MAT2377A - Probability and Statistic for Engineers

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Sets

Set A set is a collection of items.

$A = \{2, 3, 4\}$

$A = B$ implies that any $x \in A \to x \in B$

Operations on Sets

Let A and B be two sets.

$A \cap B = A$ intersects $B = A$ and $B = \{x : x \in A \text{ and } x \in B\}$

$A \cup B = A$ union $B = A$ or $B = \{x : x \in A \text{ or } x \in B\}$

(1) Example:

$A = \{2, 3, 4\}$ and $B = \{3, 4, 5\} \to A \cap B = \{3, 4\}, A \cup B = \{2, 3, 4, 5\}$

Sample Space

Sample Space All the possible outcomes in a random experiment.

$S =$ the set of outcomes in a random experiment.

(2) Example: Flip a coin

$S = \{H, T\}$

(3) Example: Roll a die

$S = \{1, 2, 3, 4, 5, 6\}$

(4) Example: Flip a coin until a heads appears.

$S = \{H, TH, TTH, \ldots\}$

(5) Example: For a circle of radius 1, any point within the circle is what set?

$S = \{(x \times y) : x^2 + y^2 \leq 1\}$

Countable Set A set in which you can list the entries, and an uncountable set is one which will not let you list the entries.

Event

Event Any subset of S (sample space) is called an event.

(6) Example:

Coin is heads: $E = \{H\} \subset \{H, T\}$

So E is an event.

(7) Example:

Die is on even number: $E = \{2, 4, 6\}$

(8) Another example:

$E = \{(x, y) : x^2 + y^2 \leq \frac{1}{4}\}$

Enumeration (Counting)

1. Multiplication Rule.

   If task 1 can be completed in n_1 different ways and task 2 can be completed in n_2 different ways, then both tasks can be completed by $n_1 \times n_2$ ways.

   (9) Example: How many 5 digit numbers are there?

Task 1: 1,2,3,4,5,6,7,8,9. $n_1 = 9$.

Task 2: 0-9. $n_2 = 10$.

...

Task 5: 0-9. $n_5 = 10$.

$9 \times 10^4 = 90,000$

(10) Example: How many 5 digit numbers with no repeats do we have?

$n_1 = 9, n_2 = 9, n_3 = 8, n_4 = 7, n_5 = 6$
$= 9 * 9 * 8 * 7$

(11) Example: In how many different ways can we write 1,2,3,4,5 in unique order?

$5 * 4 * 3 * 2 * 1 = 5!$

(12) Example: How many different orders of these letters is possible, where repeat letters are to be treated as non-unique: INDEPENDENT?

$11!/(3! * 2! * 3!)$

2. Permutation

(13) Example: With letters a, b, c, d, write all possible permutations with 2 letters.

ab, ac, ad,
ba, bc, bd,
ca, cb, cd,
da, db, dc.

$P_2^4 = 4 \times 3 = 12$

$P_r^n = ?$

n letters, $\{a_r \ldots a_n\}$, $n \leq r$ r letters.

n = number of ways for completing task 1.

n-1 = number of ways for completing task 2.

... until task r:

$n - (r - 1)$ = number of ways for completing task r.

Rule of Permutations $P_r^n = n(n-1) \times \ldots \times (n - r + 1)$

Permutations of r letters from n letters. ($r \leq n$)

$P_r^n = n(n-1)\dots(n-r+1)$

(14) Example

$P_2^4 = 4 \times 3 = 12 \to \{a, b, c, d\}$

In general for Square Permutations

$P_n^n = n(n-1)\dots(3)(2)(1) = n!$

In general for Any Permutations

$P_r^n = \frac{n(n-1)\dots(n-r+1)(n-r)(n-r-1)\dots1}{(n-r)(n-r-1)\dots1} = \frac{n!}{(n-r)!}$

Combinations

$C_r^n = \binom{n}{r}$

Combination # of combination of r letters from n leters.

(15) Example: Let $n = 4, r = 2$

$a, b, c, d \to ab, ac, ad, bc, bd, cd$

$\binom{4}{2} = 6$

$P_2^4 = 2\binom{4}{2}$

(16) Example

$\binom{4}{3} =?$

$S = \{a, b, c, d\}$

$abc, abd, acd, bcd \to \binom{4}{3} = 4$

$P_3^4 = 3!\binom{4}{3}$

General Combination Formula

$\therefore P_r^n = r!\binom{n}{r} \to \binom{n}{r} = \frac{P_r^n}{r!} = \frac{\frac{n!}{(n-r)!}}{r!} = \frac{n!}{r!(n-r)!}$

$\binom{n}{r} = \frac{n!}{r!(n-r)!}$

$P_r^n = \frac{n!}{(n-r)!}$

(17) Example: How many hands of 5 cards do we have?

$\binom{52}{5} = \frac{52!}{5!47!} = \frac{52\times51\times50\times49\times48}{120}$

(18) Example: How many diagonals do we have in a polygone with n sides?

$\binom{5}{2} - 5 = 5$

In general, the solution is:

$\binom{n}{2} - n = \frac{n(n-1)}{2} - n$

$= \frac{n(n-1)-2n}{n} = \frac{n(n-1-2)}{2} = \frac{n(n-3)}{2}$

(19) Example: A city has m by n streets in a grid. How many ways can someone traverse from one corner of the city to the other?

$\frac{(m+n)!}{m!n!} = \binom{m+n}{m} = \binom{m+n}{n}$

(20) Example: How many ways can we write out $(a+b)^5$ expanded?

$(a+b)^5 = a^5 + \frac{5!}{4!1!}a^4b + \frac{5!}{3!2!}a^3b^2 + \frac{5!}{2!3!}a^2b^3 + \frac{5!}{1!4!}ab^4 + b^5$

A General Polynomial Combinational Formula

$$(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^i b^{n-i}$$

$$(a+b+c)^n = \sum\sum \binom{n}{i,j} a^i b^j c^{n-i-j}$$

Properties of Combinations

(i) $\binom{n}{r} = \binom{n}{n-r}$

$\frac{n!}{r!(n-r)!} = \frac{n!}{r!(n-r)!(n-(n-r))!}$

(ii) $\binom{n}{r-1} + \binom{n}{r} = \binom{n+1}{r}$

$\frac{n!}{(r-1)!(n-r+1)1} + \frac{n!}{r!(n-r)!}$

$= \frac{rn!+n!(n-r+1)}{r!(n-r+1)!}$

$= \frac{(n+1)!}{r!(n-r+1)!}$

$= \binom{n+1}{r}$

(21) Example: How many subsets do we have if the set has n elements?

$S = \{a, b, c\}$

$\{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}$

You are effectively doing, in this example, n choose 0, n choose 1, n choose etc until n.

$\binom{n}{0} + \binom{n}{1} + \ldots + \binom{n}{n} = 2^n$

To explain this, we can use the multiplication rule and say, "For task i, choose the i^{th} letter or do not choose it."