2012 MTH 140 MID I



1. [3 marks] (Multiple choice question). The domain of the function $f(x) = \dfrac{1}{1-\sqrt{x-2}}$ is the set:

   A) $(2,\infty)$ B) $[2,\infty)$ C) $(2,3)\cup(3,\infty)$ D) $[2,3)\cup(3,\infty)$ E) None of these

   Write the (capital) letter of the answer in this box. **Only** the answer in the box will be marked. $\boxed{D)}$

   $x-2 \geqslant 0$ so $x \geqslant 2$

   $1-\sqrt{x-2}=0 \iff \sqrt{x-2}=1 \iff x=3$

   Domain: $\{x \geqslant 2 : x \neq 3\}$ or $\boxed{[2,3)\cup(3,\infty)}$

2. [3 marks] (Multiple choice question). If $\cot\beta = 3$ and $\pi < \beta < 2\pi$, then $\cos\beta$ is equal to:

   A) $1/3$ B) $3/\sqrt{10}$ C) $-1/\sqrt{10}$ D) $-\sqrt{10}/3$ E) None of these

   Write the (capital) letter of the answer in this box. **Only** the answer in the box will be marked. $\boxed{E)}$

   - If $\cot\beta = 3 > 0$ and $\pi < \beta < 2\pi$ then $\pi < \beta < \frac{3\pi}{2}$, so $\cos\beta < 0$.
   - $\cot\beta = 3 \Rightarrow \frac{\cos\beta}{\sin\beta} = 3 \Rightarrow \sin\beta = \frac{\cos\beta}{3}$
   - $\cos^2\beta = 1-\sin^2\beta = 1-\frac{\cos^2\beta}{9} \Rightarrow \cos^2\beta\left(1+\frac{1}{9}\right)=1$

   $\Rightarrow \cos^2\beta = \frac{9}{10} \Rightarrow \boxed{\cos\beta = -\frac{\sqrt{3}}{10}}$

3. [3 marks] (Multiple choice question). Let $f(x) = |(x+1)(x+2)| + |x-3|$ and consider a number $x$ belonging to the interval $(-2,-1)$. Then $f(x)$ is equal to:

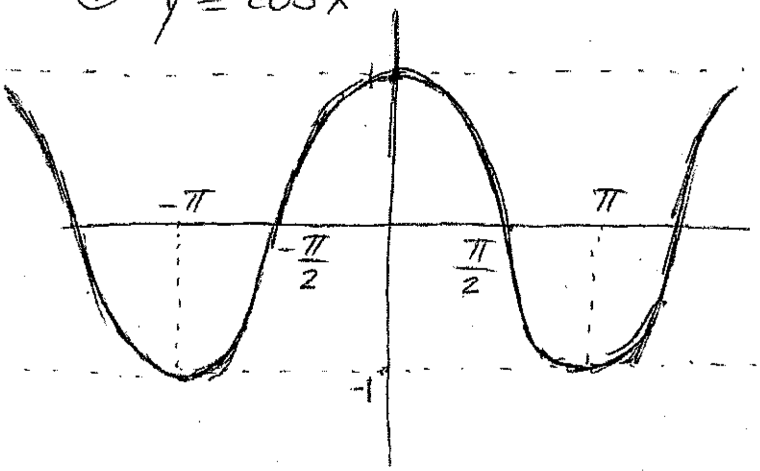
   A) $x^2+2x+5$ B) $x^2+4x-1$ C) $-x^2-2x-5$ D) $-x^2-2x-1$
   E) None of these

   Write the (capital) letter of the answer in this box. **Only** the answer in the box will be marked. $\boxed{E)}$

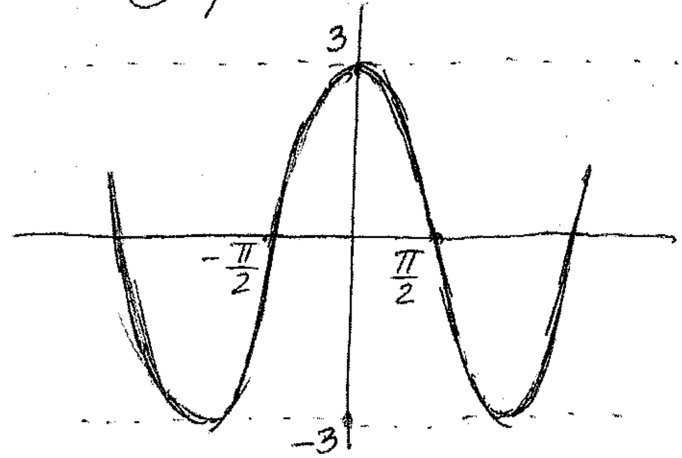
   $$f(x) = |x+1|\cdot|x+2| + |x-3|$$
   $$= (-x-1)(x+2) - (x-3)$$
   $$= -x^2-4x+1$$

4. [5 marks] Starting with the graph of $g(x) = \cos x$, apply the appropriate transformations to sketch the graph of $f(x) = |2 - 3\cos x|$. Show clearly all your steps.

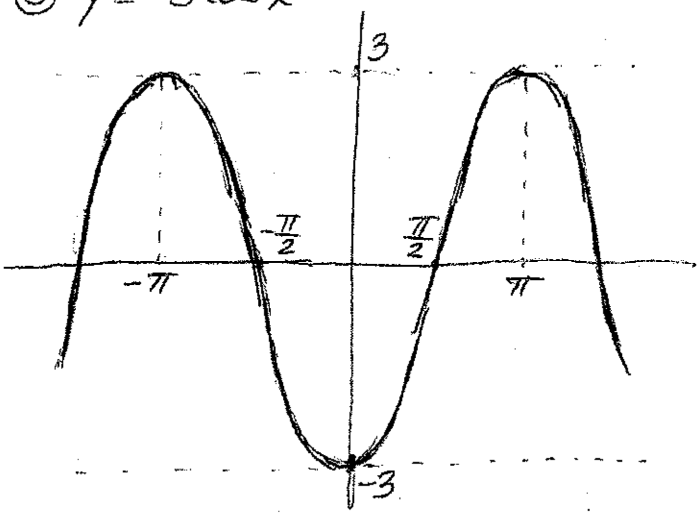
① $y = \cos x$



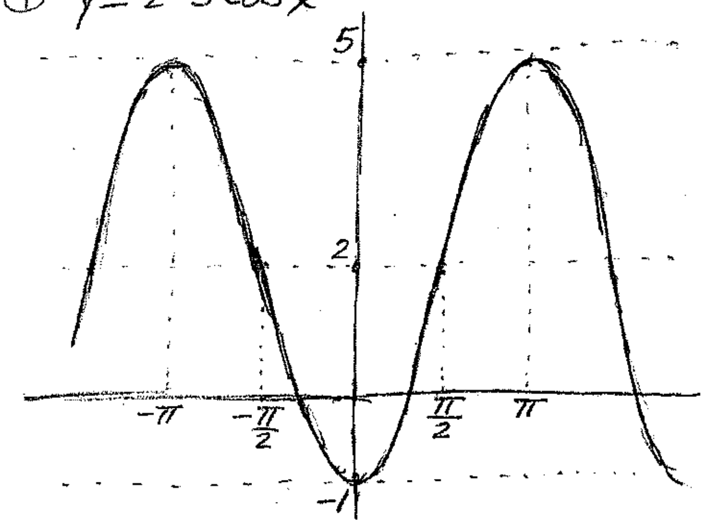

② $y = 3\cos x$



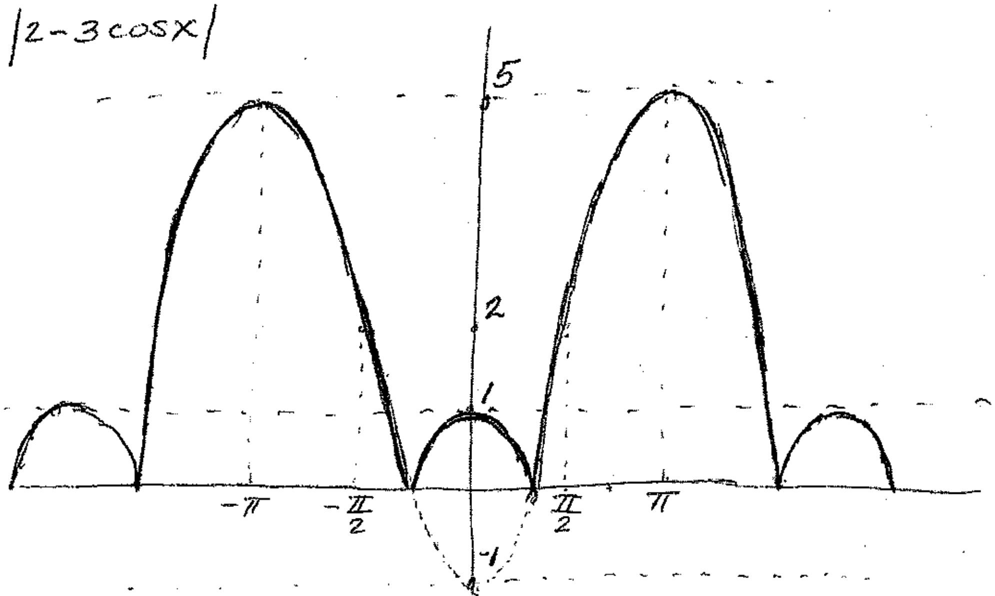

③ $y = -3\cos x$




④ $y = 2 - 3\cos x$




⑤ $f(x) = |2 - 3\cos x|$



5. [6 marks] Prove the trigonometric identity

$$\frac{\sin\theta}{1-\cos\theta} = \csc\theta + \cot\theta$$

Show clearly all your steps.

Solution.

$$\frac{\sin\theta}{1-\cos\theta} = \frac{\sin\theta}{1-\cos\theta} \times \frac{(1+\cos\theta)}{(1+\cos\theta)}$$

$$= \frac{\sin\theta + \sin\theta\cos\theta}{1-\cos^2\theta} = \frac{\sin\theta + \sin\theta\cos\theta}{\sin^2\theta}$$

$$= \frac{1}{\sin\theta} + \frac{\cos\theta}{\sin\theta}$$

$$= \csc\theta + \cot\theta$$

6. [7 marks] Let $f(x) = \sin^{-1}(x)$ and $g(x) = 2x^2 - \frac{1}{2}$.

a) Evaluate $(f \circ g)(0)$

b) Evaluate $(g \circ f)(0)$

c) Find the domain of the composite function $(f \circ g)(x)$

Solution

a) $(f \circ g)(0) = f(g(0))$

$= f(-\frac{1}{2}) = \sin^{-1}(-\frac{1}{2}) = \boxed{-\frac{\pi}{6}}$

b) $(g \circ f)(0) = g(f(0))$

$= g(0) = 2 \cdot 0^2 - 1/2 = \boxed{-\frac{1}{2}}$

c) Domain of $\sin^{-1}(2x^2 - 1/2)$

$2x^2 - 1/2$ must belong to $[-1, 1]$

As $2x^2 \geqslant 0$ then $2x^2 - 1/2 \geqslant -1/2 > -1$ is always true

$2x^2 - 1/2 \leqslant 1 \iff 2x^2 \leqslant \frac{3}{2} \iff x^2 \leqslant \frac{3}{4}$

So $-\frac{\sqrt{3}}{2} \leqslant x \leqslant \frac{\sqrt{3}}{2}$

Domain of $\sin^{-1}(2x^2 - 1/2)$: $\boxed{\left[-\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}\right]}$

7. [8 marks] Consider the function

$$f(x) = 2\ln x - \ln(x^2+1)$$

a) Find a formula for the inverse function $f^{-1}$

b) What is the range of $f^{-1}$

c) What is the range of $f$

Solution.

a) $y = 2\ln x - \ln(x^2+1) = \ln\left(\frac{x^2}{1+x^2}\right)$

$\Longleftrightarrow \frac{x^2}{1+x^2} = e^y \Longleftrightarrow x^2 = e^y + e^y x^2$

$\Longleftrightarrow x^2 = \frac{e^y}{1-e^y} \Longleftrightarrow x = \sqrt{\frac{e^y}{1-e^y}}$

Remark: The negative root is excluded because $x > 0$.

Interchanging $x$ and $y$ : $y = \boxed{\sqrt{\frac{e^x}{1-e^x}} = f^{-1}(x)}$

b) Range of $f^{-1}$ is equal to domain of $f$: $\boxed{(0,\infty)}$

c) Range of $f$ is equal to domain of $f^{-1}$

As $e^x > 0$ then $1-e^x$ must be positive

$1-e^x > 0 \Longleftrightarrow 1 > e^x$

$\Longleftrightarrow x < 0$

Range of $f$: $\boxed{(-\infty, 0)}$

8. [9 marks] Consider the function

$$f(x) = \frac{x^2 - 4}{|x| - 2}$$

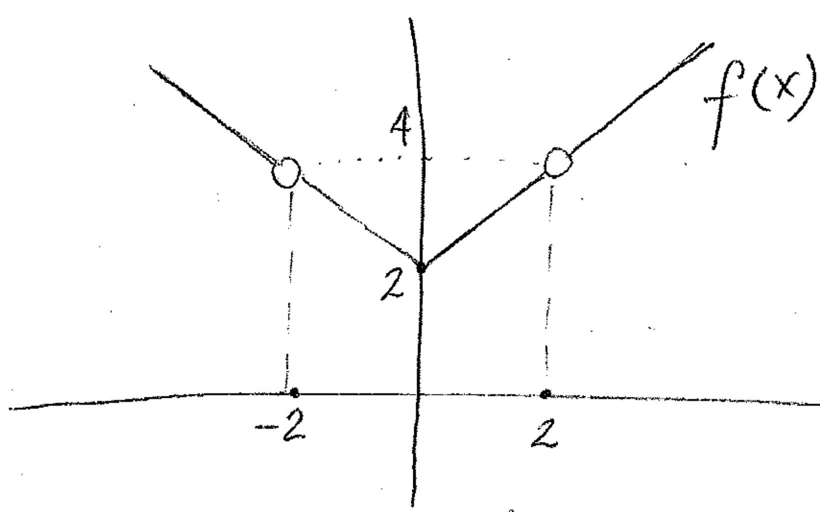
a) Evaluate $\lim_{x \to -2^-} f(x) = \lim_{x \to -2^-} \frac{x^2 - 4}{-x - 2} = \lim_{x \to -2^-} \frac{(x-2)\cancel{(x+2)}}{-\cancel{(x+2)}} = \boxed{4}$

b) Evaluate $\lim_{x \to -2^+} f(x) = \lim_{x \to -2^+} \frac{x^2 - 4}{-x - 2} = \lim_{x \to -2^+} \frac{(x-2)\cancel{(x+2)}}{-\cancel{(x+2)}} = \boxed{4}$

c) Evaluate $\lim_{x \to -2} f(x) = \boxed{4}$ because both lateral limits are equal to 4.

d) Sketch the graph of $f(x)$

$$f(x) = \begin{cases} \dfrac{(x-2)(x+2)}{-(x+2)} = -(x-2) & \text{if } x < 0,\ x \neq -2 \\[2ex] \dfrac{(x-2)(x+2)}{x-2} = x+2 & \text{if } x > 0,\ x \neq 2 \end{cases}$$



9. [6 marks] Let the function

$$f(t) = \frac{1}{t\sqrt{1+t}} - \frac{1}{t}$$

a) Is $t = 0$ a vertical asymptote of $f(t)$? Show your work.

b) Is $t = -1$ a vertical asymptote of $f(t)$? Show your work.

a) $$\lim_{t \to 0} f(t) = \lim_{t \to 0} \frac{1}{t}\left(\frac{1}{\sqrt{1+t}} - 1\right) = \lim_{t \to 0} \frac{1}{t}\left(\frac{1-\sqrt{1+t}}{\sqrt{1+t}}\right)$$

$$= \lim_{t \to 0} \frac{1}{t}\left(\frac{1-\sqrt{1+t}}{\sqrt{1+t}}\right) \cdot \left(\frac{1+\sqrt{1+t}}{1+\sqrt{1+t}}\right)$$

$$= \lim_{t \to 0} \frac{1}{t} \cdot \frac{1-(1+t)}{\sqrt{1+t} \cdot (1+\sqrt{1+t})} = \lim_{t \to 0} -\frac{1}{\sqrt{1+t}\,(1+\sqrt{1+t})}$$

$$= -\frac{1}{\sqrt{1+0}\,(1+\sqrt{1+0})} = \boxed{-\frac{1}{2}}$$

so $\boxed{t = 0 \text{ is NOT a vertical asymptote.}}$

b) $$\lim_{t \to -1^+} \frac{1}{t}\left(\frac{1}{\sqrt{1+t}} - 1\right)$$

$$= -\lim_{t \to -1^+} \left(\frac{1}{\sqrt{1+t}} - 1\right) = \boxed{-\infty}$$

Therefore there is a vertical asymptote at $t = -1$