Université d'Ottawa · University of Ottawa
Faculté des sciences
Mathématiques et de statistique
Faculty of Science
Mathematics and Statistics

Calculus III for Engineers
MAT 2322A - Fall 2010
Final Exam
Professor: Victor G. LeBlanc
Time limit: 3 hours. Closed books. No calculators.

Name: Solutions

ID Number:

Instructions

- This exam has 15 pages and you have 3 hours to complete it.
- This is a closed book exam. Furthermore, all calculators, cell phones, pagers or any other electronic or communication devices are forbidden.
- Read each question carefully before answering.
- Questions 1 to 10 are multiple choice questions. These questions are worth 2 points each and no partial marks are possible. **Please write your answers in the corresponding boxes in the grid below entitled "Answers to multiple choice Qs".**
- Questions 11 to 16 are long answer questions and are worth 5 marks each, so organize your time accordingly. **A correct answer requires a full, clearly-written and detailed solution.** Answer each question in the space provided, using backs of pages or the extra pages at the end if necessary.
- Do not unstable the test.
- Good luck!

Answers to multiple choice Qs

1	2	3	4	5	6	7	8	9	10
A	F	A	D	E	D	B	B	E	C

Grid below is used for grading (do not write in this grid)

MCQ	11	12	13	14	15	16	Total
/20	/5	/5	/5	/5	/5	/5	/50

585, av. King-Edward C.P. 450, Succ. A
Ottawa (Ontario) K1N 6N5 Canada

585 King Edward Ave., P.O. Box 450, Stn. A
Ottawa, Ontario K1N 6N5 Canada

(613) 562-5864 • Téléc./Fax (613) 562-5776
Courriel/Email: uomaths@science.uottawa.ca

1. Which of the following expressions corresponds to the integral $\int_0^8 \int_0^{y^{\frac{1}{3}}} f(x,y)\,dx\,dy$ with order of integration reversed?

A. $\int_0^2 \int_{x^3}^8 f(x,y)\,dy\,dx$

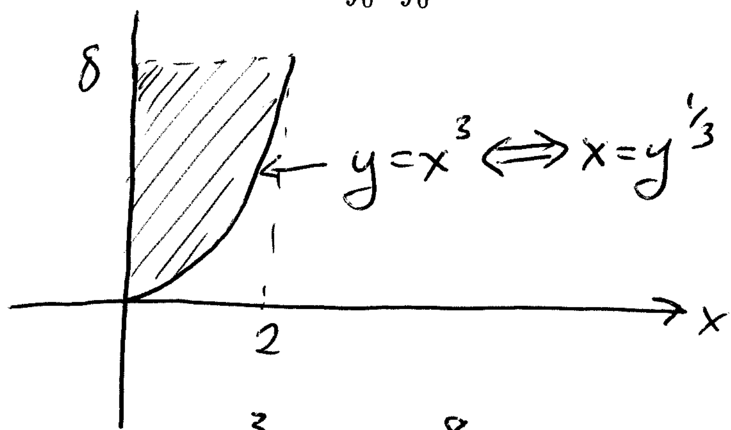
B. $\int_0^{y^{\frac{1}{3}}} \int_0^8 f(x,y)\,dx\,dy$

C. $\int_0^{y^{\frac{1}{3}}} \int_0^8 f(x,y)\,dy\,dx$

D. $\int_0^2 \int_8^{x^3} f(x,y)\,dy\,dx$

E. $\int_0^2 \int_8^{x^3} f(y,x)\,dy\,dx$

F. $\int_0^8 \int_{x^3}^2 f(x,y)\,dy\,dx$



$x^3 \le y \le 8$

$0 \le x \le 2$

2. If $f(x,y)$ is a differentiable function such that $\vec{\nabla} f(1,2) = \vec{\jmath}$, only one of the following curves can be the level curve for f through the point $(1,2)$. Which one?

A. $y = 1 + x$

B. $y = 1 + e^{x-1}$

C. $y = 2e^{x-1}$

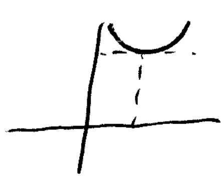
D. $y = \dfrac{2}{x}$

E. $x = 1$

F. $y = 2 + (x-1)^2$ ←

$\vec{\nabla} f(1,2) \perp$ to level curve of f at $(1,2)$

$\Rightarrow$ level curve of f at $(1,2)$ must have horizontal tangent



3. Consider the parametrized curve $\vec{r}(t) = t\,\vec{i} + t^2\,\vec{j} + t^3\,\vec{k}$, $t \in [1,2]$. Which of the following expressions leads to the total arclength of this curve?

A. $\int_1^2 \sqrt{1+4t^2+9t^4}\, dt$

B. $\int_1^2 \sqrt{t^2+t^4+t^6}\, dt$

C. $\sqrt{1+4t^2+9t^4}$

D. $\sqrt{t^2+t^4+t^6}$

E. $\sqrt{1+4\cdot 2^2+9\cdot 2^4} - \sqrt{1+4\cdot 1^2+9\cdot 1^4}$

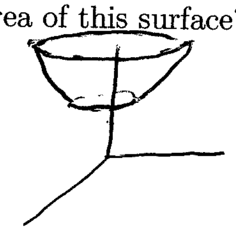
F. $\sqrt{(2-1)^2+(4-1)^2+(8-1)^2}$

$$\int_1^2 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt =$$

$$\int_1^2 \sqrt{1^2 + (2t)^2 + (3t^2)^2}\, dt =$$

$$\int_1^2 \sqrt{1+4t^2+9t^4}\, dt$$

4. Consider the surface which corresponds to the paraboloid $z = x^2+y^2$ between the planes $z=1$ and $z=4$. Which of the following expressions leads to the total area of this surface?



A. $\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (x^2+y^2)\, dy\, dx - \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (x^2+y^2)\, dy\, dx$

B. $\int_{\theta=0}^{2\pi}\int_{r=1}^{2} r^3\, dr\, d\theta$

C. $\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \sqrt{2x+2y}\, dy\, dx - \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \sqrt{2x+2y}\, dy\, dx$

D. $\int_{\theta=0}^{2\pi}\int_{r=1}^{2} \sqrt{4r^4+r^2}\, dr\, d\theta$

E. $\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (x^2+y^2)\, dy\, dx$

F. $\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (x^2+y^2)\, dy\, dx$

$$\vec{r}(r,\theta) = r\cos\theta\,\vec{i} + r\sin\theta\,\vec{j} + r^2\,\vec{k}$$

$$1 \le r \le 2$$

$$0 \le \theta \le 2\pi$$

$$\vec{r}_r \times \vec{r}_\theta = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos\theta & \sin\theta & 2r \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} = \vec{i}(-2r^2\cos\theta) + \vec{j}(-2r^2\sin\theta) + r\,\vec{k}$$

$$\|\vec{r}_r \times \vec{r}_\theta\| = \sqrt{4r^4+r^2} \Rightarrow$$

$$\text{Area} = \iint \|\vec{r}_r + \vec{r}_\theta\|\, dr\, d\theta$$

5. If C is the straight line segment starting at the point $(0,0)$ and ending at the point $(1,1)$, and $\vec{F}(x,y) = xy\,\vec{i} + y^2\,\vec{j}$, then which of the following corresponds to the value of the line integral $\int_C \vec{F}\cdot d\vec{r}$?

$\vec{r}(t) = t\vec{i} + t\vec{j} \qquad 0 \le t \le 1$

$\vec{r}\,'(t) = \vec{i} + \vec{j}$

$\vec{F}(\vec{r}(t)) = t^2\,\vec{i} + t^2\,\vec{j}$

$$\int_C \vec{F}\cdot d\vec{r} = \int_0^1 (t^2\vec{i} + t^2\vec{j})\cdot(\vec{i}+\vec{j})\,dt = \int_0^1 2t^2\,dt$$

$$= \frac{2t^3}{3}\Big|_0^1 = \frac{2}{3}$$

A. $\frac{-2}{3}$

B. 0

C. $\vec{i} + \vec{j}$

D. $\frac{1}{3}$

E. $\frac{2}{3}$

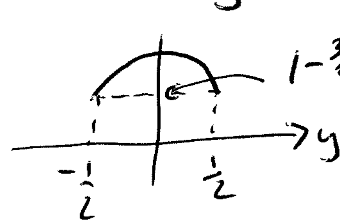
F. $\frac{2}{3}\vec{i}$

6. If $f(x,y) = x^2 + y^2$, then which of the following numbers corresponds to the global minimum value of f subject to the constraint $x^2 + 4y^2 = 1$?

$x^2 = 1 - 4y^2 \qquad -\frac{1}{2} \le y \le \frac{1}{2}$

$f\big|_{\text{constraint}} = 1 - 4y^2 + y^2$

$= 1 - 3y^2$



$1 - \frac{3}{4} = \frac{1}{4}$

A. $\frac{-1}{2}$

B. $\frac{5}{4}$

C. That global minimum value does not exist

D. $\frac{1}{4}$

E. 0

F. 1

7. Which of the following vector fields is conservative?

A. $\vec{F}(x,y,z) = yz\,\vec{i} + xz\,\vec{j} + xyz\,\vec{k}$

B. $\vec{F}(x,y,z) = x^2\,\vec{i} + e^y\,\vec{j} + \cos(z)\,\vec{k}$

C. $\vec{F}(x,y,z) = (x+y+z)\,\vec{i}$

D. $\vec{F}(x,y,z) = -z\vec{i} + x\vec{k}$

E. All of the above

F. None of the above

$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2 & e^y & \cos z \end{vmatrix} = \vec{0}$$

All other $\vec{F}$ in the list have non-zero curl.

8. If S is the disk $x^2+y^2 \le 1$, $z=1$ oriented upwards (i.e. unit normal parallel to $\vec{k}$), and $\vec{F}(x,y,z) = x\,\vec{i} + y\,\vec{j} + z\,\vec{k}$, then which of the following corresponds to the value of the surface integral $\iint_S \vec{F}\cdot d\vec{S}$?

$\vec{r}(r,\theta) = r\cos\theta\,\vec{i} + r\sin\theta\,\vec{j} + \vec{k}$ $\quad 0 \le r \le 1$, $0 \le \theta \le 2\pi$

A. 1

B. π

C. $\pi\,\vec{k}$

D. 2π

E. 0

F. $-2\pi\,\vec{k}$

$$\vec{r}_r \times \vec{r}_\theta = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos\theta & \sin\theta & 0 \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} = r\,\vec{k}$$

$$\vec{F}(\vec{r}(r,\theta)) = r\cos\theta\,\vec{i} + r\sin\theta\,\vec{j} + \vec{k}$$

$$\iint_S \vec{F}\cdot d\vec{S} = \int_0^{2\pi}\int_0^1 \vec{F}(\vec{r}(r,\theta))\cdot(\vec{r}_r\times\vec{r}_\theta)\,dr\,d\theta = \int_0^{2\pi}\int_0^1 r\,dr\,d\theta = \pi$$

9. Consider the two-dimensional region D drawn below, whose boundary is the oriented curve C, also drawn. Let $\vec{F}(x,y) = P(x,y)\,\vec{\imath} + Q(x,y)\,\vec{\jmath}$ be a vector field with continuous partial derivatives. Then which of the following equations corresponds to Green's theorem?

A. $\displaystyle\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial y} - \frac{\partial P}{\partial x}\right) dx\,dy$

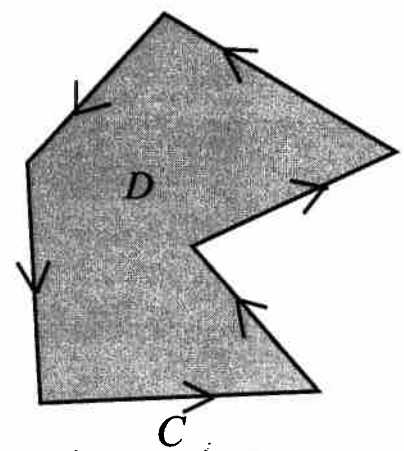
B. $\displaystyle\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial P}{\partial x} - \frac{\partial Q}{\partial y}\right) dx\,dy$

C. $\displaystyle\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) dx\,dy$

D. $\displaystyle\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} + \frac{\partial P}{\partial y}\right) dx\,dy$

E. $\displaystyle\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$

F. $\displaystyle\int_C Q\,dx + P\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$



SEE Class notes or text.

10. If $z = f(x,y)$ and $x = x(u,v)$, $y = y(u,v)$, which of the following formulas corresponds to the chain rule for the partial derivative $\dfrac{\partial z}{\partial v}$?

A. $\dfrac{\partial z}{\partial v} = \dfrac{\partial z}{\partial x}\dfrac{\partial x}{\partial u} + \dfrac{\partial z}{\partial y}\dfrac{\partial y}{\partial v}$

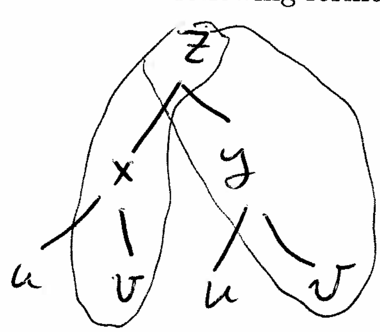
B. $\dfrac{\partial z}{\partial v} = \dfrac{\partial z}{\partial u}\dfrac{\partial u}{\partial v} + \dfrac{\partial z}{\partial v}$

C. $\dfrac{\partial z}{\partial v} = \dfrac{\partial z}{\partial x}\dfrac{\partial x}{\partial v} + \dfrac{\partial z}{\partial y}\dfrac{\partial y}{\partial v}$

D. $\dfrac{\partial z}{\partial v} = \dfrac{\partial z}{\partial x}\dfrac{\partial x}{\partial v}$

E. $\dfrac{\partial z}{\partial v} = \dfrac{\partial z}{\partial y}\dfrac{\partial y}{\partial v}$

F. $\dfrac{\partial z}{\partial v} = \dfrac{\partial z}{\partial x}\dfrac{\partial x}{\partial u}\dfrac{\partial u}{\partial v} + \dfrac{\partial z}{\partial y}\dfrac{\partial y}{\partial v}$



$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}$$

$$\overbrace{}^{P(x,y)} \qquad \overbrace{}^{Q(x,y)}$$

11. Consider the vector field $\vec{F}(x,y) = (2xy+1)\,\vec{\imath} + (x^2+1)\,\vec{\jmath}$. Show that the vector field is conservative, and then find a scalar function $f(x,y)$ such that $\vec{F}(x,y) = \vec{\nabla} f(x,y)$. Finally, compute

$$\int_C \vec{F} \cdot d\vec{r},$$

where C is some continuous curve that starts at the point $(0,0)$ and ends at the point $(3,1)$.

$$\frac{\partial Q}{\partial x} = 2x = \frac{\partial P}{\partial y} = 2x$$

$\vec{F}$ is conservative.

$$\frac{\partial f}{\partial x} = P(x,y) = 2xy+1 \Rightarrow f(x,y) = x^2y + x + h(y)$$

$$\Downarrow$$

$$\frac{\partial f}{\partial y} = x^2 + h'(y) = Q(x,y) = x^2+1$$

$$\Rightarrow h'(y) = 1$$

$$\Rightarrow h(y) = y + K.$$

So, all potentials are of the form

$$f(x,y) = x^2y + x + y + K$$

By the fundamental theorem for line integrals, we have

$$\int_C \vec{F} \cdot d\vec{r} = \int_C (\vec{\nabla} f) \cdot d\vec{r} = f(3,1) - f(0,0)$$

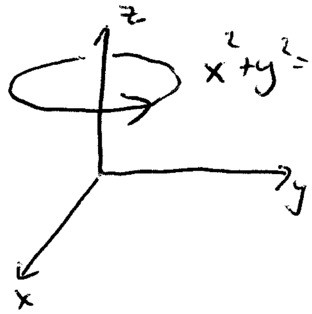
$$= 3^2 \cdot 1 + 3 + 1 - 0 = 13$$

12. Consider the vector field $\vec{F}(x,y,z) = 2y\,\vec{\imath} + 3x\,\vec{\jmath} - z^2\,\vec{k}$.

(a) Compute the curl of $\vec{F}$, i.e. compute $\vec{\nabla} \times \vec{F}(x,y,z)$.

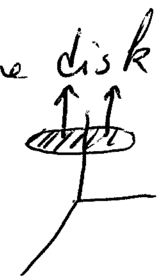
(b) **Using Stokes' theorem**, compute the line integral $\int_C \vec{F} \cdot d\vec{r}$, where C is the circle $x^2 + y^2 = 9$ in the plane $z = 1$, oriented counter-clockwise when viewed from above.

a) $$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2y & 3x & -z^2 \end{vmatrix} = \vec{\imath}\cdot 0 - \vec{\jmath}\cdot 0 + \vec{k}\,(3-2) = \vec{k}$$

b) 

$x^2+y^2=9,\ z=1$

Let D denote the disk $x^2+y^2 \le 9$, $z=1$ oriented upwards. Then $\partial D = C$.



By Stokes' thm., we have $\displaystyle\int_C \vec{F}\cdot d\vec{r} = \iint_D (\vec{\nabla}\times\vec{F})\cdot d\vec{S}$

Parametrize D: $\vec{r}(r,\theta) = r\cos\theta\,\vec{\imath} + r\sin\theta\,\vec{\jmath} + \vec{k}$ $\quad 0 \le r \le 3$, $0 \le \theta \le 2\pi$

$\vec{r}_r \times \vec{r}_\theta = \dots = r\vec{k}$

$$\Rightarrow \iint_D (\vec{\nabla}\times\vec{F})\cdot d\vec{S} = \int_0^{2\pi}\int_0^3 (r\vec{k})\cdot\vec{k}\,dr\,d\theta = \int_0^{2\pi}\int_0^3 r\,dr\,d\theta = \text{Area of } D$$
$$= \pi\cdot 3^2$$
$$= 9\pi$$

13. Consider the vector field $\vec{F}(x,y,z) = (y-x)\,\vec{\imath} + (z-y)\,\vec{\jmath} + (y-x)\,\vec{k}$.

(a) Compute the divergence of $\vec{F}$, i.e. compute $\vec{\nabla} \cdot \vec{F}(x,y,z)$.

(b) Let E denote the solid cube $0 \le x \le 1$, $0 \le y \le 1$, $0 \le z \le 1$, and let S denote the surface which is the boundary of E, oriented outwards. **Use Gauss' divergence theorem** to compute the flux integral $\iint_S \vec{F} \cdot d\vec{S}$

a) $\vec{\nabla} \cdot \vec{F} = \frac{\partial}{\partial x}(y-x) + \frac{\partial}{\partial y}(z-y) + \frac{\partial}{\partial z}(y-x) = -2$

b) By Gauss' divergence theorem, we have

$$\iint_S \vec{F} \cdot d\vec{S} = \iiint_E (\vec{\nabla} \cdot \vec{F})\, dV = \iiint_E (-2)\, dV = -2 \text{ Volume}(E)$$
$$= -2 \cdot 1^3$$
$$= -2.$$

14. Find and classify the critical points of the function $\frac{x^3}{3} + y^2 - 3x - 2xy$.

$$f(x,y) = \frac{x^3}{3} + y^2 - 3x - 2xy$$

$f_x = x^2 - 3 - 2y$ $\qquad$ $\left.f_y = 2y - 2x\right\}$ $\quad f_x = 0,\ f_y = 0 \Rightarrow$

$x = y$

$x^2 - 2x - 3 = 0$

$(x-3)(x+1) = 0$

$x = 3,\ x = -1$

There are two critical points: $(3,3)$ and $(-1,-1)$.

$f_{xx} = 2x$, $\quad f_{xy} = f_{yx} = -2$, $\quad f_{yy} = 2$

$D = f_{xx} f_{yy} - f_{xy}^2 = (2x) \cdot 2 - (-2)^2 = 4x - 4$

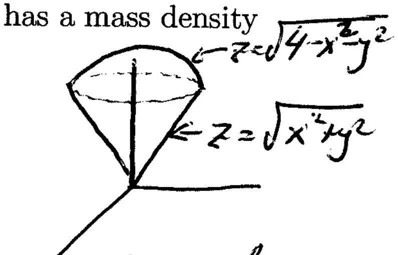
For $(3,3)$: $D = 4 \cdot 3 - 4 = 12 - 4 = 8 > 0$ and $f_{xx} = 2 \cdot 3 = 6 > 0$

So $(3,3)$ is a local minimum

For $(-1,-1)$: $D = 4 \cdot (-1) - 4 = -8 < 0$

So $(-1,-1)$ is a saddle

15. Consider the solid region E which is bounded below by the cone $z = \sqrt{x^2+y^2}$ and bounded above by the hemisphere $z = \sqrt{4-x^2-y^2}$. Suppose this solid has a mass density given by $\sigma(x, y, z) = (x^2 + y^2 + z^2)^3$. Find the total mass of this solid.



$$\text{Mass} = \iiint_E \underbrace{\sigma}_{\text{density}} \, dV = \int_0^{2\pi}\int_0^{\pi/4}\int_0^{2} \rho^6 \cdot \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta$$

In spherical coordinates

$0 \le \rho \le 2$

$0 \le \phi \le \pi/4$

$0 \le \theta \le 2\pi$

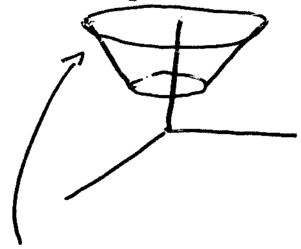
$$= 2\pi \left(\frac{\rho^9}{9} \Big|_0^2 \right) \int_0^{\pi/4} \sin\phi \, d\phi$$

$$= \frac{2\pi}{9} \cdot 2^9 \cdot (-\cos\phi) \Big|_0^{\pi/4} = \frac{2\pi \cdot 2^9}{9} \left(1 - \frac{\sqrt{2}}{2}\right)$$

or

$$= \frac{2\pi}{9} \cdot 2^8 (2-\sqrt{2}) \quad \text{or} \quad \frac{512}{9}\pi(2-\sqrt{2}).$$

16. Consider the scalar function $f(x,y,z) = 4z$. Compute the surface integral $\iint_S f\,dS$, where S is the portion of the cone $z = \sqrt{x^2+y^2}$ that lies between the planes $z = 1$ and $z = 4$.



$$\vec{r}(r,\theta) = r\cos\theta\,\vec{i} + r\sin\theta\,\vec{j} + r\vec{k}$$

$$1 \le r \le 4$$

$$0 \le \theta \le 2\pi$$

$$\vec{r}_r \times \vec{r}_\theta = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos\theta & \sin\theta & 1 \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix}$$

$$= \vec{i}(-r\cos\theta) - \vec{j}(r\sin\theta) + r\vec{k}$$

$$\|\vec{r}_r \times \vec{r}_\theta\| = \sqrt{r^2(\cos^2\theta + \sin^2\theta) + r^2} = \sqrt{2r^2} = \sqrt{2}\,r$$

$$\iint_S f\,dS = \int_0^{2\pi}\int_1^4 \underbrace{4\cdot r}_{f} \cdot \underbrace{\sqrt{2}\,r}_{\|\vec{r}_r + \vec{r}_\theta\|}\,dr\,d\theta = \cdots = 168\sqrt{2}\,\pi$$