Carleton University, School of Math & Stats
Math 2004-C [F13] — TEST 1

Dr. ℜJ Cova

September 2013

(only non-programmable, non-graphing calculators allowed)

QUESTIONS [60 points]

§1. (24 marks) (i)Calculate the area of the triangle whose vertices are located at the points $A(1,0,0)$, $B(0,1,0)$, $C(0,0,1)$. (ii)Write an equation of the plane that contains A, B, C. (iii)Determine whether (3, -3, 2) lies on this plane.

(i) $\text{Area} = \frac{1}{2}|\vec{AB} \times \vec{AC}|$, $\vec{AB} = \langle -1, 1, 0\rangle$, $\vec{AC} = \langle 0, -1, 1\rangle$

$\to \vec{AB} \times \vec{AC} = \langle 1, 1, 1\rangle$

7 marks

$\text{Area} = \frac{\sqrt{3}}{2}$

(ii) $\vec{n} = \vec{AB} \times \vec{AC} = \langle 1, 1, 1\rangle$, choose $A(1,0,0)$ to get:

$\vec{n} \cdot \vec{AP} = 0 \to \langle 1, 1, 1\rangle \cdot \langle x-1, y-0, z-0\rangle = 0$

12 marks

Π: $x + y + z = -1$ [$P(x, y, z)$]

(iii) The point $(3, -3, 2) \notin \Pi$ because

$3 + (-3) + 2 \neq -1$

5 marks

§2. (22 marks) (i)Write the parametric and, if possible, (ii)the symmetric equations of the line that passes through the point (1, 2, -1) and is parallel to the line with parametric equations $x = -1 + t, \quad y = 2 + 2t, \quad z = -2 - 3t$. (iii)Find the point where this line intersects the xy-plane.

The given line has direction vec $\vec{v} = \langle 1, 2, -3 \rangle$, which may be used for the line we seek. Whence:

(i) $\langle x, y, z \rangle = \langle 1, 2, -1 \rangle + t \langle 1, 2, -3 \rangle \rightarrow$

9 marks

ℓ: $x = 1 + t$, $y = 2 + 2t$, $z = -1 - 3t$.

4 marks

(ii) $\dfrac{x-1}{1} = \dfrac{y-2}{2} = \dfrac{z+1}{-3}$

(iii) We require $z = 0$ in (i), ie, $t = -1/3$.

Wherefore:

$$\begin{cases} x = 1 - 1/3 = 2/3 \\ y = 2 - 2/3 = 4/3 \end{cases} \rightarrow$$

9 marks

$\ell \cap xy$-plane @ $\left(\frac{2}{3}, \frac{4}{3}, 0\right)$

§3. (14 marks) Compute the volume of the parallelepiped determined by the vectors

$$\vec{u} = < 0, 1, -2 >, \quad \vec{v} = < 1, 1, 0 >, \quad \vec{w} = < 1, 2, 3 >$$

Just volume $= |(\vec{u} \times \vec{v}) . \vec{w}|$ (or $|\vec{u} . (\vec{v} \times \vec{w})|$)

$$(\vec{u} \times \vec{v}) . \vec{w} = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 1 & 2 \\ -2 & 0 & 3 \end{vmatrix} = -5 \rightarrow \text{Vol} = 5$$

Carleton University, School of Math & Stats
Math 2004-B [F13] — TEST 2
*****TUTORIAL SECTION → [ ]

Dr. ℜJ Cova

October 2013

(only non-programmable, non-graphing calculators allowed)

QUESTIONS [60 points]

§1. (25 marks) Let a certain parametric curve be defined by $x = e^{-t}$, $y = e^{2t}$. (i)Obtain the slope of the tangent line of this curve at the point P_0 given by $t = 0$. (ii)Write an equation of the tangent line at the point P_0. (iii)Get $\frac{d^2y}{dx^2}$. (iv)Evaluate $\frac{d^2y}{dx^2}$ at $t = 1/4$.

5 marks

(i) $x_t = -e^{-t}$, $y_t = 2e^{2t}$ $\rightarrow$ $y' = y_x \equiv y_t/x_t = -2e^{3t}$

$[x_t \equiv dx/dt, \text{etc}]$ $\rightarrow$ $\boxed{y'\big|_{t=0} = -2}$ 4 marks

(ii) $y - y_0 = y'(x - x_0)$, $P_0\,(x(0), y(0)) = (1, 1)$; x_0, y_0

$\rightarrow$ $y - 1 = -2(x-1)$ $\Rightarrow$ $\boxed{y = -2x + 3}$ 7 marks

(iii) $y'' = \dfrac{\frac{d}{dt}y'}{x_t} = \dfrac{-6e^{3t}}{-e^{-t}}$ $\rightarrow$ $\boxed{y'' = 6e^{4t}}$

6 marks

(iv) $\boxed{y''\big|_{t=\frac{1}{4}} = 6e}$

3 marks

§2. (35 marks) Consider the curve C given by $r = \frac{1}{2}\cos\theta$ in polar coordinates. Determine, if possible, the points on the curve where the tangent line is (i)horizontal or (ii)vertical. (iii)Compute the length of the curve C between $\theta = -\pi/2$ and $\theta = \pi/2$. (iv)Calculate the area of the region bounded by C, $\theta = -\pi/2$ and $\theta = 0$. (v)Please write the equation $r = \frac{1}{2}\cos\theta$ in cartesian coordinates.

$$x = r\cos\theta, \quad y = r\sin\theta \rightarrow x_\theta = r_\theta\cos\theta - r\sin\theta, \quad y_\theta = r_\theta\sin\theta + r\cos\theta \qquad [x_\theta \equiv dx/d\theta, \text{ etc}]$$

$$\rightarrow \begin{cases} x_\theta = -\sin\theta\cos\theta = -\sin(2\theta)/2 \\ y_\theta = \frac{1}{2}(\cos^2\theta - \sin^2\theta) = \cos(2\theta)/2 \end{cases}$$

6 marks

$$\Rightarrow y' = \frac{y_\theta}{x_\theta} \Rightarrow y' = -\frac{\cos(2\theta)}{\sin(2\theta)}$$

(i) H.T.L: We require $y_\theta = 0 \wedge x_\theta \neq 0$.

3 marks

$y_\theta = 0 \rightarrow \cos(2\theta) = 0 \rightarrow 2\theta = \pm\frac{\pi}{2} \rightarrow \theta = \pm\frac{\pi}{4}$

where indeed $x_\theta \neq 0$. Hence, we've got HTL @

$(r,\theta) = \left(\frac{\sqrt{2}}{4}, \pm\frac{\pi}{4}\right)$ ← 3 marks

(ii) V.T.L: We need $x_\theta = 0 \wedge y_\theta \neq 0$.

$x_\theta = 0 \rightarrow \sin(2\theta) = 0 \rightarrow 2\theta = 0, \pi \rightarrow \theta = 0, \frac{\pi}{2}$

where we verify that indeed $y_\theta \neq 0$.

3 marks

Whence, $\exists$ VTL @

$(r,\theta) = \left(\frac{1}{2}, 0\right), \left(0, \frac{\pi}{2}\right)$

3 marks

JG

(iii) $$L = \int_{-\pi/2}^{\pi/2} \sqrt{x_\theta^2 + y_\theta^2}\, d\theta = \int_{-\pi/2}^{\pi/2} \sqrt{r_\theta^2 + r^2}\, d\theta$$

$$\Rightarrow L = \int_{-\frac{\pi}{2}}^{\pi/2} \left\{ \left(-\tfrac{1}{2}\sin\theta\right)^2 + \left(\tfrac{1}{2}\cos\theta\right)^2 \right\}^{1/2} d\theta$$

$$= \frac{1}{2}\int_{-\pi/2}^{\pi/2} d\theta \qquad [\sin^2\theta + \cos^2\theta = 1]$$

$L = \frac{\pi}{2}$

6 marks

(iv) $$A = \int_{-\pi/2}^{0} \frac{1}{2} r^2 d\theta = \frac{1}{8}\int_{-\pi/2}^{0} \cos^2\theta\, d\theta$$

$$A = \frac{1}{8}\int_{-\pi/2}^{0} \frac{1+\cos(2\theta)}{2}\, d\theta$$

6 marks

$$= \frac{1}{8}\left\{ \frac{1}{2}\int_{-\pi/2}^{0} d\theta + \frac{1}{2}\int_{-\pi/2}^{0} \cancel{\cos(2\theta)\, d\theta} \right\} \quad 0$$

$\Rightarrow$ $A = \frac{\pi}{32}$

(v) $r = \frac{1}{2}\cos\theta \rightarrow r^2 = \frac{1}{2} r\cos\theta \rightarrow x^2 + y^2 = \frac{1}{2}x$

$x^2 - \frac{1}{2}x + y^2 = 0 \rightarrow \left(x - \frac{1}{4}\right)^2 + y^2 = \frac{1}{16}$ [by completing squares]

5 marks

JG

Carleton University, School of Math & Stats
Math 2004-C [F13] — TEST 3
****************PLEASE...TUTORIAL SECTION → [ ]

Dr. ℜJ Cova

October 2013

(only non-programmable, non-graphing calculators allowed)

QUESTIONS [60 marks]

§1. (*a*) (16 marks) Find the first and second partial derivatives f_x, f_y, f_{xx}, f_{yx} of

$$f(x,y) = x^3y^2 + xy^3 - 2x + 3y + 1.$$

(*b*) [10 marks] Given $\ln(x^2+z^2) + yz^3 = -2x^2$, use implicit differentiation to compute $\partial_x z$ and $\partial_y z$.

(a) $f_x = 3x^2y^2 + y^3 - 2$; $f_y = 2x^3y + 3xy^2 + 3$

$f_{xx} = 6xy^2$; $f_{yx} = 6x^2y + 3y^2$

(4 marks each) = 16 m

(b) ∂_x : $\dfrac{2x + 2zz_x}{x^2+z^2} + 3yz^2z_x = -4x$

$\to z_x\left(\dfrac{2x+2z+3yz^2[x^2+y^2]}{x^2+z^2}\right) = -4x$

$\to z_x = \dfrac{-4x\,(x^2+z^2)}{2x+2z+3yz^2(x^2+y^2)}$

(5 marks each) = 10 m

∂_y : $\dfrac{2zz_y}{x^2+z^2} + z^3 + 3yz^2z_y = 0 \to$

$z_y = \dfrac{z^2\,(x^2+z^2)}{2+3yz\,(x^2+z^2)}$

§2. (20 marks) Use the chain rule to calculate $\partial_t w$, where

$$w \equiv w[x(t), y(t), z(t)] = x e^{xyz}; \quad x = t,\ y = \sin(2t),\ z = \frac{t}{t^2+1}.$$

$$W_t = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}, \quad (*)$$

where:

$$\frac{\partial w}{\partial x} = e^{xyz} + xyz\, e^{xyz}, \quad \frac{\partial w}{\partial y} = x^2 z\, e^{xyz}$$

$$\frac{\partial w}{\partial z} = x^2 y\, e^{xyz},$$

(3 marks each = 9m)

and

(2 marks each = 6)

$$\frac{dx}{dt} = 1, \quad \frac{dy}{dt} = 2\cos(2t), \quad \frac{dz}{dt} = \frac{1-t^2}{(1+t^2)^2}.$$

Inserting these derivatives into $(*)$ one gets:

$$W_t = e^{xyz} + e^{xyz}\, 2x^2 z \cos(2t) + e^{xyz} x^2 y \frac{1-t^2}{(1+t^2)^2}$$

or, re-arranging:

(5 marks)

$$W_t = e^{xyz}\left\{1 + x^2\left[2z\cos(2t) + y\frac{1-t^2}{(1+t^2)^2}\right]\right\}$$

§3. (14 marks) Given the point in spherical coordinates $Q = (\rho, \theta, \phi) = (1, \frac{\pi}{4}, \frac{\pi}{3})$, provide two representations of Q in cylindrical coordinates (r, θ, z).

Try sphericals $\to$ cartesians $\to$ cylindricals:

$$x = \rho\cos\theta\sin\varphi = \frac{\sqrt{6}}{4}$$
$$y = \rho\sin\theta\sin\varphi = \frac{\sqrt{6}}{4}$$
$$z = \rho\cos\varphi = \frac{1}{2}$$

$$\Rightarrow (x, y, z) = \left(\frac{\sqrt{6}}{4}, \frac{\sqrt{6}}{4}, \frac{1}{2}\right) \quad \text{3m}$$

$$r^2 = x^2 + y^2 = \frac{3}{4} \to r = \pm\frac{\sqrt{3}}{2} \quad \text{2m}$$

$$\tan\theta = \frac{y}{x} = 1 \to \theta = \frac{\pi}{4}, \frac{5\pi}{4} \quad \text{4m}$$

$$z = z = 1/2 \quad \text{1m}$$

4m

$$\Rightarrow (r, \theta, z) = \begin{cases} \left(\frac{\sqrt{3}}{2}, \frac{\pi}{4}, \frac{1}{2}\right) \\ (-\sqrt{3}/2, 5\pi/4, 1/2) \end{cases}$$

Carleton University, School of Math & Stats
Math 2004-B [F13] — TEST 4
***TUTORIAL SECTION → [ ]

Dr. ℜJ Cova

November 2013

(may use a basic calculator and blank sheets)

QUESTIONS [60 marks]

§1. (30 marks) Consider the function $f(x,y) = e^{-2x}\cos y$ and the point $P = (0, \pi/4)$. (a)Calculate the directional derivative of $f(x,y)$ along $\vec{v} = < -1, -1 >$ at P. (b)Find a vector giving the direction in which $f(x,y)$ increases the most rapidly at P. (c)What is the maximum rate of increase of $f(x,y)$ at P? And the minimum? (d)Obtain equations [in the form $y = mx+b$] of the normal and tangent lines to the level curve $f(x,y) = \sqrt{2}/2$ at P.

(a) $D_{\hat{v}} f(x,y) = \vec{\nabla} f \cdot \hat{v}$, where $\vec{\nabla} f = \langle -2e^{-2x}\cos y, -e^{-2x}\sin y \rangle$

$\hat{v} = \dfrac{\vec{v}}{|\vec{v}|} = \dfrac{\sqrt{2}}{2}\langle -1, -1 \rangle$

$\rightarrow D_{\hat{v}} f(x,y) = \sqrt{2}e^{-2x}(\cos y + \frac{1}{2}\sin y)$

$D_{\hat{v}} f(0, \pi/4) = \dfrac{3}{2}$

(c) $|\vec{\nabla} f(0, \frac{\pi}{4})| = \dfrac{\sqrt{5}}{\sqrt{2}}$ MAX

MIN $= -\sqrt{5/2}$

(b) $\vec{\nabla} f(0, \frac{\pi}{4}) = \langle -\sqrt{2}, -\frac{\sqrt{2}}{2} \rangle$

(d) Normal l_N : $\vec{\nabla} f(0, \pi/4)$ serves as the direction vector. Hence:

$\vec{r} = \langle 0, \frac{\pi}{4} \rangle + t\langle -\sqrt{2}, -\frac{\sqrt{2}}{2} \rangle \rightarrow x = -\sqrt{2}t \wedge y = \frac{\pi}{4} - \frac{\sqrt{2}}{2}t$

$\rightarrow \dfrac{x}{\sqrt{2}} = \dfrac{y - \pi/4}{\sqrt{2}/2} \longrightarrow y = \dfrac{x}{2} + \dfrac{\pi}{4}$

§2. (20 marks) Consider the function $f(x,y) = y^3 + x^2 - 4x - 3y + 17$. Find its local minima/~~maxima~~ and saddle points (if any) using $D(x,y) = f_{xx}f_{yy} - [f_{xy}]^2$.

$$\begin{cases} f_x = 2x - 4 = 0 \\ f_y = 3y^2 - 3 = 0 \end{cases} \rightarrow \begin{cases} x = 2 \\ y = \pm 1 \end{cases} \rightarrow \text{CP}: (2,1),\ (2,-1)$$

Now, $f_{xx} = 2$, $f_{yy} = 6y$, $f_{xy} = 0$ $\rightarrow$

$D(x,y) = 12y$

$D(2,1) > 0$ ∧ $f_{xx} > 0$ → local min @ $(2,1)$

$D(2,-1) < 0$ → saddle point @ $(2,-1)$

It's OK if students don't calculate the actual values $f(2,1)$ ∧ $f(2,-1)$.

§3. (10 marks) Determine the critical points of $f(x,y) = x^2 - 6x - x\sqrt{y} + y$

$f_x = 2x - 6 - \sqrt{y}$, $f_y = -\dfrac{x}{2\sqrt{y}} + 1$

CP for $y = 0$
'cos $f_y \nexists$ @ $y=0$
∧ $y = 0 \in \text{Dom} f$

$$\begin{cases} f_x = 0 \\ f_y = 0 \end{cases} \rightarrow \begin{array}{l} 2x - \sqrt{y} - 6 = 0 \quad (1) \\ -\dfrac{x}{2\sqrt{y}} + 1 = 0 \rightarrow \sqrt{y} = \dfrac{x}{2} \quad (2) \end{array}$$

Plugging (2) in (1) we get: $2x - \dfrac{x}{2} = 6 \rightarrow \begin{cases} x = 4 \\ y = 4 \end{cases}$

$\Rightarrow$ CP: $(4,4)$ ∧ $(x,0)$

Carleton University, School of Math & Stats
Math 2004-C [F13] — TEST 5
***TUTORIAL SECTION → [   ]

Dr. ℜJ Cova

November 2013

(may use a basic calculator and blank sheets)

QUESTIONS [60 marks]

§1. (25 marks) Use the method of Lagrange multipliers to find the minimum value of the function $f(x,y,z) = x + 2y + z$ subject to the contraint $x^2 + 4y^2 - z = 0$.

$\vec{\nabla} f = \langle 1, 2, 1 \rangle$ 3m

3m

$\vec{\nabla} g = \langle 2x, 8y, -1 \rangle$, $g = x^2 + 4y^2 - z = 0$

3m $\vec{\nabla} f = \lambda g \rightarrow \langle 1, 2, 1 \rangle = \lambda \langle 2x, 8y, -1 \rangle \rightarrow$

3m $\begin{cases} 2\lambda x = 1 & (1) \\ 8\lambda y = 2 & (2) \\ \lambda = -1 & (3) \end{cases}$ Insert (3) in (1), (2): $x = -1/2$, $y = -1/4$ 3m each

And using the values of x & y in the constraint:

$z = x^2 + 4y^2 = \frac{1}{4} + \frac{1}{4} \Rightarrow z = 1/2$ 3m

So the critical point is: $\left(-\frac{1}{2}, -\frac{1}{4}, \frac{1}{2}\right)$.

The minimum value is

$f\left(-\frac{1}{2}, -\frac{1}{4}, \frac{1}{2}\right) = -\frac{1}{2} - \frac{1}{2} + \frac{1}{2} \Rightarrow f_{min} = -\frac{1}{2}$

Award 1 mark by hand, please.

3m

§2. (20 marks) Evaluate the integral by changing to polar coordinates:

$$\int_{-1}^{1}\int_{0}^{\sqrt{1-y^2}} \frac{1}{1+x^2+y^2}\,dx\,dy$$

8m

$$\rightarrow \int_{-\pi/2}^{\pi/2}\int_{0}^{1} \frac{1}{r^2+1}\, r\,dr\,d\theta \quad (1)$$

where: $\displaystyle\int_0^1 \frac{1}{r^2+1}\, r\,dr = \frac{1}{2}\int_1^2 \frac{du}{u} = \frac{\ln 2}{2} \quad (2)$

$u = r^2+1$
$du = 2r\,dr$

8m

Plugging (2) into (1):

$$\frac{\ln 2}{2}\int_{-\pi/2}^{\pi/2} d\theta = \frac{\pi \ln 2}{2}$$

4m

§3. (15 marks) Compute the integral

$$\int_0^{2\pi}\int_0^{\pi/2}\int_0^2 \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta;$$

4m

The integral over ρ is $\displaystyle\int_0^2 \rho^2 d\rho = \frac{1}{3}\rho^3\Big|_0^2 = \frac{8}{3} \quad (1)$

the integral over φ is: $\displaystyle\int_0^{\pi/2} \sin\varphi\, d\varphi = -\cos\varphi\Big|_0^{\pi/2} = 1 \quad (2)$

The integral over θ is:

4m

$\displaystyle\int_0^{2\pi} d\theta = 2\pi \quad (3)$ 4m

Putting (1), (2), (3) together we get: $\frac{16\pi}{3}$

3m