-4 p. ~~299-300~~ 292-3

be careful - only use L'Hôpital's Rule when it applies (and when you have to)

see Ex 5 p. 300 : 293

what do we do if we have $\lim_{x\to a} f(x)g(x)$, but $f(x)\to 0$ and $g(x)\to\infty$?

(p ~~300~~ 294) this is called an <u>indeterminate product</u> (or form of type $0\cdot\infty$)

we could rewrite $fg = \frac{f}{1/g}$ and have a $0/0$

or $fg = \frac{g}{1/f}$ and have an ∞/∞

example:

$$\lim_{x\to\infty} e^{-x}\ln x = \lim_{x\to\infty} \frac{\ln x}{e^x} = \lim_{x\to\infty} \frac{1/x}{e^x} = 0 \quad (\S 4.5\, p\, 305\, \#23)$$

see also Ex 6 p ~~301~~ 294

if $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = \infty$, then the limit

$\lim_{x\to a} (f(x) - g(x))$ is called an <u>indeterminate difference</u> (p ~~301~~ 294) or form of type $\infty - \infty$

and this limit could be anything between $-\infty$ and ∞

example : (#34) $\lim_{x\to 1}\left(\frac{1}{\ln x}-\frac{1}{x-1}\right) = \lim_{x\to 1}\frac{(x-1)-\ln x}{(x-1)\ln x}$ $\left(\frac{0}{0}\right)$

$$= \lim_{x\to 1}\frac{1-1/x}{\ln x+\frac{1}{x}(x-1)} = \lim_{x\to 1}\frac{1-1/x}{-\frac{1}{x}+1+\ln x} \quad \left(\frac{0}{0}\right)$$

$$= \lim_{x\to 1}\frac{x-1}{-1+x+x\ln x} = \lim_{x\to 1}\frac{1}{+1+\ln x+1} = \frac{1}{2}$$

see also Ex 7 p ~~301-2~~ 295

<u>indeterminate powers</u> (p ~~302~~) 295 arise when we do $\lim_{x\to a}(f(x))^{g(x)}$

and

$\lim_{x\to a} f(x)$	$\lim_{x\to a} g(x)$	type
0	0	0^0
∞	0	∞^0
1	$\pm\infty$	1^∞

but we've already seen that taking the log of both sides simplifies differentiation in a case like this, so we could try this here as well

example : . . . $\lim_{x\to 0^+}(-\ln x)^x$

as $x\to 0^+$ $-\ln x\to\infty$, so this is ∞^0

if $y=(-\ln x)^x$, then $\ln y = \ln((-\ln x)^x) = x\ln(-\ln x)$

so $\lim_{x\to 0^+}\ln y = \lim_{x\to 0^+} x\ln(-\ln x)$ (now we have $0\cdot\infty$)

$$= \lim_{x\to 0^+}\frac{\ln(-\ln x)}{1/x} \quad \left(\frac{\infty}{\infty}\right)$$

$$= \lim_{x\to 0^+}\frac{(1/-\ln x)(-1/x)}{-1/x^2} = \lim_{x\to 0^+}\frac{-x}{\ln x} = 0$$

so $\lim_{x\to 0^+} (-\ln x)^x = \lim_{x\to 0^+} y = \lim_{x\to 0^+} e^{\ln y} = e^0 = 1$

see also Ex 8 and 9 p ~~302-3~~ 295-6

more examples:

i, ~~#8~~, $\lim_{x\to 0} \frac{x+\tan x}{\sin x} = \lim_{x\to 0} \frac{1+\sec^2 x}{\cos x} = 2$

ii, $\lim_{x\to \pi} \frac{\tan x}{x} = 0$ (don't need L'Hôpital!)

iii, ~~#26~~, 28 $\lim_{x\to -\infty} x^2 e^{+x} = \lim_{x\to -\infty} \frac{x^2}{e^{-x}} = 0$ (why?)

iv, ~~#32~~, 34 $\lim_{x\to 0} (\csc x - \cot x) = \lim_{x\to 0} \left(\frac{1}{\sin x} - \frac{\cos x}{\sin x}\right) = \lim_{x\to 0} \left(\frac{1-\cos x}{\sin x}\right)$

$= \lim_{x\to 0} \frac{\sin x}{\cos x} = 0$

v, #~~40~~, 44 $\lim_{x\to\infty} x^{\ln 2/1+\ln x}$

$y = x^{\ln 2/1+\ln x}$

$\ln y = \frac{\ln 2}{1+\ln x} \ln x$

so $\lim_{x\to\infty} \ln y = \lim_{x\to\infty} \frac{\ln 2}{1+\ln x} \ln x = \lim_{x\to\infty} \frac{\ln 2 \; 1/x}{1/x} = \lim_{x\to\infty} \ln 2 = \ln 2$

$\therefore \lim_{x\to\infty} x^{\ln 2/1+\ln x} = \lim_{x\to\infty} e^{\ln y} = e^{\ln 2} = 2$

(38)

§4.6: Optimization Problems

299-300

the text offers guidelines for solving optimization problems on p. ~~306~~ basically, we'll apply what we learned in §4.2 on finding the max and min values of a function on a closed interval to "real-world" problems

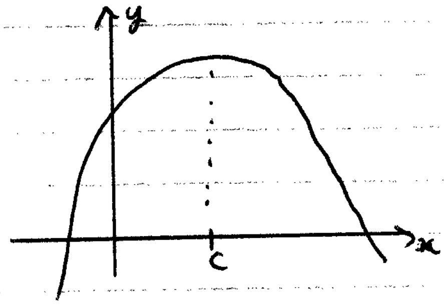
sometimes, we won't have a closed interval for the variable what do we do?

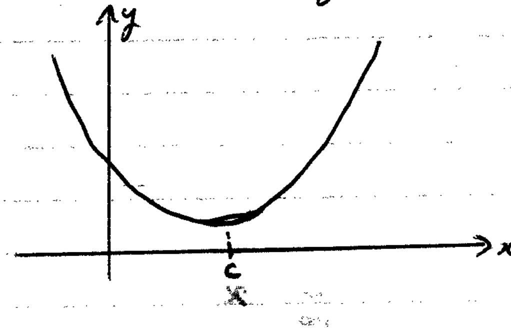
the <u>First Derivative Test for Absolute Extrema</u> (p ~~308~~) 302

suppose that c is a critical number of a continuous function f

a) if $f'(x) > 0$ for all $x < c$ and $f(x) < 0$ for all $x > c$, then $f(c)$ is the absolute maximum value of f

b) if $f'(x) < 0$ for all $x < c$ and $f'(x) > 0$ for all $x < c$, then $f(c)$ is the absolute minimum value of f



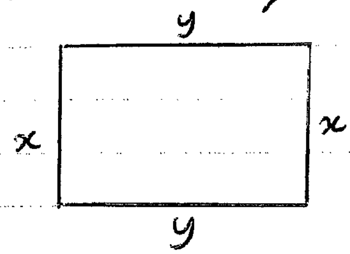


ie if we don't have endpoints to check, we can still determine if we have found the max or min

see Ex. 1-5 on pages ~~306-11~~ 300-4

example : suppose a farmer has 500 m of fence and he wants to enclose a rectangular area with it
(1) such that none of the sides is less than 50m long
what are the dimensions of the rectangle that will enclose the largest area ?

rectangle



let the length of the sides be x and y

so we have the constraint that $2x + 2y = 500$
(the perimeter of the rectangle must equal the amount of fence available)

so $x + y = 250$

we know that $x \geq 50$, $y \geq 50$

but $y = 250 - x$

so $250 - x \geq 50 \Rightarrow x \leq 200$

so the range on x is $50 \leq x \leq 200$

(and similarly $50 \leq y \leq 200$)

now the area of the rectangle is $A = xy$
but we can replace y with $250 - x$
to get $A = xy = x(250 - x) = 250x - x^2$
(a function of one variable, x, only)

so let's put this all into context :
we're trying to find the value of x in the interval $50 \leq x \leq 200$ that gives the maximum value of $A(x) = 250x - x^2$
(sounds familiar, doesn't it ?)

so we can apply our extreme value algorithm :

if $A(x) = 250x - x^2$

then $\dfrac{dA}{dx} = A'(x) = 250 - 2x = 2(125 - x)$

so $A'(x) = 0$ if $x = 125$, which is in the interval so we need to calculate the area of the rectangle for $x = 50, 125$ and 200

$A(50) = 250(50) - (50)^2 = 200(50) = 10000\ m^2$

$A(125) = 250(125) - (125)^2 = (125)^2 = 15625\ m^2$

$A(200) = 250(200) - (200)^2 = 50(200) = 10000\ m^2$

$\therefore$ the maximum area is achieved if $x = 125\,m$ but then $y = 125\,m$ as well and the rectangle is really a square

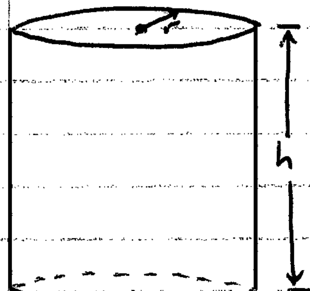
② #7 cylindrical can must have $V = 1000\ cm^3$

like Ex #2 p ~~307~~ 301

a) find the dimensions of the can that require the minimum amount of tin

diameter ≥ 6 cm and height ≥ 4 cm

let r be the radius and h the height



amount of tin = surface area of can

$A = 2\pi r^2 + 2\pi r h$

$V = \pi r^2 h = 1000 \Rightarrow h = \dfrac{1000}{\pi r^2}$

so $A = 2\pi r^2 + 2\pi r \dfrac{1000}{\pi r^2} = 2\pi r^2 + \dfrac{2000}{r}$

$r \geq 3$ cm and $h = \dfrac{1000}{\pi r^2} \geq 4$ cm $\Rightarrow r^2 \leq \dfrac{1000}{4\pi} = 79.577$

so $3 \leq r \leq 8.9$ cm

$\dfrac{dA}{dr} = 4\pi r - \dfrac{2000}{r^2}$ $\qquad \dfrac{dA}{dr} = 0$ if $4\pi r - \dfrac{2000}{r^2} = 0$

or if $4\pi r = \dfrac{2000}{r^2}$ or $r^3 = \dfrac{2000}{4\pi}$ or $r = 5.42$ cm

$A(3) = 2\pi(3)^2 + 2000/3 = 723.22\ cm^2$

$A(5.4) = 2\pi(5.42)^2 + 2000/5.42 = 553.58\ cm^2$ $\quad\therefore$ min when

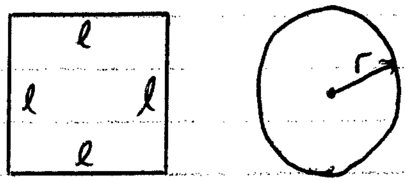
$A(8.9) = 2\pi(8.9)^2 + 2000/8.9 = 722.41\ cm^2$ $\quad r = 5.42$ cm $h = 10.84$ cm

b) notice that $d = 2r = h$

#14 100 cm piece of wire is cut into 2 pieces – one is bent to make a square, the other a circle

(4) how should the wire be cut to a) maximize the total area enclosed? b) minimize

100 cm



let l be the length of the sides of the square and r the radius of the circle

then $4l + 2\pi r = 100$ $\qquad l = (100 - 2\pi r)/4 = 25 - \frac{\pi r}{2}$

total area $A = l^2 + \pi r^2$

$= \left(25 - \frac{\pi r}{2}\right)^2 + \pi r^2$ $\qquad 0 \le r \le \frac{100}{2\pi} = 15.9$ cm

$= 625 - 25\pi r + \frac{\pi^2}{4} r^2 + \pi r^2$

$A'(r) = -25\pi + \frac{\pi^2}{2} r + 2\pi r$ $\quad A'(r) = 0$ if $r = \frac{25\pi}{\frac{\pi^2}{2} + 2\pi} = 7$ cm

$A(0) = 625 - 25\pi(0) + \frac{\pi^2}{4}(0)^2 + \pi(0)^2 = 625 \text{ cm}^2$ $\underset{=l^2}{}$ (no circle)

$A(7) = 625 - 25\pi(7) + \frac{\pi^2}{4}(7)^2 + \pi(7)^2 = 350 \text{ cm}^2$

$A(15.9) = \pi(15.9)^2 = 794 \text{ cm}^2$ (no square)

a) so the area is maximized if there is no square

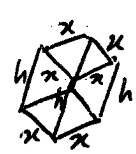
b) the area is minimized if $r = 7$ cm and $l = 14$ cm

i) a closed box has a fixed surface area A and a square base of length x

(5) what is the maximum value of V?

let the height of the box be h

then the surface area is $A = 2x^2 + 4xh$



$h = \frac{A - 2x^2}{4x}$

and the volume is $V = x^2 h = x^2 \frac{A - 2x^2}{4x} = \frac{Ax - 2x^3}{4}$

obviously we must have $x > 0$ but since $V > 0$, need

$Ax - 2x^3 > 0$ or $2x^2 < A$ or $x < \sqrt{\frac{A}{2}}$

so the interval is $0 \le x \le \sqrt{\frac{A}{2}}$

at either endpoint, $V=0$, so max occurs at critical point inside interval

$$\frac{dV}{dx} = \frac{A}{4} - \frac{6}{4}x^2 = \frac{1}{4}(A-6x^2) \quad \text{so} \quad \frac{dV}{dx}=0 \text{ if } x=\sqrt{\frac{A}{6}}$$

then the max volume is $V\left(\sqrt{\frac{A}{6}}\right) = \frac{A}{4}\sqrt{\frac{A}{6}} - \frac{1}{2}\left(\sqrt{\frac{A}{6}}\right)^3$

$$= \frac{1}{4}\frac{A^{3/2}}{6^{1/2}} - \frac{1}{2}\frac{A^{3/2}}{6^{3/2}}$$

$$= \left(\frac{6}{4}-\frac{1}{2}\right)\left(\frac{A}{6}\right)^{3/2} = \left(\frac{A}{6}\right)^{3/2}$$

what is the shape?

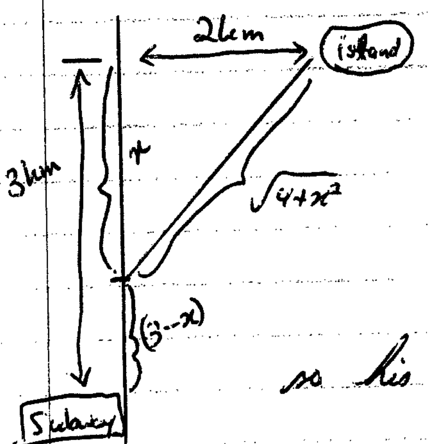
if $x=\sqrt{\frac{A}{6}}$, $h = \dfrac{A-2\left(\sqrt{A/6}\right)^2}{4\sqrt{\frac{A}{6}}} = \dfrac{A-2(A/6)}{4\sqrt{\frac{A}{6}}} = \dfrac{\frac{2}{3}A}{4\sqrt{\frac{A}{6}}} = \frac{2}{3}A\frac{1}{4}\sqrt{\frac{6}{A}}$

$$= \sqrt{\frac{A}{6}}$$

ie the box is a cube

(6) ii, Jared has moved to an island 2 km from the shore
the closest Subway is located 3 km down the coast
if he can walk at 6 km per hour and row at 4 km/h,
what route should he take to minimize his travel time?

like ex 4 p307 303



let x (km) be the distance down the coast where Jared will row to
then he will have to row a distance of $\sqrt{4+x^2}$ km
and walk a distance of $(3-x)$ km

so his travel time is $T(x) = \dfrac{\sqrt{4+x^2}}{4} + \dfrac{(3-x)}{6}$

$$= \frac{1}{4}(x^2+4)^{1/2} - \frac{1}{6}x + \frac{1}{2} \quad \text{(hours)}$$

the ~~range~~ domain for x is $0 \le x \le 3$ km ...

$T'(x) = \frac{1}{8}(x^2+4)^{-1/2}(2x) - \frac{1}{6} = \frac{x}{4\sqrt{x^2+4}} - \frac{1}{6}$

so $T'(x) = 0$ if $\frac{x}{4\sqrt{x^2+4}} - \frac{1}{6} = 0$ $\quad \frac{x}{\sqrt{x^2+4}} = \frac{4}{6} = \frac{2}{3}$

or $3x = 2\sqrt{x^2+4}$ or $9x^2 = 4(x^2+4) = 4x^2+16$

$5x^2 = 16 \qquad x = 1.79$ km

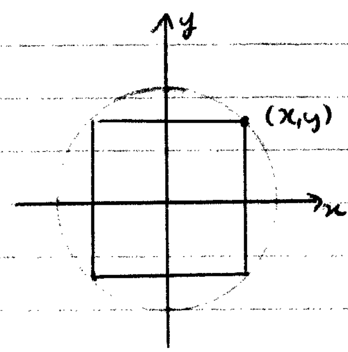
$T(0) = \frac{1}{2} + \frac{1}{2} = 1$ hr

$T(1.79) = \frac{1}{4}\left(\frac{16}{5}+4\right)^{1/2} - \frac{1}{6}(1.79) + \frac{1}{2} = 0.87$ hr $\simeq 52.3$ min

$T(3) = \frac{1}{4}(9+4)^{1/2} - \frac{1}{6}(3) + \frac{1}{2} = \frac{\sqrt{13}}{4} \approx 0.90$ hr $\simeq 54.1$ min

(7) find the largest area of a rectangle that can be inscribed in the circle $x^2+y^2=1$

like Ex #5 p ~~310~~ 303



let x and y be half of the lengths of the sides

then $A = (2x)(2y) = 4xy$

let $x^2+y^2=1 \Rightarrow y = \sqrt{1-x^2}$

so $A(x) = 4x(1-x^2)^{1/2}$

the domain for x is $0 \le x \le 1$ (and $A(0) = A(1) = 0$)

$$\begin{aligned} A'(x) &= 4(1-x^2)^{1/2} + (4x)\left(\tfrac{1}{2}\right)(1-x^2)^{-1/2}(-2x) \\ &= 4(1-x^2)^{1/2} - 4x^2(1-x^2)^{-1/2} \\ &= 4(1-x^2)^{-1/2}\left[(1-x^2) - x^2\right] \\ &= 4(1-x^2)^{-1/2}(1-2x^2) \end{aligned}$$

$A'(x) = 0$ if $x = 1/\sqrt{2}$ then $y = 1/\sqrt{2}$

and the rectangle is a square with area $A = 2$

A10

4.7 § ~~4.8~~: Newton's Method

when we solve $f(x)=0$ for x, we are root finding

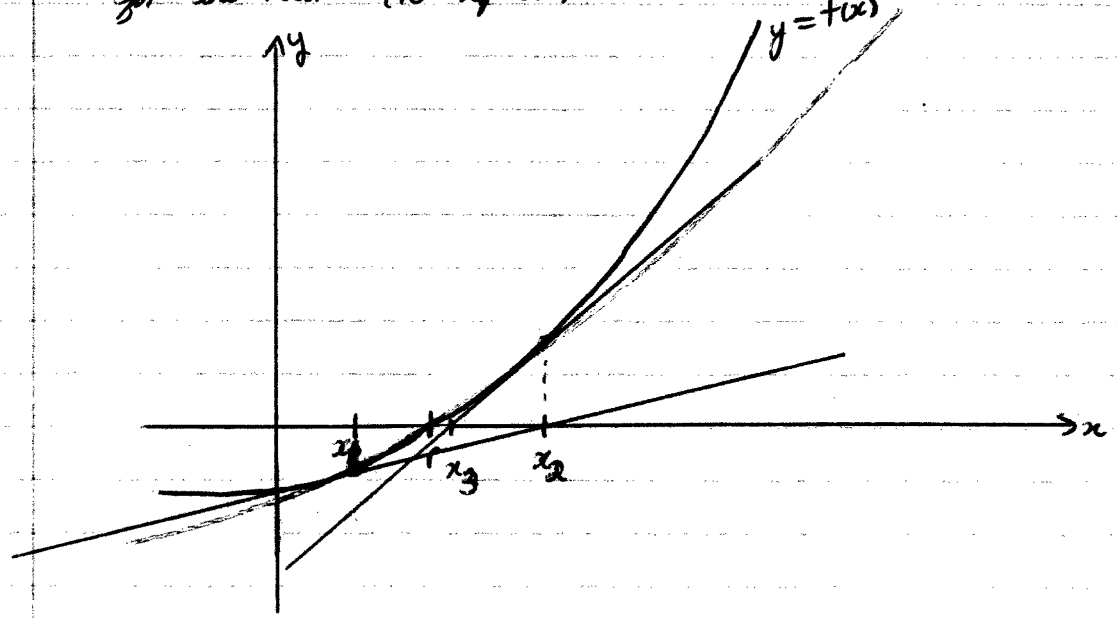
solving $f(x)=0$ is very easy if $f(x)=mx+b$ or $f(x)=ax^2+bx+c$ because we have formulas for the roots

there are formulas for the roots of 3rd and 4th degree polynomials, but we would not want to use them

there are no formulas for the roots of polynomials of degree ≥ 5 and there are no formulas for general $f(x)$ like $f(x)=x^2-3\cos x$

Newton's Method is a numerical approximation that will allow us to find the roots of $f(x)$ to any desired accuracy

assume that $f(r)=0$ and let's say that we have a guess x_1 for the root (ie $x_1 \approx r$)



locate the point $(x_1, f(x_1))$ ~~was~~ on the curve and draw the tangent line $y = f(x_1) + f'(x_1)(x-x_1)$

take the point where this line crosses the x axis as x_2

ie $x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)}$ (we'll have a problem if $f'(x_1)=0$ and we can run into trouble if it's too close to 0) then repeat the procedure

the tangent line at $(x_2, f(x_2))$ is $y = f(x_2) + f'(x_2)(x - x_2)$

where this line crosses the axis is taken as x_3

so $x_3 = x_2 - \dfrac{f(x_2)}{f'(x_2)}$

and so on...

ie $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$ (p ~~323~~) 313

and we generate a sequence of approximations $x_1, x_2, x_3, \ldots$

where $x_n \to r$ as $n \to \infty$ (ie $\lim_{n\to\infty} x_n = r$)

and the sequence is said to converge to r

(more on sequences and convergence in Cal II)

examples:

i) use Newton's Method to find $\sqrt[3]{9}$ to 6 decimal places

$\sqrt[3]{9}$ is a root of $f(x) = x^3 - 9$

and we'll take $x_1 = 2$

$f'(x) = 3x^2$, so $x_{n+1} = x_n - \dfrac{x_n^3 - 9}{3x_n^2} = \dfrac{2x_n^3 + 9}{3x_n^2}$

$$x_2 = \frac{2x_1^3 + 9}{3x_1^2} = \frac{2(2)^3 + 9}{3(2)^2} = \frac{25}{12} = 2.083\ 333\ 333$$

$$x_3 = \frac{2x_2^3 + 9}{3x_2^2} = \frac{2(2.083\ 333\ 333)^3 + 9}{3(2.083\ 333\ 333)^2} = 2.080\ 088\ 889$$

$$x_4 = \frac{2x_3^3 + 9}{3x_3^2} = \frac{2(2.080\ 088\ 889)^3 + 9}{3(2.080\ 088\ 889)^2} = 2.080\ 083\ 823$$

$$x_5 = \frac{2x_4^3 + 9}{3x_4^2} = \frac{2(2.080\ 083\ 823)^3 + 9}{3(2.080\ 083\ 823)^2} = 2.080\ 083\ 823$$

we stop because $x_4 = x_5$ to 6 decimal places (9 actually)

$\therefore \sqrt[3]{9} \approx 2.080\ 084$ (to 6 dec places)

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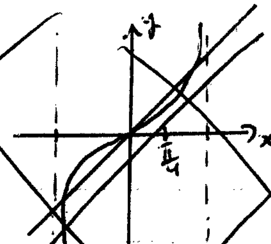
ii, solve $\tan x = 2x$ $(x>0)$ (to 6 decimal places)

write $f(x) = \tan x - 2x$ and take $x_1 = \pi/4$



then $f'(x) = \sec^2 x - 2$

so $x_{n+1} = x_n - \dfrac{\tan x_n - 2x_n}{\sec^2 x_n - 2} = \dfrac{x_n \sec^2 x_n - \tan x_n}{\sec^2 x_n - 2} = \dfrac{x_n - \sin x_n \cos x_n}{1 - 2\cos^2 x_n}$

then $x_2 = \dfrac{x_1 - \sin x_1 \cos x_1}{1 - 2\cos^2 x_1} = \dfrac{\pi/4 - \sin(\pi/4)\cos(\pi/4)}{1 - 2\cos^2(\pi/4)} \simeq 0.$

$x_3 = \dfrac{x_2 - \sin x_2 \cos x_2}{1 - 2\cos^2 x_2} = \dfrac{0.5\ldots - \sin(0.\ldots)\cos(0.\ldots)}{1 - 2\cos^2(0.\ldots)}$

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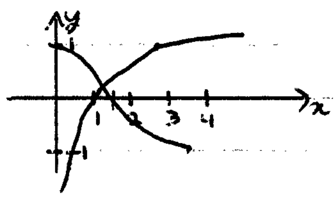
ii, solve $\ln x = \cos x$ (to 6 decimal places)



take $x_1 = 1$

let $f(x) = \ln x - \cos x$

so $f'(x) = \frac{1}{x} + \sin x$

and $x_{n+1} = x_n - \left(\dfrac{\ln x_n - \cos x_n}{\frac{1}{x_n} + \sin x_n} \right)$

$x_2 = x_1 - \left(\dfrac{\ln x_1 - \cos x_1}{\frac{1}{x_1} + \sin x_1} \right) = 1 - \left(\dfrac{\ln(1) - \cos(1)}{\frac{1}{1} + \sin(1)} \right) = 1.293\,407\,993$

$x_3 = x_2 - \left(\dfrac{\ln x_2 - \cos x_2}{\frac{1}{x_2} + \sin x_2} \right) = 1.293407993 - \left(\dfrac{\ln 1.293407993 - \cos(1.293407993)}{\frac{1}{1.293407993} + \sin(1.293407993)} \right.$

$= 1.302\,656\,109$

$x_4 = 1.302\,963\,992$

$x_5 = 1.302\,964\,001$ $\therefore$ to 6 dec places, $x = 1.302964$

$\ln(1.302964) = 0.264641669$ (angle is $\approx 74.7°$)

$\cos(1.302964) = 0.264641671$

See also Ex 1-3 p ~~323-5~~ 313-5

4.8 ~~§4.9~~ : Antiderivatives

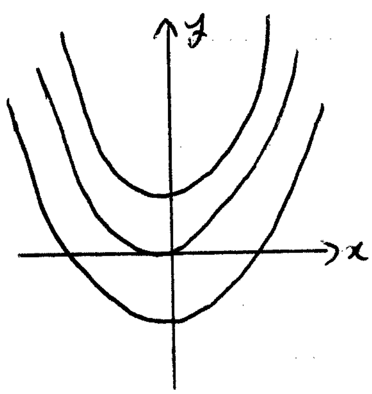
up until now, we've basically been doing the following:
given $f(x)$, what is $f'(x)$?
now, we'll do the opposite: ie given $f'(x)$, what's $f(x)$?
since $f'(x)$ is the derivative of $f(x)$, we call $f(x)$ an antiderivative of $f'(x)$

(~~p 327~~) 317
a function F is called an antiderivative of f on an interval I if $F'(x) = f(x)$ for all x in I

simple example: if $f(x) = 2x$, then $F(x) = x^2$ is an antiderivative
but then so is $G(x) = x^2+2$, $H(x) = x^2-7$, etc...
ie anything of the form x^2+C (C is an arbitrary constant)
is an antiderivative of $f(x) = 2x$

(p ~~327~~ 317) (thm) if F is an antiderivative of f on an interval I, then the most general antiderivative of f on I is $F(x)+C$

ie $F(x)+C$ represents all antiderivatives of $f(x)$



this family of parabolas has form x^2+C and they are all antiderivatives of $2x$

we can use our differentiation rules/formulas to get the following:

Table 2 p ~~329~~ 318

Function $f(x)$	Antiderivative $F(x)$
$x^n \quad (n \neq -1)$	$\frac{x^{n+1}}{n+1} + C$
x^{-1}	$\ln\lvert x\rvert + C$

e^x	$e^x + C$
$\cos x$	$\sin x + C$
$\sin x$	$-\cos x + C$
$\sec^2 x$	$\tan x + C$
$(1+x^2)^{-1}$	$\arctan x + C$, etc...

examples:

i, $f'(x) = x + 2$ $\quad f(x) = \frac{1}{2}x^2 + 2x + C$

ii, $g'(t) = (t+1)^{1/2}$ $\quad g(t) = \frac{2}{3}(t+1)^{3/2} + C$

iii, $\frac{dy}{dx} = 2e^{-x} + x - \cos x$ $\quad y = -2e^{-x} + \frac{1}{2}x^2 - \sin x + C$

we know that the general antiderivative has the form $F(x)+C$ which represents all antiderivatives - a whole family of curves with the ~~th~~ same derivative

but how do we pick a specific one?

by specifying an additional condition

example: find f if $f'(x) = \frac{1}{x+2} + 2$ and $f(0) = 3$

$f(x) = \ln|x+2| + 2x + C$

then $f(0) = \ln 2 + C = 3 \Rightarrow C = 3 - \ln 2$

so $f(x) = \ln|x+2| + 2x + 3 - \ln 2$

(p ~~324~~) 319

this is an example of a <u>differential equation</u> with an initial condition (called an <u>initial value problem</u>)

(more on this in Cal II)

just as we can differentiate more than once, we can also antidifferentiate more than once

example: find f if $f''(x) = 3x^2 + 7x - 1$

if $f''(x) = 3x^2 + 7x - 1$, then $f'(x) = x^3 + \frac{7}{2}x^2 - x + C$
and so $f(x) = \frac{1}{4}x^4 + \frac{7}{6}x^3 - \frac{1}{2}x^2 + Cx + K$

we can get C and K by adding the conditions $f(0) = 2$, $f'(0) = \pi$
then $f(0) = K = 2$ $\quad f'(0) = C = \pi$
so $f(x) = \frac{1}{4}x^4 + \frac{7}{6}x^3 - \frac{1}{2}x^2 + \pi x + 2$

what would do if we were asked to find f if $f'(x) = \sqrt{1+x^4}$?
we have to be careful:
$f(x)$ is NOT $f(x) = \frac{2}{3}(1+x^4)^{3/2} + C$

because then $f'(x) = (1+x^4)^{1/2}(4x^3)$

and it is NOT $f(x) = \frac{2}{3}\frac{(1+x^4)^{3/2}}{4x^3} + C$

because then $f'(x) = \frac{2}{3}\left[\frac{\frac{3}{2}(1+4x^4)^{1/2}(4x^3)^2 - 12x^2(1+x^4)^{3/2}}{(4x^3)^2}\right]$

we would have no luck trying to guess an $f(x)$
See also Ex 1 – 4 p ~~328 – 30~~ 318 – 20

skip { there is a graphical way of getting at least the shape of $f(x)$ using <u>direction fields</u> (p 330)

(direction field example) }

No — if an object moves in a straight line (Rectilinear Motion p ~~331~~ 320), with position function $s(t)$, then its velocity is $v(t)$ and acceleration is $a(t)$
but $a(t) = v'(t) = s''(t)$ $\quad (v(t) = s'(t))$
so $v(t)$ is the antiderivative of $a(t)$, $s(t)$ that of $v(t)$

example: falling object has acceleration $a(t) = -9.8\ m/s^2$
(down is typically taken as the negative direction)

then $v(t) = -9.8t + C$, but $C = v(0) = v_0$ initial velocity so $v(t) = -9.8t + v_0$

and then $s(t) = -4.9t^2 + v_0 t + K$

but $s(0) = K = s_0$ initial position, so $s(t) = -4.9t^2 + v_0 t + s_0$

example: § 4.9 p 335 # ~~40~~ 42

particle moves with $a(t) = 5 + 4t - 2t^2$

initial velocity is $v(0) = 3$ m/s

initial displacement is $s(0) = 10$ m

find its position function

$$v(t) = 5t + 2t^2 - \frac{2}{3}t^3 + v_0 = 5t + 2t^2 - \frac{2}{3}t^3 + 3$$

$$s(t) = \frac{5}{2}t^2 + \frac{2}{3}t^3 - \frac{1}{6}t^4 + 3t + s_0 = -\frac{1}{6}t^4 + \frac{2}{3}t^3 + \frac{5}{2}t^2 + 3t + 10$$

see also Ex ~~6 & 7 p 331-2~~

5 & 6 p 320-1

Chapter 5: Integrals

§5.1 : Areas and Distance

there are 2 central problems in Calculus

i, given $y = f(x)$, what is the slope of the tangent line at $x = a$?

this leads to Differentiation

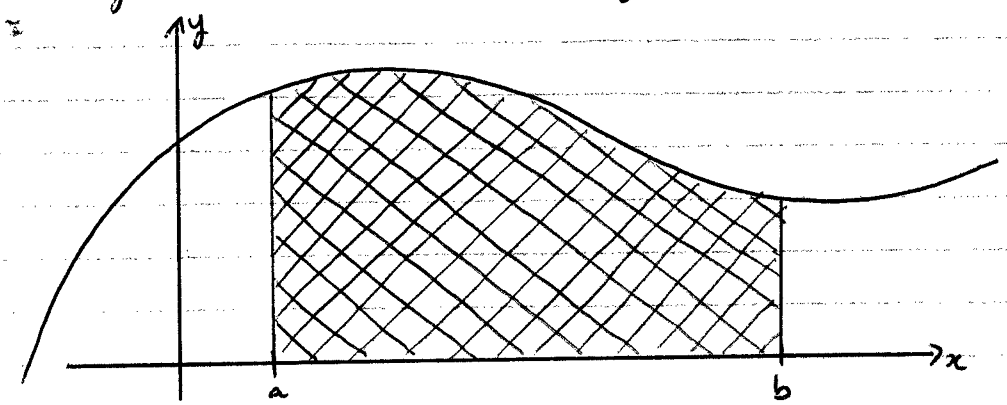
ii, what is the area under the curve $y = f(x)$ between $x = a$ and $x = b$?

this leads to Integration

we shall start studying the 2nd problem now

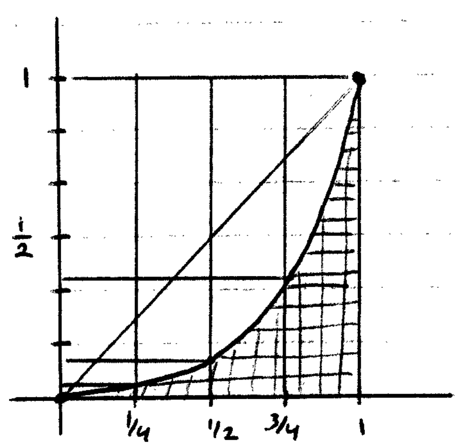
and we'll see later that there is a connection between the 2

through the Fundamental theorem of Calculus (§5.4)



example: what is the area under the curve $y = f(x) = x^3$

between 0 and 1?



we can see that the area, A

is $0 < A < 1/2$

we can make a better estimate

by using rectangles

chop $[0,1]$ into 4 pieces

use the right-hand endpoints of

the subintervals

then we have the following estimate of the area:

$R_4 = \frac{1}{4}\left[\left(\frac{1}{4}\right)^3 + \left(\frac{1}{2}\right)^3 + \left(\frac{3}{4}\right)^3 + (1)^3\right] = 0.390625$ which is an ~~[illegible]~~ overestimate if we used the function value of f at the left hand endpoints, we'd have

$L_4 = \frac{1}{4}\left[(0)^3 + \left(\frac{1}{4}\right)^3 + \left(\frac{1}{2}\right)^3 + \left(\frac{3}{4}\right)^3\right] = 0.140625$ an underestimate

we have $L_4 = 0.140625 < A < R_4 = 0.390625$

what if we did it again with 8 rectangles?
we should get a better approximation because each rectangle would be a better approx

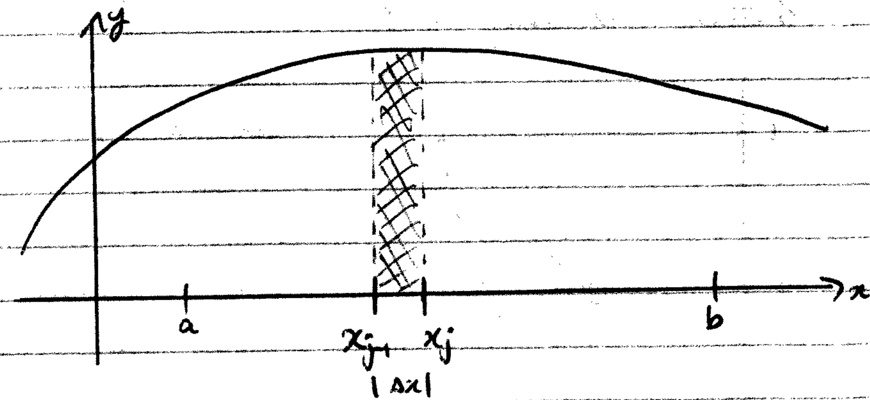
then $R_8 = \frac{1}{8}\left[\left(\frac{1}{8}\right)^3 + \left(\frac{1}{4}\right)^3 + \left(\frac{3}{8}\right)^3 + \left(\frac{1}{2}\right)^3 + \left(\frac{5}{8}\right)^3 + \left(\frac{3}{4}\right)^3 + \left(\frac{7}{8}\right)^3 + (1)^3\right] = 0.31640625$

and $L_8 = \frac{1}{8}\left[(0)^3 + \left(\frac{1}{8}\right)^3 + \left(\frac{1}{4}\right)^3 + \left(\frac{3}{8}\right)^3 + \left(\frac{1}{2}\right)^3 + \left(\frac{5}{8}\right)^3 + \left(\frac{3}{4}\right)^3 + \left(\frac{7}{8}\right)^3\right] = 0.19140625$

so $0.19140625 < A < 0.31640625$
we'd get better approximations with 16 rectangles, etc...
the true value is 0.25 (we'll see how to get this later)

see also Ex 1 & 2 p. 343-6 332-4

let's generalize what we've done:



take the interval $[a,b]$ and chop it into n subintervals of length $\Delta x = \frac{b-a}{n}$

then we have subintervals $[x_0, x_1], [x_1, x_2], \ldots, [x_{j-1}, x_j], \ldots, [x_{n-1}, x_n]$

where $x_0 = a$, $x_1 = x_0 + \Delta x = a + \Delta x$, $x_2 = x_1 + \Delta x = a + 2\Delta x$, etc...
$x_j = a + j\Delta x$, ..., $x_n = a + n\Delta x = a + (b-a) = b$

if we use the right-hand endpoints, we'll have

$$R_n = f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + \cdots + f(x_n)\Delta x$$
$$= \sum_{j=1}^{n} f(x_j)\Delta x$$

or with the left-hand endpoints,

$$L_n = f(x_0)\Delta x + f(x_1)\Delta x + \cdots + f(x_{n-1})\Delta x$$
$$= \sum_{j=0}^{n-1} f(x_j)\Delta x$$

or we could use any <u>sample point</u> x_j^* in the subinterval $[x_{j-1}, x_j]$ and have $f(x_1^*)\Delta x + f(x_2^*)\Delta x + \cdots + f(x_n^*)\Delta x = \sum_{j=1}^{n} f(x_j^*)\Delta x$

the point is that all three of these approximations will get better and better as $n \to \infty$ (or $\Delta x \to 0$)
and in the limit, they will all give the true area:

(p ~~348~~ 337) the <u>area</u> A of the region S that lies under the graph of the continuous function f is the limit of the sum of the areas of the approximating rectangles

ie $$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} \sum_{j=1}^{n} f(x_j)\Delta x$$

or $$A = \lim_{n\to\infty} L_n = \lim_{n\to\infty} \sum_{j=0}^{n-1} f(x_j)\Delta x$$

or $$A = \lim_{n\to\infty} \sum_{j=1}^{n} f(x_j^*)\Delta x$$

let's look at our example again: $y = f(x) = x^3$ between 0 and 1
we'll use R_n

$\Delta x = \frac{1-0}{n} = \frac{1}{n}$ so $x_j = \frac{j}{n}$, $f(x_j) = \left(\frac{j}{n}\right)^3$

so $$R_n = \sum_{j=1}^{n} f(x_j)\Delta x = \sum_{j=1}^{n} \left(\frac{j}{n}\right)^3\left(\frac{1}{n}\right) = \frac{1}{n^4}\sum_{j=1}^{n} j^3$$

from appendix F p A~~47~~ 44 (or p 346 357) $\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$

so $R_n = \frac{1}{n^4}\sum_{j=1}^{n} j^3 = \frac{1}{n^4}\left(\frac{n(n+1)}{2}\right)^2 = \frac{1}{n^4}\,\frac{n^2(n+1)^2}{4} = \frac{1}{4}\left(\frac{n+1}{n}\right)^2 = \frac{1}{4}\left(1+\frac{1}{n}\right)$

then $A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} \frac{1}{4}\left(1+\frac{1}{n}\right)^2 = \frac{1}{4}$

(we'll see a much simpler way to get this later in the Chapter)

if a car is travelling at a constant velocity, we know that it travels a distance equal to velocity × time

how can we estimate distance travelled if velocity is not constant?

let's say we have the data:

time (s)	0	2	4	6	8
velocity (m/s)	20	12	6	2	0

how far did the car travel before it came to a stop?

for $0 \le t \le 2$ s, the car went at most 20 m/s so at most 40 m

$2 \le t \le 4$ s, at most 12 m/s, so at most 24 m

$4 \le t \le 6$ s, " " 6 m/s, " " " 12 m

$6 \le t \le 8$ s, " " 2 m/s, " " " 4 m

ie the car travelled at most $40 + 24 + 12 + 4 = 80$ m

but, on $0 \le t \le 2$s, the car went at least 12 m/s ⇒ at least 24 m

$2 \le t \le 4$s, " " " " " 6 m/s ⇒ " " 12 m

$4 \le t \le 6$s, " " " " " 2 m/s ⇒ " " 4 m

($6 \le t \le 8$s, " " " " " 0 m/s ⇒ " " 0 m)

ie the car travelled at least $24+12+4+0 = 40$ m

how can we get a better estimate?

what if we had the velocity measured every second?

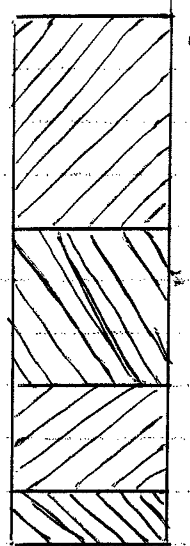
time (s)	0	1	2	3	4	5	6	7	8
velocity (m/s)	20	15	12	8	6	3	2	0.8	0

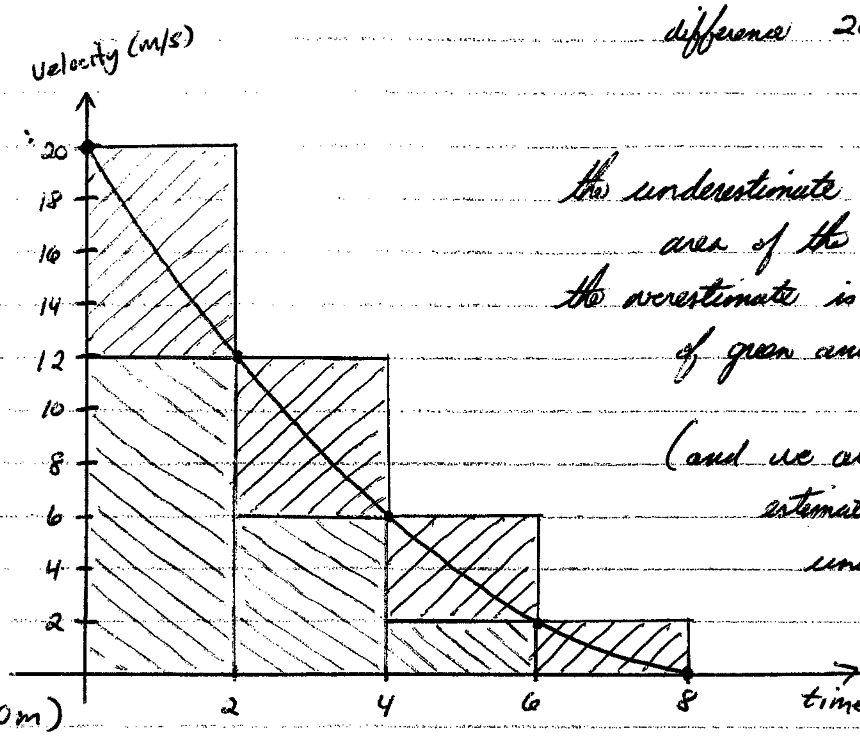
then we'll get at most $20+15+12+8+6+3+2+0.8 = 66.8$ m
and at least $15+12+8+6+3+2+0.8+0 = 46.8$ m

so with the 2 s data : $40\text{ m} \le \text{distance} \le 80\text{ m}$ difference 40 m

and with the 1 s data : $46.8\text{ m} \le \text{distance} \le 66.8\text{ m}$
difference 20 m

graphically :

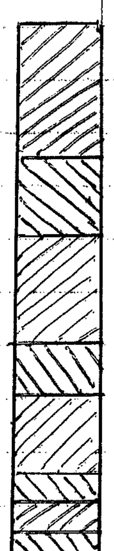


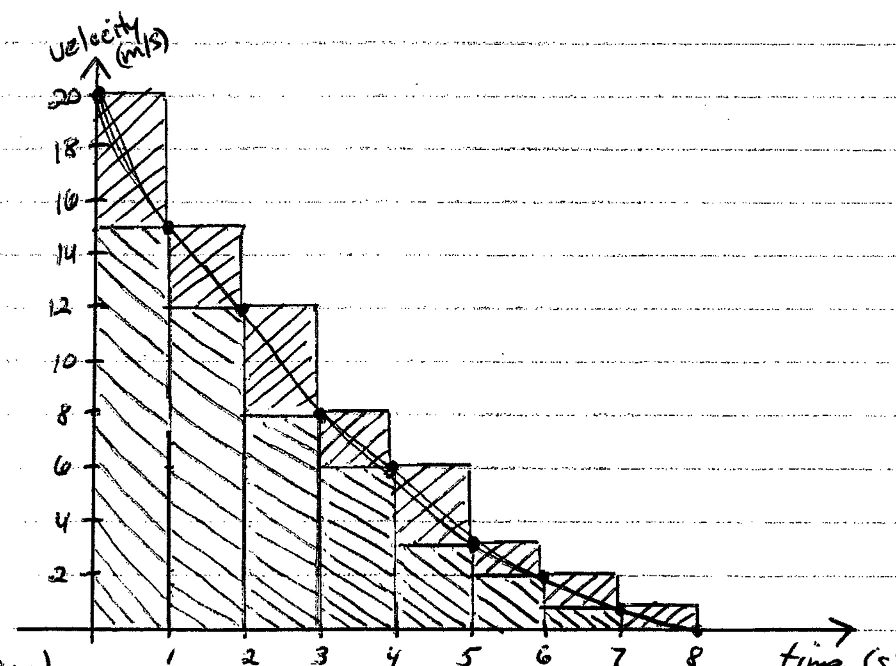


the underestimate is the area of the green rectangles
the overestimate is the area of green and red rectangles

(and we are trying to estimate the area under the curve)

(difference is $\underset{\text{m/s}}{20} \times \underset{\text{s}}{2} = 40\text{ m}$)

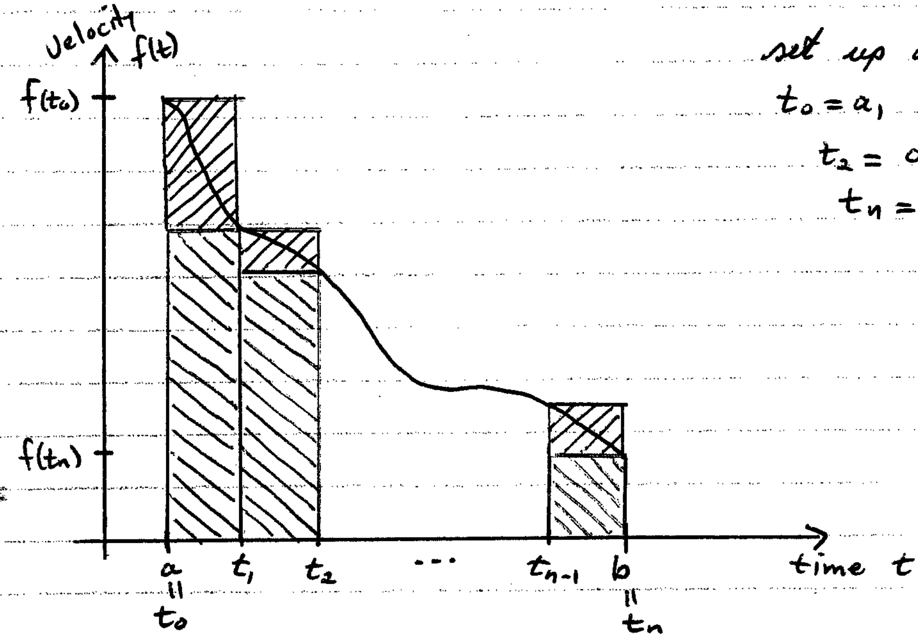




(difference is $\underset{\text{m/s}}{20} \times \underset{\text{s}}{1} = 20\text{ m}$)

and we could narrow the difference even more if we had velocity measurements every 0.5 s, 0.1 s, etc...

now, let's generalize to velocity function $f(t)$ on $a \le t \le b$ with time interval $\Delta t = \frac{b-a}{n}$



set up our time subintervals:
$t_0 = a, \quad t_1 = a + \Delta t,$
$t_2 = a + 2\Delta t, \ldots,$
$t_n = a + n\Delta t = b$

then our overestimate is $f(t_0)\Delta t + f(t_1)\Delta t + \cdots + f(t_{n-1})\Delta t$
$= \sum_{i=0}^{n-1} f(t_i)\Delta t$ called the <u>left hand sum</u>

(p 213)

and our underestimate is $f(t_1)\Delta t + f(t_2)\Delta t + \cdots + f(t_n)\Delta t$
$= \sum_{i=1}^{n} f(t_i)\Delta t$ called the <u>right hand sum</u>

the difference between the 2 sums is
$|\text{overestimate} - \text{underestimate}| = |f(b) - f(a)|\Delta t$

how do we make this difference smaller? by shrinking $\Delta t \to 0$,
or, equivalently, letting $n \to \infty$
in the limit, the difference goes to zero and the sums become equal
and so distance travelled = area under curve $f(t)$ on $a \le t \le b$

$$= \lim_{\Delta t \to 0} \sum_{i=0}^{n-1} f(t_i)\Delta t = \lim_{n\to\infty} \sum_{i=0}^{n-1} f(t_i)\Delta t \qquad (LHS)$$

$$= \lim_{\Delta t \to 0} \sum_{i=1}^{n} f(t_i)\Delta t = \lim_{n\to\infty} \sum_{i=1}^{n} f(t_i)\Delta t \qquad (RHS)$$

See also Ex 4 p 350-1

§ 5.2 : the Definite Integral

we'll give a name to what we were doing in the last section:
if f is a continuous function defined for $a \le x \le b$ and if we have divided the interval $[a,b]$ into n subintervals of length $\Delta x = \frac{b-a}{n}$, so that the subintervals are $[\underset{a}{\overset{\shortparallel}{x_0}}, x_1], [x_1, x_2], \ldots$

$[x_{j-1}, x_j], \ldots, [x_{n-1}, x_n]$ (where $x_j = x_0 + j\Delta x$)
and we have sample points x_j^* in subinterval $[x_{j-1}, x_j]$
(like the endpoints, for example)
then the definite integral of f from a to b is $\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{j=1}^{n} f(x_j^*)\Delta x$

(p ~~354~~ 343)

$$\underset{\text{integral sign}}{\int}_{a \,(\text{lower limit})}^{b \,(\text{upper limit})} \underset{\text{integrand}}{f(x)}\,dx = \lim_{n\to\infty} \underbrace{\sum_{j=1}^{n} f(x_j^*)\Delta x}_{\text{Riemann Sum}}$$

~~(see pages 357-358)~~

$\Sigma \to \int$
$\Delta x \to dx$
$x_j^* \to x$

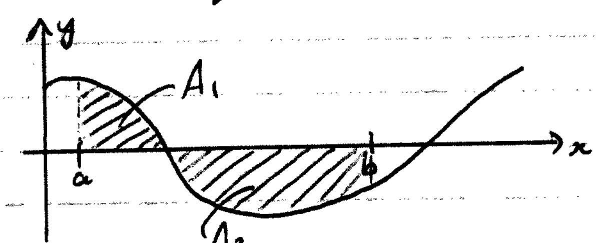
and $\int_a^b f(x)\,dx$ is a number, not a function

the value of the number is independent of the name given to the variable, ie $\int_a^b f(x)\,dx = \int_a^b f(w)\,dw$

we've already seen that if $f(x) \ge 0$, the definite integral has the meaning of the area under the curve from a to b

what if $f(x) < 0$ for some interval in $[a,b]$?

eg.



then $\int_a^b f(x)\,dx$ is the net area, ie $\int_a^b f(x)\,dx$ = area above axis − area below $= A_1 - A_2$

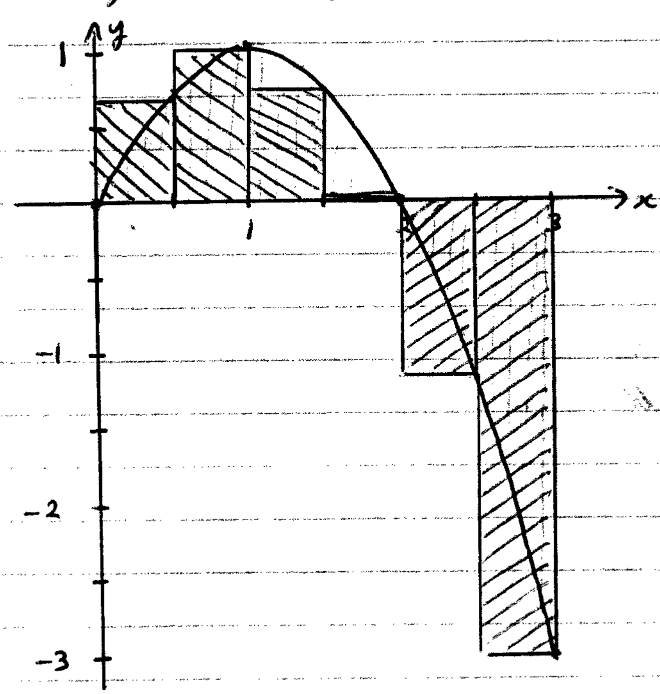
(so $\int_a^b f(x)\,dx < 0$ for this example) also, we can do this for $f(x)$ with a finite number of jump discontinuities or if the subintervals have unequal lengths

there are some useful formulas on page ~~357~~: 346

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}, \quad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}, \quad \sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$$

$$\sum_{i=1}^{n} c = nc, \quad \sum_{i=1}^{n} ca_i = c\sum_{i=1}^{n} a_i, \quad \sum_{i=1}^{n} (a_i \pm b_i) = \sum_{i=1}^{n} a_i \pm \sum_{i=1}^{n} b_i$$

example: evaluate the Riemann sum for $f(x) = 2x - x^2$ using right-hand endpoints with $a = 0$, $b = 3$ with $n = 6$



with $n = 6$, $\Delta x = \frac{b-a}{n} = \frac{3-0}{6} = \frac{1}{2}$

so $x_0 = 0$, $x_1 = 0.5$, $x_2 = 1$, $x_3 = 1.5$, $x_4 = 2$, $x_5 = 2.5$, $x_6 = 3$

the Riemann sum is $R_6 = \sum_{j=1}^{6} f(x_j)\Delta x$

$$= \Delta x\big(f(x_1) + f(x_2) + f(x_3) + f(x_4) + f(x_5) + f(x_6)\big)$$

$= (0.5)\big(f(0.5) + f(1) + f(1.5) + f(2) + f(2.5) + f(3) \big)$
$= (0.5)(0.75 + 1 + 0.75 + 0 - 1.25 - 3)$
$= -0.875$

example : evaluate $\int_0^3 (2x - x^2)\,dx$ (the true value)

ie find $\lim_{n\to\infty} R_n = \lim_{n\to\infty} \sum_{j=1}^{n} f(x_j)\,\Delta x$

with n subintervals, $\Delta x = \frac{3}{n}$, so $x_0 = 0$, $x_1 = \frac{3}{n}$, $x_2 = \frac{6}{n}$, etc ...
ie $x_j = \frac{3j}{n}$

so $\int_0^3 (2x - x^2)\,dx = \lim_{n\to\infty} \sum_{j=1}^{n} f(x_j)\,\Delta x = \lim_{n\to\infty} \sum_{j=1}^{n} f\left(\frac{3j}{n}\right)\left(\frac{3}{n}\right)$

$$= \lim_{n\to\infty} \frac{3}{n} \sum_{j=1}^{n} \left[2\left(\frac{3j}{n}\right) - \left(\frac{3j}{n}\right)^2 \right]$$

$$= \lim_{n\to\infty} \frac{3}{n} \sum_{j=1}^{n} \left[\frac{6}{n} j - \frac{9}{n^2} j^2 \right]$$

$$= \lim_{n\to\infty} \frac{3}{n} \left[\frac{6}{n} \sum_{j=1}^{n} j - \frac{9}{n^2} \sum_{j=1}^{n} j^2 \right]$$

$$= \lim_{n\to\infty} \left[\frac{18}{n^2} \sum_{j=1}^{n} j - \frac{27}{n^3} \sum_{j=1}^{n} j^2 \right]$$

$$= \lim_{n\to\infty} \left[\frac{18}{n^2}\left(\frac{n(n+1)}{2}\right) - \frac{27}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right) \right]$$

$$= \lim_{n\to\infty} \left[9\left(\frac{n(n+1)}{n^2}\right) - \frac{9}{2}\left(\frac{n(n+1)(2n+1)}{n^3}\right) \right]$$

$$= \lim_{n\to\infty} \left[9\left(\frac{n+1}{n}\right) - \frac{9}{2}\left(\frac{(n+1)(2n+1)}{n^2}\right) \right]$$

$$= \lim_{n\to\infty} \left[9\left(1 + \frac{1}{n}\right) - \frac{9}{2}\left(1 + \frac{1}{n}\right)\left(2 + \frac{1}{n}\right) \right]$$

$$= 9 - \frac{9}{2}(2)$$

$= 0$ ie the area of the region above the x axis is equal to the area of the region below

(see Ex 2 p ~~357-8~~ 346-7)

we could have used any sample points we liked
in addition to the endpoints, an obvious choice is to use
the midpoints of the subintervals
the midpoint of $[x_{j-1}, x_j]$ is $x_j^* = \frac{1}{2}(x_{j-1} + x_j) = \bar{x}_j$

so then the Midpoint Rule is $\int_a^b f(x)\,dx \approx \sum_{j=1}^{n} f(\bar{x}_j)\,\Delta x$
(p ~~360~~) 349

example: let's redo our example with the rule
the midpoints are $\bar{x}_1 = 0.25$, $\bar{x}_2 = 0.75$, $\bar{x}_3 = 1.25$, $\bar{x}_4 = 1.75$,
$\bar{x}_5 = 2.25$ and $\bar{x}_6 = 2.75$

so $\int_0^3 (2x - x^2)\,dx \approx \sum_{j=1}^{6} f(\bar{x}_j)\,\Delta x$

$$= (0.5)[f(0.25) + f(0.75) + f(1.25) + f(1.75) + f(2.25) + f(2.75)$$
$$= (0.5)[0.4375 + 0.9375 + 0.9375 + 0.4375 + 0.5625 - 2.0625]$$
$$= 0.0625 \ (= M_6)$$

(which is closer to the true value of 0)
(see Ex 5 p ~~360~~) 349

350 − 2
the Definite Integral satisfies some properties (p ~~361−3~~)

if $a = b$, then $\Delta x = 0$ so $\int_a^a f(x)\,dx = 0$ (ie area under single point is 0)

if we reverse the order of integration to be from b to a,
then $\Delta x = \frac{a-b}{n} = -\left(\frac{b-a}{n}\right)$, so $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$

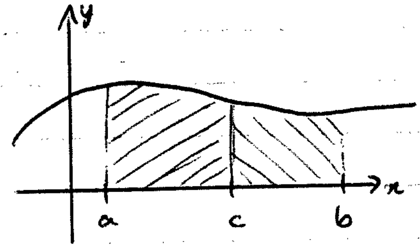
some of the properties follow immediately from the properties of sums:

$\int_a^b c\,dx = c(b-a)$ c is any constant

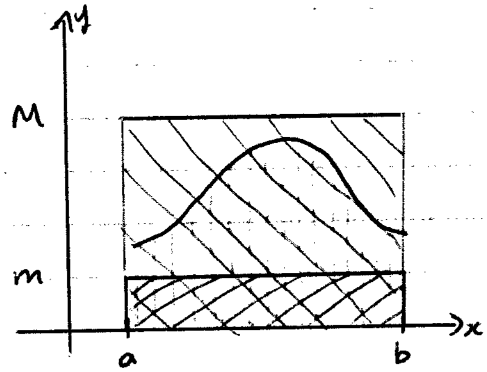
$\int_a^b c\,f(x)\,dx = c\int_a^b f(x)\,dx$ " "

$\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

if $f(x) \geq 0$ on $a \leq x \leq b$, $\int_a^b f(x)\,dx \geq 0$ (area under curve)



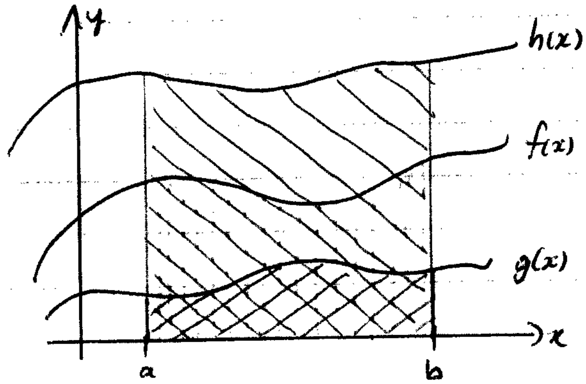
$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$



if $m \leq f(x) \leq M$ for all x on $a \leq x \leq b$,

then $\int_a^b m\,dx \leq \int_a^b f(x)\,dx \leq \int_a^b M\,dx$

or $m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$



if $g(x) \leq f(x) \leq h(x)$ for all x on $a \leq x \leq b$

then $\int_a^b g(x)\,dx \leq \int_a^b f(x)\,dx \leq \int_a^b h(x)\,dx$

A examples:

i) if $\int_0^{10} f(x)\,dx = 5$ and $\int_0^6 f(x) = 7$, what is a) $\int_6^{10} f(x)\,dx$?

$$\int_6^{10} f(x)\,dx = \int_0^{10} f(x)\,dx - \int_0^6 f(x)\,dx = 5 - 7 = -2$$

b) $\int_0^6 3(f(x))\,dx = 21$

ii) $\int_0^1 (4 + x^3)\,dx = \int_0^1 4\,dx + \int_0^1 x^3\,dx = 4(1-0) + \frac{1}{4} = 4\frac{1}{4}$

c) $\int_6^{10} (f(x) + 2)\,dx = -2 + 2(4) = 6$

§ 5.3 : Evaluating Definite Integrals

when we were looking at the example of distance travelled between times $t=a$ and $t=b$, we found that this was equal to the area under the velocity graph

ie $\int_a^b v(t)\,dt$ = distance travelled = $s(b)-s(a)$
or displacement

or $\int_a^b s'(t)\,dt = s(b)-s(a)$ (since $v(t)=s'(t)$)

this is a special case of the Evaluation Theorem (p ~~366~~ 356)
if f is continuous on the interval $[a,b]$,
then $\int_a^b f(x)\,dx = F(b)-F(a)$
where F is any antiderivative of f (ie $F'=f$)

this is also expressed as the ~~Total~~ Net Change theorem (p ~~374~~ 360)
the integral of a rate of change is the total change
$\int_a^b F'(x)\,dx = F(b)-F(a)$

these statements say the same thing (of course) and are part of the Fundamental Theorem of Calculus (§5.4)

examples :

i, $\int_0^1 x^3\,dx = F(1)-F(0)$ where $F(x)$ is any antiderivative of x^3
$= \frac{1}{4}(1)^4 - \frac{1}{4}(0)^4$ take $F(x)=\frac{1}{4}x^4$
$= \frac{1}{4}$

can you see how easy this has become?

why don't we put $F(x) = \frac{1}{4}x^4 + C$?
because then $F(1)-F(0) = \frac{1}{4}(1)^4 + C - \left(\frac{1}{4}(0)^4 + C\right) = \frac{1}{4}$
ie the C's will just cancel anyways

ii, $\int_0^3 (2x-x^2)\,dx = F(3)-F(0)$ $F(x) = x^2 - \frac{1}{3}x^3$
$= (3)^2 - \frac{1}{3}(3)^3 - 0 = 0$

the integral sign gives us a convenient notation for antiderivatives
we can represent an antiderivative of $f(x)$ by the <u>indefinite integral</u> $\int f(x)\,dx$ (p ~~368~~ 358) (no limits)

examples: (see p ~~369~~) 358

i) $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1)$

ii) $\int \frac{1}{x}\,dx = \ln|x| + C$

iii) $\int \cos x\,dx = \sin x + C$

iv) $\int \frac{1}{1+x^2}\,dx = \arctan x + C$, etc...

we must be careful to realize that since $\int f(x)\,dx$ is an antiderivative of $f(x)$, the indefinite integral is a function of x

for example: $\int x^3\,dx = \frac{1}{4}x^4 + C$ (function)

but $\int_0^1 x^3\,dx = \frac{1}{4}x^4\Big|_0^1 = \frac{1}{4}$ (number)

more examples:

i) $\int_0^\pi \sin x\,dx = -\cos x\Big|_0^\pi = -(\cos(\pi) - \cos(0)) = -(-1-1) = 2$
ie the area under the sine curve from 0 to π is 2

ii) $\int (x^2 + 4x + e^x)\,dx = \frac{1}{3}x^3 + 2x^2 + e^x + C$ (general antiderivative or indef. int.)

iii) $\int_0^1 \sqrt{t}\,dt = \int_0^1 t^{1/2}\,dt = \frac{2}{3}t^{3/2}\Big|_0^1 = \frac{2}{3}$

iv) $\int_1^5 \frac{1}{x}\,dx = \ln x\Big|_1^5 = \ln 5 - \ln 1 = \ln 5$

v) $\int \left(\frac{1}{x+1} - \frac{2}{x^2+1}\right)dx = \ln|x+1| - 2\arctan x + C$

(64)

vi, the linear density of an object $\rho(x) = \frac{dm}{dx} = m'(x)$,
the rate of change of the mass,

so the mass of the object is $\int_a^b \rho(x)\,dx$

so if we have a rod of length 3m with density $\rho(x) = \sqrt{x} + e^{-x}$ kg/m
lying on the x axis between $x = 2$ m and $x = 5$ m,
its mass is

$$\text{mass} = \int_2^5 \rho(x)\,dx = \int_2^5 (\sqrt{x} + e^{-x})\,dx = \frac{2}{3}x^{3/2} - e^{-x}\Big|_2^5$$

$$= \frac{2}{3}(5)^{3/2} - e^{-5} - \left(\frac{2}{3}(2)^{3/2} - e^{-2}\right)$$

$$\simeq 5.7 \text{ kg}$$

vii, $\int 2e^{4x}\,dx = \frac{1}{2}e^{4x} + C$

viii, $\int_1^{\pi} (e^{-2x} + 2)\,dx = -\frac{1}{2}e^{-2x} + 2x\Big|_1^{\pi}$

$$= -\frac{1}{2}e^{-2\pi} + 2\pi - \left(-\frac{1}{2}e^{-2} + 2\right) \simeq 4.35$$

ix, we can show that $\int \frac{2t}{1+t^2}\,dt = \ln(1+t^2) + C$

by differentiating : $\frac{d}{dt}\left(\ln(1+t^2) + C\right) = \frac{1}{1+t^2}\frac{d}{dt}(t^2) + 0$

$$= \frac{2t}{1+t^2}$$

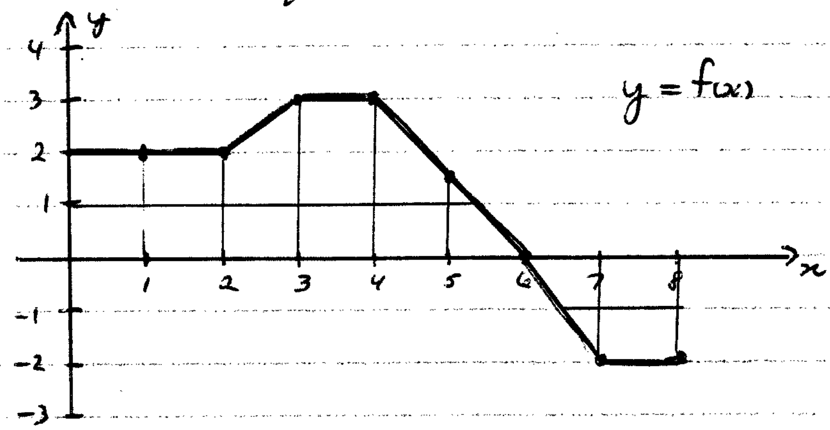
x, $\int \cos(2x)\,dx = \frac{1}{2}\sin(2x) + C$

xi, $\int_2^3 \sec^2 y\,dy = \tan y\Big|_2^3 = \tan(3) - \tan(2) \simeq 2.0425$

See also Ex 1-8 p. ~~363-73~~ 357-63

§5.4 : the Fundamental Theorem of Calculus

suppose we have this function :



and we define $g(x) = \int_0^x f(t)\,dt$, ie $g(x)$ is the definite integral of f from 0 to x

then $g(0) = \int_0^0 f(t)\,dt = 0$

$$g(1) = \int_0^1 f(t)\,dt = (1)(2) = 2$$

$$g(2) = \int_0^2 f(t)\,dt = \int_0^1 f(t)\,dt + \int_1^2 f(t)\,dt = 2 + (1)(2) = 4$$

$$g(3) = \int_0^3 f(t)\,dt = \int_0^2 f(t)\,dt + \int_2^3 f(t)\,dt = 4 + 2 + \tfrac{1}{2}(1)(1) = 6.5$$

$$g(4) = \int_0^4 f(t)\,dt = \int_0^3 f(t)\,dt + \int_3^4 f(t)\,dt = 6.5 + (1)(3) = 9.5$$

$$g(5) = \int_0^5 f(t)\,dt = \int_0^4 f(t)\,dt + \int_4^5 f(t)\,dt = 9.5 + (1)(1.5) + \tfrac{1}{2}(1)(1.5) = 11.75$$

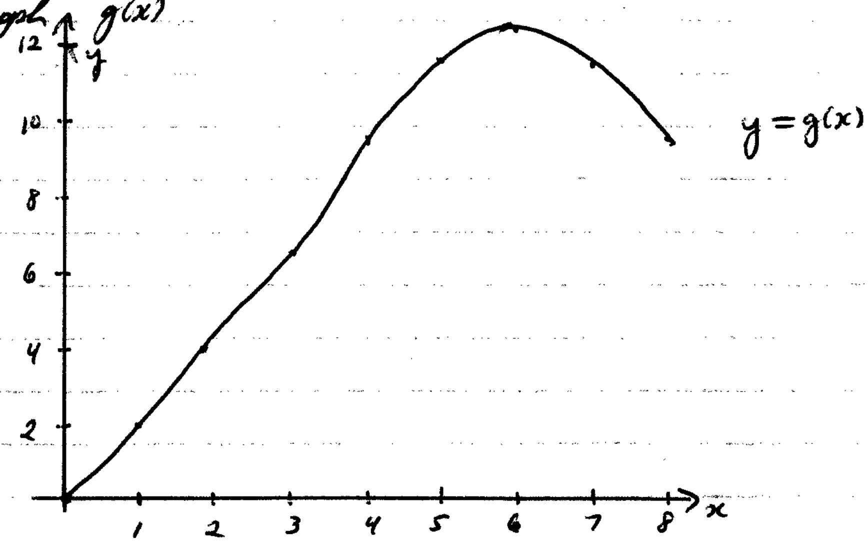
$$g(6) = \int_0^6 f(t)\,dt = \int_0^5 f(t)\,dt + \int_5^6 f(t)\,dt = 11.75 + \tfrac{1}{2}(1)(1.5) = 12.5$$

$$g(7) = \int_0^7 f(t)\,dt = \int_0^6 f(t)\,dt + \int_6^7 f(t)\,dt = 12.5 - \tfrac{1}{2}(1)(2) = 11.5$$

$$g(8) = \int_0^8 f(t)\,dt = \int_0^7 f(t)\,dt + \int_7^8 f(t)\,dt = 11.5 - (1)(2) = 9.5$$

(6)

let's graph $g(x)$



notice that $g(x)$ is a function of x

$g(x)$ is increasing until $x=6$ and then it decreases

ie $g(x)$ has a max at $x=6$

and that is when $f(x)$ changes sign from + to −

what do you suppose $g'(x)$ is?

let's look at another example of these strange functions:

let $f(t) = t^3+2$

and let $g(x) = \int_2^x (t^3+2)\,dt = \frac{1}{4}t^4 + 2t \Big|_2^x =$

$$= \frac{1}{4}x^4 + 2x - \left(\frac{1}{4}(2)^4 + 2(2)\right)$$

$$= \frac{1}{4}x^4 + 2x - 8$$

but notice that $g(x) = F(x) = \frac{1}{4}x^4 + 2x - 8$

and that $F(x)$ is an antiderivative of $f(x) = x^3+2$

ie $g'(x) = F'(x) = x^3+2 = f(x)$

ie $g(x)$ is an antiderivative of $f(x)$

this is part of the Fundamental Theorem of Calculus:

(p ~~379~~ 369 and ~~381~~ 371)

i, if f is continuous on $[a,b]$, then the function g defined by
$$g(x) = \int_a^x f(t)\,dt \qquad a \le x \le b$$
is an antiderivative of f, ie $g'(x) = f(x)$ for $a < x < b$

or $\frac{d}{dx}\left(\int_a^x f(t)\,dt\right) = f(x)$

ii, $\int_a^b f(x)\,dx = F(b) - F(a)$ where F is any antiderivative of f
(Evaluation theorem from §5.3)

it is very easy to see why part i, is true :
$$\frac{d}{dx}\left(\int_a^x f(t)\,dt\right) = \frac{d}{dx}\left(F(t)\Big|_a^x\right) = \frac{d}{dx}\Big(F(x) - \underset{\substack{\uparrow \\ \text{constant}}}{F(a)}\Big) = F'(x) = f(x)$$

what is the Fundamental Theorem telling us?
it says that differentiation and integration are inverse processes
ie they "undo" what each other does

it may be hard to imagine defining functions via integrals,
but they actually do pop up in applications
some examples are :

the <u>Fresnel integral</u> $\quad S(x) = \int_0^x \sin(\pi t^2/2)\,dt$

the <u>Fresnel function</u> $\quad F(x) = \int_0^x \cos(t^2)\,dt$

the <u>sine - integral</u> $\quad Si(x) = \int_0^x \frac{\sin t}{t}\,dt$

the <u>error function</u> $\quad erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\,dt$

Note : it is necessary to write these functions as integrals because
we cannot find the antiderivatives exactly

examples :

i, if $g(x) = \int_0^x \sqrt{1+t^3}\, dt$, then $g'(x) = \sqrt{1+x^3} = (1+x^3)^{1/2}$

and $g''(x) = \frac{1}{2}(1+x^3)^{-1/2}(3x^2) = \frac{3}{2}\frac{x^2}{\sqrt{1+x^3}}$, etc ...

ii, if $g(x) = \int_a^x f(t)\, dt$, and $f(t) = \int_c^t h(u)\, du$

then $g'(x) = f(x) = \int_c^x h(u)\, du$

so $g''(x) = f'(x) = h(x)$

iii, $g(x) = \int_1^x \ln(t^2+1)\, dt$, then $g'(x) = \ln(x^2+1)$

so $g'(2) = \ln 5$

iv, find $\frac{d}{dx}\left(\int_2^{\sqrt{x}} \cos t\, dt\right)$

we have to be careful here because of the $\sqrt{x}$, we'll need the Chain Rule

if we let $g(x) = \int_2^x \cos t\, dt$, then $g(\sqrt{x}) = \int_2^{\sqrt{x}} \cos t\, dt$,

so $\frac{d}{dx}\left(\int_2^{\sqrt{x}} \cos t\, dt\right) = \frac{d}{dx}\left(g(\sqrt{x})\right) = g'(\sqrt{x})\,\frac{d}{dx}(\sqrt{x})$

we know that $g'(x) = \cos x$, so $g'(\sqrt{x}) = \cos(\sqrt{x})$

and $\frac{d}{dx}(\sqrt{x}) = \frac{1}{2}x^{-1/2}$

so $\frac{d}{dx}\left(\int_2^{\sqrt{x}} \cos t\, dt\right) = \cos(\sqrt{x})\left(\frac{1}{2}x^{-1/2}\right) = \frac{1}{2\sqrt{x}}\cos(\sqrt{x})$

(see also Ex 5 p ~~384~~) 370

or let $u = \sqrt{x}$, then $\frac{d}{dx}\left(\int_2^{\sqrt{x}} \cos t\, dt\right) = \frac{d}{dx}\left(\int_2^{u} \cos t\, dt\right) =$

$$= \frac{d}{du}\left(\int_2^u \cos t\,dt\right)\frac{du}{dx} = \cos u\,\frac{du}{dx} = \cos\sqrt{x}\left(\tfrac{1}{2}x^{-1/2}\right)$$

(since $\frac{d}{dx}(f(u)) = \frac{df}{du}\frac{du}{dx}$)

v) $\frac{d}{dx}\left(\int_1^{x^2} \ln(t+e^t)\,dt\right) = 2x\ln(x^2+e^{x^2})$

§5.5 : the Substitution Rule

in this section, we'll learn how to "reverse" the Chain Rule and find antiderivatives of more complicated functions.

back in section 4.9, we saw that we could not find an antiderivative for $\sqrt{1+x^4}$ (ie $\int\sqrt{1+x^4}\,dx$ we can't find)

but what about $\int 4x^3\sqrt{1+x^4}\,dx$?

since we know that $\frac{d}{dx}\left((1+x^4)^{3/2}\right) = \frac{3}{2}(1+x^4)^{1/2}(4x^3)$,

we can say that $\int 4x^3\sqrt{1+x^4}\,dx = \frac{2}{3}(1+x^4)^{3/2} + C$

we can see how to get this :
if we let $u = 1+x^4$, then $du = \frac{du}{dx}\,dx = 4x^3\,dx$

so $\int 4x^3\sqrt{1+x^4}\,dx = \int\sqrt{u}\,du = \frac{2}{3}u^{3/2}+C = \frac{2}{3}(1+x^4)^{3/2}+C$

another way to look at this $u = 1+x^4 = g(x)$
then $g'(x) = 4x^3$

so $\int 4x^3\sqrt{1+x^4}\,dx = \int g'(x)\sqrt{g(x)}\,dx = \int f(g(x))\,g'(x)\,dx$

where $f(u) = \sqrt{u}$

if $F'(x) = f(x)$, then we have $\int F'(g(x))g'(x)\,dx$

but we know that $\frac{d}{dx}\left(F(g(x))\right) = F'(g(x))g'(x)$
(Chain Rule)

so $\int F'(g(x))g'(x)\,dx = F(g(x)) + C$

so we have the Substitution Rule ~~(p 387)~~ : 376
if $u=g(x)$ is a differentiable function whose range is an interval I and f is continuous on I, then

$$\int f(g(x))\,g'(x)\,dx = \int f(u)\,du = F(u) + C = F(g(x)) + C$$

(since $u = g(x)$, $\frac{du}{dx} = g'(x)$, so $du = g'(x)dx$)

examples:

i) $\int \frac{2t}{t^2+1}\,dt$ if we let $u = t^2+1$, then $du = 2t\,dt$

$= \int \frac{du}{u} = \ln|u| + C = \ln|t^2+1| + C = \ln(t^2+1) + C$

ii) $\int e^{-3x}\,dx$ let $u = -3x$, then $du = -3\,dx$

$= -\frac{1}{3}\int e^{-3x}(-3)\,dx = -\frac{1}{3}\int e^{u}\,du = -\frac{1}{3}e^{u} + C = -\frac{1}{3}e^{-3x} + C$

iii) $\int \cot x\,dx = \int \frac{\cos x}{\sin x}\,dx$ $u = \sin x$, $du = \cos x\,dx$

$= \int \frac{du}{u} = \ln|u| + C = \ln|\sin x| + C$

iv) $\int x^2\cos(x^3+\pi)\,dx$ $u = x^3+\pi$ $du = 3x^2\,dx$

$= \frac{1}{3}\int \cos(u)\,du = \frac{1}{3}\sin u + C = \frac{1}{3}\sin(x^3+\pi) + C$

see also examples 1-5 on pages ~~387-9~~ 376-8

What do we do with definite integrals?
we have a choice - find the antiderivative with substitution
or change the limits of integration

example: $\int_1^3 \frac{3x}{\sqrt{1+x^2}}\,dx$

to find $\int \frac{3x}{\sqrt{1+x^2}}\,dx$ let $u = 1+x^2$, $du = 2x\,dx$

$$= \frac{3}{2}\int \frac{du}{\sqrt{u}} = \frac{3}{2}(2)\,u^{1/2} + C = 3u^{1/2} + C = 3\sqrt{1+x^2} + C$$

so $\int_1^3 \frac{3x}{\sqrt{1+x^2}}\,dx = 3\sqrt{1+x^2}\Big|_1^3 = 3\sqrt{10} - 3\sqrt{2} = 3(\sqrt{10}-\sqrt{2})$

or if $u = 1+x^2$, then when $x=1$ $u=2$, and when
$x = 3$, $u = 10$, so

$$\int_1^3 \frac{3x}{\sqrt{1+x^2}}\,dx = \frac{3}{2}\int_2^{10}\frac{du}{\sqrt{u}} = \frac{3}{2}(2)\sqrt{u}\Big|_2^{10} = 3\sqrt{u}\Big|_2^{10} = 3(\sqrt{10}-\sqrt{2})$$

ie we have the <u>Substitution Rule for Definite Integrals</u> (p 389):
if g' is continuous on $[a,b]$ and f is continuous on the range of $u = g(x)$, then

$$\int_a^b f(g(x))\,g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$$

example: $\int_0^{\pi/2} \sin x \cos x\,dx$ $\quad u = \sin x \quad du = \cos x\,dx$
$x=0 \;\; u=0, \; x = \pi/2, \; u=1$

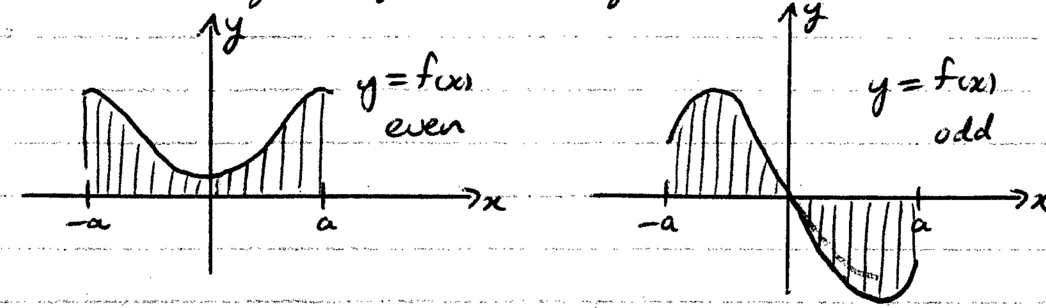
$$= \int_0^1 u\,du = \frac{1}{2}u^2\Big|_0^1 = \frac{1}{2}(1-0) = \frac{1}{2}$$

see also examples 6, 7, 8 on pages ~~378~~ 379-80

A09 A08 A10

back in chapter ~~1~~, we met the idea of symmetry:
 f is an even function if $f(-x) = f(x)$ for all x in domain
 f " " odd " " $f(-x) = -f(x)$ " " " " "

look at the graphs of functions of these kinds:



then we can see what the Integrals of Symmetric Functions over symmetric limits must be (p ~~391~~): 380

suppose f is continuous on $[-a, a]$

i, if f is even, $\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$

ii, if f is odd, $\int_{-a}^{a} f(x)\,dx = 0$

examples:

i, $\int_{-2}^{2} (x^3 + x^5)\,dx = 0$ (because $x^3 + x^5$ is odd)

or $\int_{-2}^{2} (x^3 + x^5)\,dx = \frac{1}{4}x^4 + \frac{1}{6}x^6 \Big|_{-2}^{2} = \frac{1}{4}(2)^4 + \frac{1}{6}(2)^6 - \left(\frac{1}{4}(-2)^4 + \frac{1}{6}(-2)^6\right)$
$= 0$

ii, $\int_{-1}^{1} (x^2+2)\,dx = 2\int_0^1 (x^2+2)\,dx = 2\left(\frac{1}{3}x^3 + 2x\right)\Big|_0^1 = 2\left(\frac{1}{3}(1)^3 + 2(1)\right) - 0$
$= 14/3$

(because x^2+2 is even)

another example:

See also Ex 9 & 10 p ~~391-2~~ 381

$\int_0^1 \frac{2x}{1+x^4}\,dx = \int_0^1 \frac{2x}{1+(x^2)^2}\,dx$ let $u = x^2$, $du = 2x$
$x=0, u=0$ $x=1, u=1$

$= \int_0^1 \frac{du}{1+u^2} = \arctan u \Big|_0^1 = \arctan(1) - \arctan(0) = \frac{\pi}{4} - 0 = \frac{\pi}{4}$

§ 5.6 : Integration by Parts

in this section, we'll see how to "reverse" the Product Rule and hence find antiderivatives of products

let u and v be functions of x
by the Product Rule $\frac{d}{dx}(uv) = \left(\frac{d}{dx}(u)\right)v + u\frac{d}{dx}(v)$

$$= u'v + uv'$$

rewrite this as $uv' = \frac{d}{dx}(uv) - u'v$

now, let's integrate $\int uv'\,dx = \int \frac{d}{dx}(uv)\,dx - \int u'v\,dx$

but we know that $\int \frac{d}{dx}(uv)\,dx = uv$, so we have

$$\boxed{\int uv'\,dx = uv - \int u'v\,dx}$$ the formula for Integration by Parts

or, equivalently, $\boxed{\int u\,dv = uv - \int v\,du}$ (p ~~394~~) 383, since $dv = v'dx$ and $du = u'dx$

so, how do we use it? by identifying the integral on the LHS with the one we're trying to solve

so, how do we pick u and v to use the formula $\int u\,dv = uv - \int v\,du$?

first of all, the integral on the RHS should be easier to find than the one on the left (our original problem)

here's some tips to make this happen:

i, when we pick dv, we need to know what v is (otherwise, we'll get nowhere)

ii, we want du to be nicer (or at least no worse) than u

iii, we want v to be nicer (or at least no worse) than dv

examples :

i, $\int xe^x dx = xe^x - \int e^x dx$ $\qquad u = x \quad du = dx$
$\qquad dv = e^x dx \quad v = e^x$

$= xe^x - e^x + C$

check : $\frac{d}{dx}(xe^x - e^x + C) = e^x + xe^x - e^x = xe^x$

ii, $\int x \cos x \, dx = x \sin x - \int \sin x \, dx$ $\qquad u = x \quad du = dx$
$\qquad dv = \cos x \, dx \quad v = \sin x$

$= x \sin x + \cos x + C$

See Ex 1 p ~~394~~ 383-4 check: $\frac{d}{dx}(x \sin x + \cos x + C) = \sin x + x\cos x - \sin x = x \cos x$

what if we had chosen differently in i, ?

$\int xe^x dx = \frac{1}{2}x^2 e^x - \int \frac{1}{2}x^2 e^x dx$ $\qquad u = e^x \quad du = e^x dx$
$\qquad dv = x dx \quad v = \frac{1}{2}x^2$

this is true, but the integral on the RHS is "worse" than the one we started with

example : we can use Integration by Parts to get antiderivatives even when it doesn't look like there is a product present : eg $\int \arctan(x) \, dx$

the reason - we can always write $\arctan(x) = (1)(\arctan(x))$

so $\int \arctan(x) \, dx = \int (1) \arctan(x) \, dx$

but how to choose u and v ?

if we took $u = 1$ then $du = 0 \, dx = 0$

but we'd have $dv = \arctan(x) \, dx$ and $v = \int \arctan(x) dx$

ie gets us nowhere

so we'd have to take $dv = dx \Rightarrow v = x$

and $u = \arctan(x)$, so $du = \frac{1}{1+x^2} dx$

hold it a minute, doesn't this violate our guidelines
ie we want v no worse than dv ?
not really and here's why

$$\int \arctan(x)\,dx = x\arctan(x) - \int \frac{x}{1+x^2}\,dx$$

and the integral on the RHS is much nicer than LHS

so $\int \arctan(x)\,dx = x\arctan(x) - \int \frac{x}{1+x^2}\,dx$ | simple substitution let $y = 1+x^2$ then $dy = 2x\,dx$

$$= x\arctan(x) - \frac{1}{2}\int \frac{dy}{y}$$

$$= x\arctan(x) - \frac{1}{2}\ln|y| + C$$

See Ex 2 p ~~395~~ 384

$$= x\arctan(x) - \frac{1}{2}\ln|1+x^2| + C$$

$$= x\arctan(x) - \frac{1}{2}\ln(1+x^2) + C \quad (\text{since } 1+x^2 \geq 1)$$

sometimes, we have to use integration by parts more than once:
examples:

i, Ex 3 p ~~395~~ 384-5

$$\int x^2 e^x\,dx = x^2e^x - \int 2xe^x\,dx$$

$u = x^2$ $du = 2x\,dx$
$dv = e^x dx$ $v = e^x$

$$= x^2e^x - 2\int xe^x\,dx$$

$$= x^2e^x - 2(xe^x - e^x) + C$$
$$= x^2e^x - 2xe^x + 2e^x + C$$

ii, $\int e^x \cos x\,dx$

here's the problem - no matter how we pick u and v,
du and dv are of the same difficulty, so we'll get
nowhere, right?

$\int e^x\cos x\,dx = e^x\sin x - \int e^x \sin x\,dx$ so we're nowhere ?
$u = e^x$ $du = e^x dx$ no - do it again and

(76) $\int e^x \cos x\, dx = e^x \sin x - \int e^x \sin x\, dx$ (do Int by Parts on integral on RHS)

$u = e^x \quad dv = \sin x\, dx$

$du = e^x dx \quad v = -\cos x$

$= e^x \sin x - \left(-e^x \cos x - \int e^x(-\cos x)\, dx\right)$

See Ex 4 p ~~396~~ 385

$= e^x \sin x + e^x \cos x - \int e^x \cos x\, dx \leftarrow$ but this is what we want!

so $2\int e^x \cos x\, dx = e^x \sin x + e^x \cos x + C$

$\therefore \quad \int e^x \cos x\, dx = \frac{1}{2}\left(e^x \sin x + e^x \cos x\right) + C$

one last thing - what do we do with definite integrals?
go back to the derivation of the formula:

$\int u v'\, dx = \int \frac{d}{dx}(uv)\, dx - \int u' v\, dx$

so must have $\int_a^b u v'\, dx = \int_a^b \frac{d}{dx}(uv)\, dx - \int_a^b u' v\, dx$

ie $\boxed{\int_a^b u v'\, dx = uv\Big|_a^b - \int_a^b u' v\, dx}$

or $\boxed{\int_a^b u\, dv = uv\Big|_a^b - \int_a^b v\, du}$ p ~~399~~ 386

example: $\int_1^2 \ln x = x \ln x\Big|_1^2 - \int_1^2 (x)\left(\frac{1}{x}\right) dx$ $\quad u = \ln x \quad du = \frac{1}{x}\, dx$

$dv = dx \quad v = x$

$= x \ln x\Big|_1^2 - \int_1^2 dx$

$= x \ln x\Big|_1^2 - x\Big|_1^2$

$= (2)\ln(2) - (1)\ln(1) - (2-1) = 2\ln 2 - 1$

$(\simeq 0.3863)$

See Ex 5 p ~~397~~ 386-7
and Ex 6 p ~~397~~

§ 5.7 : Additional Techniques of Integration

often, we'll have to use identities to help us find integrals involving the trig functions

examples : i, $\int \cos^2 x \, dx$

we can't use a simple substitution like $y = \cos x$, because then $dy = -\sin x \, dx$ and we don't have the $\sin x$

what about integration by parts?

$u = \cos x \qquad du = -\sin x \, dx$

$dv = \cos x \, dx \qquad v = \sin x$

then $\int \cos^2 x \, dx = \sin x \cos x + \int \sin^2 x \, dx$ true, but not helpful

recall the double angle formula $\cos 2x = \cos^2 x - \sin^2 x$

and the identity $\cos^2 x + \sin^2 x = 1$

then $\cos 2x = \cos^2 x - \sin^2 x = \cos^2 x - (1 - \cos^2 x) = 2\cos^2 x - 1$

so $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$

and so $\int \cos^2 x \, dx = \int \frac{1}{2}(1 + \cos 2x) \, dx = \frac{1}{2}\left(x + \frac{1}{2}\sin 2x\right) + C$

$= \frac{1}{2}x + \frac{1}{4}\sin 2x + C$

ii, $\int \sin x \cos x \, dx$

there are 3 obvious ways to do this one :

a) $\int \sin x \cos x \, dx = \int u \, du = \frac{1}{2}u^2 + C = \frac{1}{2}\sin^2 x + C$

$u = \sin x \quad du = \cos x \, dx$

b, $\int \sin x \cos x \, dx = -\int u \, du = -\frac{1}{2}u^2 + C = -\frac{1}{2}\cos^2 x + C$

$u = \cos x \quad du = -\sin x \, dx$

c, $\int \sin x \cos x \, dx = \int \frac{1}{2}\sin 2x \, dx = -\frac{1}{4}\cos 2x + C$

they look different, but are they really iii, $\int \sin^3 x \, dx = \int \sin^2 x \sin x \, dx = \int (1-\cos^2 x) \sin x \, dx$

$= \int \sin x \, dx - \int \cos^2 x \sin x \, dx \qquad u = \cos x \quad du = -\sin x \, dx$

$= \int \sin x \, dx + \int u^2 \, du$

$= -\cos x + \frac{1}{3} u^3 + C = \frac{1}{3} \cos^3 x - \cos x + C$

See Ex 1&2 p ~~400~~ 389-90

sometimes we can use <u>trigonometric substitution</u>:

examples: i, $\int \frac{dx}{\sqrt{1-x^2}}$ let $x = \sin\theta$
$dx = \cos\theta \, d\theta$

So $\int \frac{dx}{\sqrt{1-x^2}} = \int \frac{\cos\theta \, d\theta}{\sqrt{1-\sin^2\theta}} = \int \frac{\cos\theta}{\cos\theta} d\theta = \int d\theta = \theta + C$
$= \arcsin x + C$

ii, $\int \frac{dx}{1+x^2}$ let $x = \tan\theta \quad dx = \sec^2\theta \, d\theta$

$= \int \frac{\sec^2\theta \, d\theta}{1+\tan^2\theta} = \int \frac{\sec^2\theta \, d\theta}{\sec^2\theta} = \int d\theta = \theta + C = \arctan x + C$

iii, $\int \frac{dx}{x^2+a^2}$ let $x = a\tan\theta \quad dx = a\sec^2\theta \, d\theta$

$= \int \frac{a\sec^2\theta \, d\theta}{a^2\tan^2\theta + a^2} = \int \frac{a\sec^2\theta \, d\theta}{a^2(\tan^2\theta+1)} = \int \frac{a\sec\theta \, d\theta}{a^2 \sec\theta}$

$= \frac{1}{a} \int d\theta = \frac{1}{a}\theta + C = \frac{1}{a} \arctan\left(\frac{x}{a}\right) + C$

see Ex 3 p ~~401-2~~ 390-1 for proof that the area of a circle is πr^2

a nice technique for finding the integrals of rational functions is called <u>partial fractions</u>

See also Appendix G (p A56) 47

examples: i, $\int \frac{x+9}{x^2-2x-3} \, dx$ here the numerator is not a multiple of the derivative of the denominator so the substitution $u = x^2 - 2x - 3$ is not going to help

notice that the denominator factors : ie $x^2 - 2x - 3 = (x+1)(x-3)$

so $\dfrac{x+9}{x^2-2x-3} = \dfrac{x+9}{(x+1)(x-3)} = \dfrac{A}{x+1} + \dfrac{B}{x-3}$

(since the degree of the numerator is smaller than that of the denominator)

we must have $A(x-3) + B(x+1) = x+9$

$(A+B)x + (B-3A) = x+9$

so $A+B = 1$ and $B-3A = 9$ $\quad A = -2,\ B = 3$

$$\therefore \int \frac{x+9}{x^2-2x-3}\,dx = \int\left(\frac{-2}{x+1} + \frac{3}{x-3}\right)dx = -2\ln|x+1| + 3\ln|x-3| + C$$

ii, $\displaystyle\int \frac{2x^3-5x^2+3x-10}{x^2-2x-3}\,dx$ here the degree of the numerator is greater than denominator, so we do long division first

$$\begin{array}{r} 2x - 1 \\ x^2-2x-3 \,\big)\, \overline{2x^3 - 5x^2 + 3x - 10} \\ \underline{2x^3 - 4x^2 - 6x} \\ -x^2 + 9x - 10 \\ \underline{-x^2 + 2x + 3} \\ 7x - 13 \end{array}$$

$$\therefore \frac{2x^3-5x^2+3x-10}{x^2-2x-3} = 2x - 1 + \frac{7x-13}{x^2-2x-3}$$

$$= 2x - 1 + \frac{7x-13}{(x+1)(x-3)}$$

$A + B = 7$

$-3A + B = -13$

$A = 5,\ B = 2$

$$= 2x - 1 + \frac{A}{x+1} + \frac{B}{x-3}$$

$$= 2x - 1 + \frac{5}{x+1} + \frac{2}{x-3}$$

and so $\displaystyle\int \frac{2x^3-5x^2+3x-10}{x^2-2x-3}\,dx$

$$= \int\left(2x-1+\frac{5}{x+1}+\frac{2}{x-3}\right)dx = x^2-x+5\ln|x+1|+2\ln|x-3|+C$$

what if we cannot factor the denominator?
try completing the square

examples: i) $\displaystyle\int \frac{2}{x^2+4x+5}\,dx = \int \frac{2}{(x+2)^2+1}\,dx$ let $y=x+2$, $dy=dx$

$$= \int \frac{2}{y^2+1}\,dy = 2\int \frac{dy}{1+y^2} = 2\arctan y + C$$
$$= 2\arctan(x+2)+C$$

ii) $\displaystyle\int \frac{3}{4t^2-8t+20}\,dt = \frac{3}{4}\int \frac{dt}{t^2-2t+5} = \frac{3}{4}\int \frac{dt}{(t-1)^2+4}$

let $y=t-1$ $dy=dt$

$$= \frac{3}{4}\int \frac{dy}{y^2+(2)^2}$$
$$= \frac{3}{4}\left(\frac{1}{2}\right)\arctan\left(\frac{y}{2}\right)+C$$
$$= \frac{3}{8}\arctan\left(\frac{t-1}{2}\right)+C$$

check: $\displaystyle\frac{d}{dt}\left(\frac{3}{8}\arctan\left(\frac{t-1}{2}\right)+C\right) = \frac{3}{8}\left(\frac{1}{1+\left(\frac{t-1}{2}\right)^2}\right)\left(\frac{1}{2}\right)$

$$= \frac{3}{16}\left(\frac{1}{1+\frac{(t-1)^2}{4}}\right) = \frac{3}{4}\left(\frac{1}{4+(t-1)^2}\right) = \frac{3}{4}\left(\frac{1}{t^2-2t+5}\right)$$

iii) $\displaystyle\int \frac{2x+6}{x^2+4x+5}\,dx = \int\left(\frac{2x+4}{x^2+4x+5}+\frac{2}{x^2+4x+5}\right)dx$

$$= \ln|x^2+4x+5| + 2\arctan(x+2)+C$$

see also Ex 4 & 5 p ~~402-4~~ 391-3

§ 5.9 : Approximate Integration

at the beginning of the chapter, we saw that we can approximate the definite integral $\int_a^b f(x)dx$ with

(p412) the left hand sum (p401) $\int_a^b f(x)\,dx \approx L_n = \sum_{j=0}^{n-1} f(x_j)\,\Delta x = \sum_{j=1}^{n} f(x_{j-1})\,\Delta x$

the right hand sum (p402) $\int_a^b f(x)\,dx \approx R_n = \sum_{j=1}^{n} f(x_j)\,\Delta x$

or the Midpoint Rule (p402) $\int_a^b f(x)\,dx \approx M_n = \sum_{j=1}^{n} f(\bar{x}_j)\,\Delta x$

where $\Delta x = \frac{b-a}{n}$, $x_j = a + j\Delta x$, $\bar{x}_j = \frac{1}{2}(x_{j-1}+x_j)$

sometimes, we'll need to use these numerical approximations because we will not be able to find an antiderivative (exactly)
eg $\int_0^1 e^{x^2}dx$, $\int_0^1 \cos(x^2)\,dx$

example : $\int_0^1 \cos(x^2)\,dx$ with M_4 to 6 decimal places

$n=4 \Rightarrow \Delta x = \frac{1-0}{4} = 0.25$, $x_0 = 0$, $x_1 = 0.25$, $x_2 = 0.5$, $x_3 = 0.75$, $x_4 = 1$

and $\bar{x}_1 = 0.125$, $\bar{x}_2 = 0.375$, $\bar{x}_3 = 0.625$, $\bar{x}_4 = 0.875$

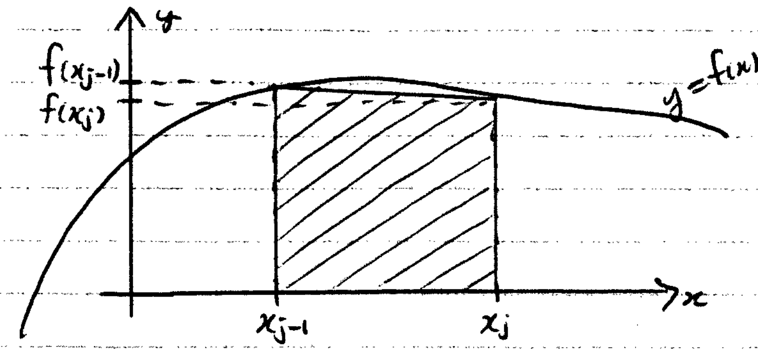
so $\int_0^1 \cos(x^2)\,dx \simeq M_4 = \sum_{j=1}^{4} \cos(\bar{x}_j^2)\,(0.25)$

$$= (0.25)\left[\cos((0.125)^2) + \cos((0.375)^2) + \cos((0.625)^2) + \cos((0.875)^2)\right]$$
$$\simeq 0.908\,907$$

the Midpoint Rule works by "averaging" the x values
what do we get if we average the y values?
ie $\frac{1}{2}(f(x_{j-1}) + f(x_j))$?

then we'll have $\int_a^b f(x)\,dx \approx \frac{1}{2}(L_n + R_n) = \frac{1}{2}\sum_{j=1}^{n}\left[f(x_{j-1}) + f(x_j)\right]\Delta x = T_n$

this is called the Trapezoidal Rule because it is
(p413) 402 equivalent to using trapezoids to estimate the area



the area of the trapezoid is
$\frac{1}{2}(f(x_{j-1}) + f(x_j))\Delta x$

example: $\int_0^1 \cos(x^2)\,dx$ with T_4

$$\int_0^1 \cos(x^2)\,dx \approx T_4 = \frac{1}{2}\sum_{j=1}^{4}\left[f(x_{j-1}) + f(x_j)\right]\Delta x$$

$$= \frac{1}{2}\sum_{j=1}^{4}\left[\cos((x_{j-1})^2) + \cos((x_j)^2)\right]\Delta x$$

$$= \frac{1}{2}(0.25)\left[\cos(x_0^2) + \cos(x_1^2) + \cos(x_1^2) + \cos(x_2^2) + \cos(x_2^2) + \cos(x_3^2) + \cos(x_3^2) + \cos(x_4^2)\right]$$

$$= \frac{1}{2}(0.25)\left[\cos(x_0^2) + 2\cos(x_1^2) + 2\cos(x_2^2) + 2\cos(x_3^2) + \cos(x_4^2)\right]$$
$$= \frac{1}{2}(0.25)\left[\cos(0^2) + 2\cos((0.25)^2) + 2\cos((0.5)^2) + 2\cos((0.75)^2) + \cos(1^2)\right]$$
$$\simeq 0.895\,759$$

see also Ex 1 p ~~413~~ 403

we define the errors of these approximations to be (p414)

$$E_M = \int_a^b f(x)\,dx - M_n\ , \qquad E_T = \int_a^b f(x)\,dx - T_n$$

so how good are our results?
since we don't know the true value of $\int_0^1 \cos(x^2)\,dx$,
we cannot find the errors exactly
but we can make estimates, called error bounds: (p415 405

suppose $|f''(x)| \le K$ for $a \le x \le b$, then

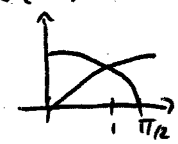
$$|E_M| \le \frac{K(b-a)^3}{24n^2} \qquad |E_T| \le \frac{K(b-a)^3}{12n^2}$$

if $f(x) = \cos(x^2)$, $f'(x) = -2x\sin(x^2)$

$f''(x) = -2\sin(x^2) - 4x^2\cos(x^2)$

so $|f''(x)| = 2\sin(x^2) + 4x^2\cos(x^2)$ on $[0,1]$

both sin & cos ≥ 0



so we'll certainly have $|f''(x)| \leq 6$ on $[0,1]$

so $|E_M| \leq \dfrac{6(1)^3}{24(4)^2} = 0.015625$

and $|E_T| \leq \dfrac{6(1)^3}{12(4)^2} = 0.03125$

our results are probably (much) better than these estimates indicate

we can also use the error bound formulas to help us figure out how many steps to do:

if we wanted to use the Midpoint Rule to get $\int_0^1 \cos(x^2)\,dx$ to 6 decimal place accuracy, how many steps are required?

so we want $|E_M| \leq 0.000\,001$

ie need $\dfrac{K(b-a)^3}{24n^2} < 0.000\,001$

or $24n^2 > \dfrac{K(b-a)^3}{0.000\,001} = \dfrac{6(1)}{0.000\,001} = 6\,000\,000$

so $n > 500$

so, using our estimate for K, we'd need $n = 500$ steps to <u>guarantee</u> 6 decimal place accuracy

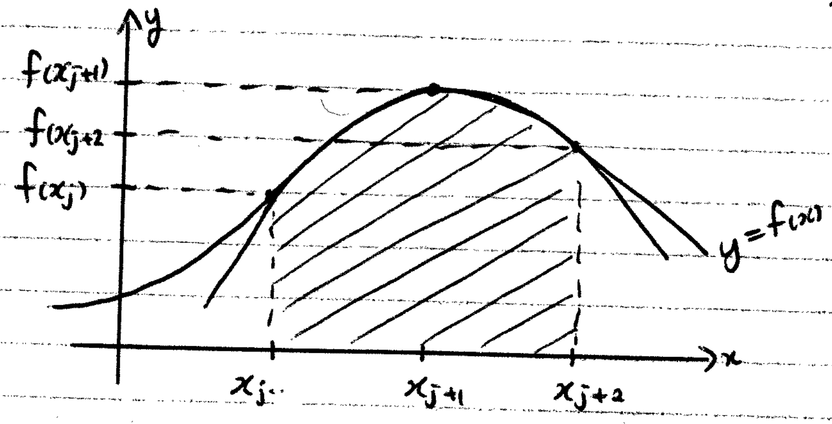
in reality, a much smaller number of steps will probably do (20, 50, 100 ?)

because we've only made a rough estimate for K and the error bound formula only gives a bound, not the true value of the error

See also Ex 2 & 3 p ~~415-6~~ 405-6

a more sophisticated numerical integration techniques is called

<u>Simpson's Rule</u> (p ~~416-8~~ 406-8) and it uses parabolas:



we take 3 points $(x_j, f(x_j))$, $(x_{j+1}, f(x_{j+1}))$, $(x_{j+2}, f(x_{j+2}))$
we find the (unique) parabola that passes through them and then use the area under the parabola to approximate the area under the curve
the area under the parabola is $\frac{\Delta x}{3}\left(f(x_j) + 4f(x_{j+1}) + f(x_{j+2})\right)$

if we sum over all of the parabolas, we get

$$\int_a^b f(x)\,dx \approx S_n = \frac{\Delta x}{3}\Big(f(x_0) + 4f(x_1) + f(x_2) + f(x_2) + 4f(x_3) + \cdots + f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\Big)$$

$$= \frac{\Delta x}{3}\Big(f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\Big)$$

Note: with Simpson's Rule, n <u>must</u> be even

the <u>error bound</u> is: (p ~~419~~ 409)
suppose $|f^{(4)}(x)| \le K$ on $a \le x \le b$
then $|E_S| \le \frac{K(b-a)^5}{180\, n^4}$

note that $|E_S|$ goes like n^{-4}
ie Simpson's Rule has smaller errors than Midpoint or Trap

example : $\int_0^1 \cos(x^2)\,dx$ with S_4

$$\int_0^1 \cos(x^2)\,dx \simeq S_4 = \frac{\Delta x}{3}\left(\cos(x_0^2) + 4\cos(x_1^2) + 2\cos(x_2^2) + 4\cos(x_3^2) + \cos(x_4^2)\right)$$

$$= \frac{0.25}{3}\left(\cos(0) + 4\cos((0.25)^2) + 2\cos((0.5)^2) + 4\cos((0.75)^2) + \cos(1^2)\right)$$

$$\simeq 0.904\,501$$

Simpson's Rule actually is related to Mid and Trap
read the section carefully and look at the examples

See also Ex 4-7 p ~~418-20~~ 408-10