1. For the complex function

$$F(s) = \frac{1}{s^2 + b\,s + 12}$$

a) which of the following values result in complex poles: b= 1 or b=20?

b) For the value of b that results in complex poles, obtain the inverse Laplace transform.

When b = 10

$$F(s) = \frac{1}{s^2 + 10s + 17}$$

$$p = \frac{-10 \pm \sqrt{100 - 68}}{2}$$

The poles are real numbers.

When b = 2

$$F(s) = \frac{1}{s^2 + 2s + 17}$$

$$p = \frac{-2 \pm \sqrt{4 - 68}}{2}$$

The poles are complex numbers.

So

$$F(s) = \frac{1}{s^2 + 2s + 17}$$

$$= \frac{1}{(s+1)^2 + 4^2}$$

$$= \frac{1}{4} \cdot \frac{4}{(s+1)^2 + 4^2}$$

$$\therefore\ f(t) = \frac{1}{4} e^{-t} \sin 4t$$

---

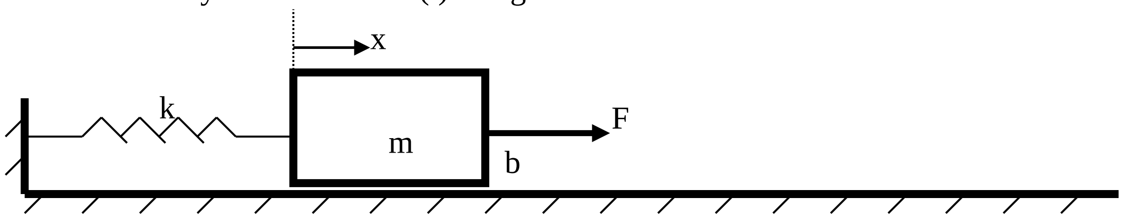
2. For the mechanical system shown below F(t) is the external force applied. For the case of viscous friction coefficient b, between the mass m and the ground, at the moment of application of the force F, its point of application has zero initial conditions, while initial conditions are

$$x(0) = 0.2 \ \ [\mathrm{m}] \qquad \qquad \dot{x}(0) = 0.02 \ \ [\mathrm{m/s}]$$

a) Obtain X(s) for a impulse force input F=6 δ(t), and m=1 [Kg], b =0.2 [N/ms^{-1}] k=9[N/m]

b) What is the damping ratio, the natural frequency and the damped natural frequency of oscillation in Hz?

c) Calculate steady state value of x(t) using final value theorem.



a)

$$F - kx = m\ddot{x}$$

$$\because F = 3\delta(t), \quad m = 1, \quad k = 4$$

$$\therefore \ddot{x} + 4x = 3\delta(t)$$

Take Laplace transform on both sides.

$$s^2 X(s) - sx(0) - \dot{x}(0) + 4X(s) = 3$$

$$\because x(0) = 0.1; \quad \dot{x}(0) = 0.01$$

$$\therefore s^2 X(s) - 0.1s - 0.01 + 4X(s) = 3$$

$$\therefore X(s) = \frac{3.01 + 0.1s}{s^2 + 4}$$

b)

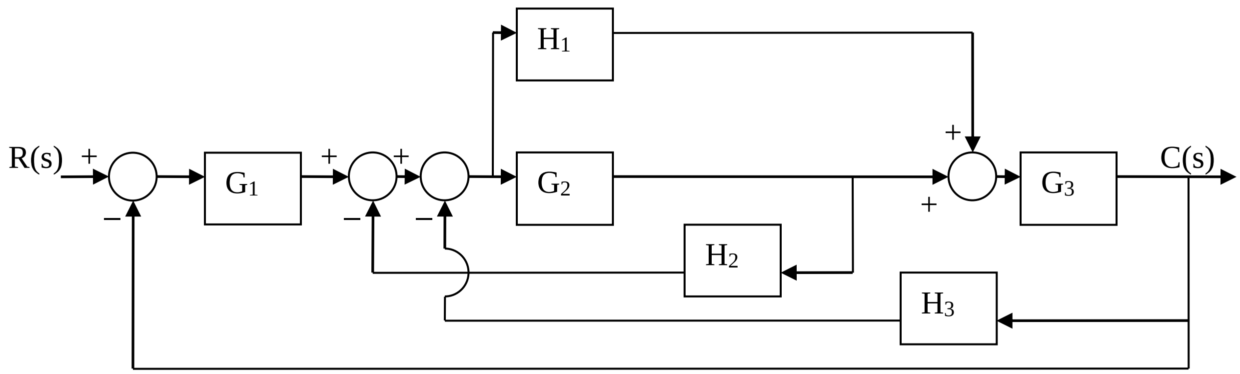
$$\omega^2 = 4 \quad \omega = 2 \qquad \qquad f = \frac{\omega}{2\pi} = \frac{1}{\pi} = 0.318(Hz)$$

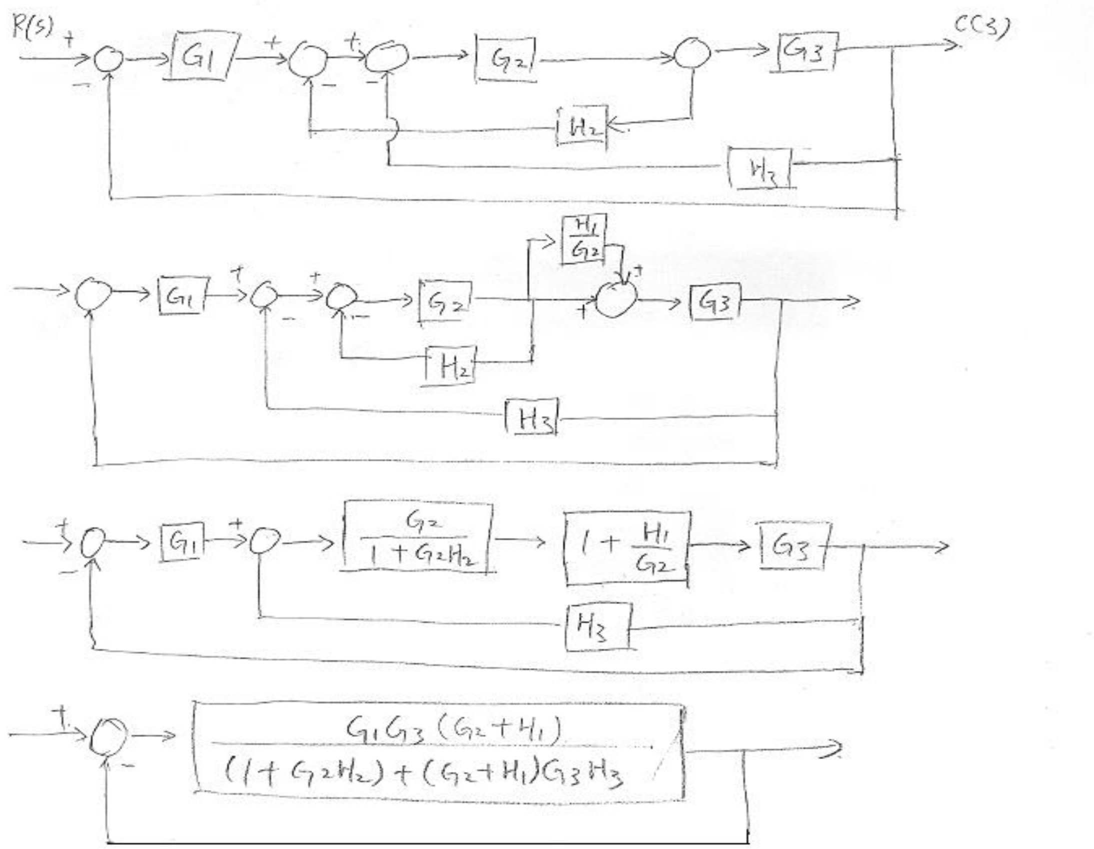
c)

$$sF(s) = \frac{0.1s^2 + 3.01s}{s^2 + 4}$$

$$p_1 = 2j; \, p_2 = -2j$$

3. Simplify the block diagram shown and obtain C(s)/R(s).





1)

$$\frac{C(s)}{R(s)} = \frac{G_1G_2G_3 + G_1G_3H_1}{1 + G_2H_2 + G_2G_3H_3 + G_3H_1H_3 + G_1G_2G_3 + G_1G_3H_1}$$

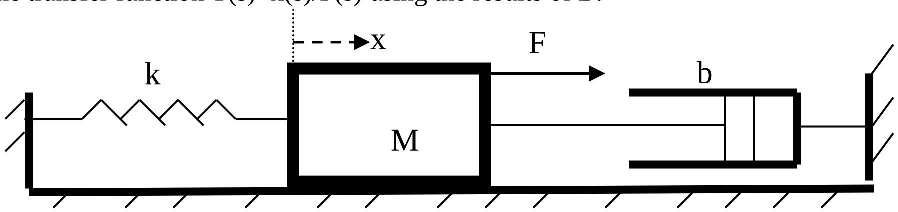
-

4. For the mechanical system shown below F [N] is the external force applied and x [m] is the position variable of the mass M. For the case of no friction, M=8[Kg] k=4 [N/m], b =6 [Ns/m] and zero initial conditions,

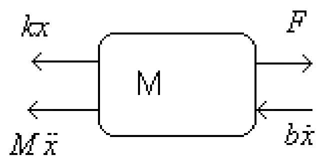
A) obtain the transfer function T(s)=x(s)/F(s) from the Laplace transform of the Newton second law.

B) obtain the state space representation of the system assuming that the output is the displacement x.

C) derive the formula for the transfer function from the state space representation matrices A, B and C. Obtain the transfer function T(s)=x(s)/F(s) using the results of B.



(a)F.B.D.



$$M\ddot{x} + b\dot{x} + kx = F$$

Take Laplace transform on both sides

$$\frac{X(s)}{F(s)} = \frac{1}{Ms^2 + bs + k} = \frac{1}{4s^2 + 3b + 2}$$

(b)

Let $\begin{matrix} X_1 = x \\ X_2 = \dot{X}_1 = \dot{x} \end{matrix}$

$$\because \ddot{x} + \frac{b}{M}\dot{x} + \frac{k}{M}x = F$$

$$\ddot{x} + 0.75\dot{x} + 0.5x = F$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -0.5 & -0.75 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} F$$

$$x = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$

(c)

$$\dot{\bar{x}} = A\bar{x} + B\bar{u}$$

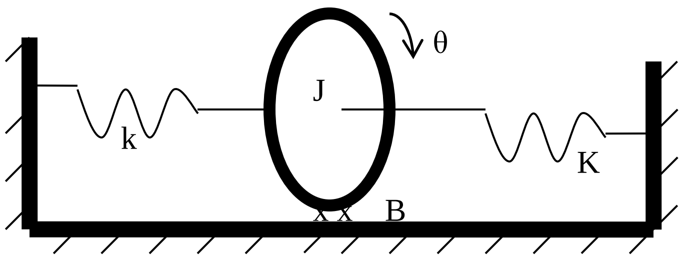
$$y = C\bar{x} + D\bar{u}$$

$$G(s) = C(sI - A)^{-1}B + D$$

$$G(s) = \frac{1}{s^2 + 0.75s + 0.5}$$

---

5. For the mechanical system shown below θ is the angular displacement of a disk with moment of inertia J connected to a fixed frame by two torsional springs with the coefficients k and K. Between the disk and the fixed frame there is viscous friction with the coefficient B. Obtain θ (s), given initial conditions θ(0)=0 and dθ(t)/dt=0.1 [rad/s] for t=0.



$$J\ddot{\theta} + B\dot{\theta} + (k+K)\theta = 0$$

Take Laplace Transform on both sides

$$J\left[s^2 - s\theta(0) - \dot{\theta}(0)\right] + B\left[s\Theta(s) - \theta(0)\right] + (k+K)\Theta(s)$$

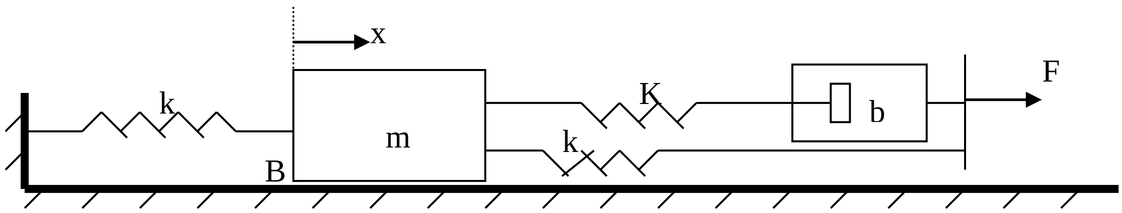
$$(Js^2 + Bs + k + K)\Theta(s) = 0.1Js + 0.5J + 0.1B$$

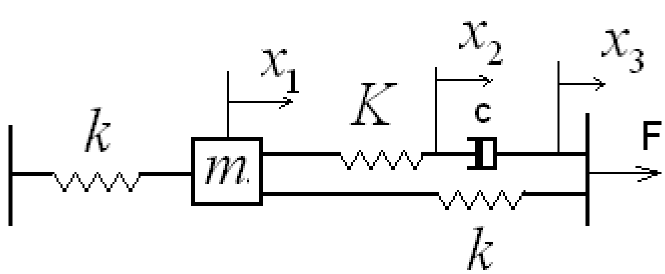
$$\Theta(s) = \frac{0.5J}{Js^2 + Bs + k + K}$$

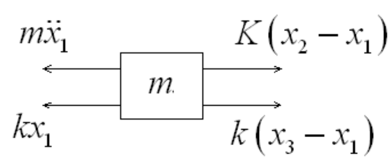
---

6. For the mechanical system shown there is a friction viscous coefficient B between the mass m and the ground. At the moment of application of the force F, its point of application has zero initial conditions and. The damper has the friction viscous coefficient b

Obtain the response x(s) for a init impulse force input F.







$$m\ddot{x}_1 + kx_1 - K(x_2 - x_1) - k(x_3 - x_1) = 0 \qquad (1)$$

$$K(x_2 - x_1) = c(\dot{x}_3 - \dot{x}_2) \qquad (2)$$

$$c(\dot{x}_3 - \dot{x}_2) + k(x_3 - x_1) = F \qquad (3)$$

Sub. (2) into (1) $\quad m\ddot{x}_1 + kx_1 - \{c(\dot{x}_3 - \dot{x}_2) + k(x_3 - x_1)\} = 0 \qquad (4)$

Sub. (3) into (4) $\quad m\ddot{x}_1 + kx_1 - F = 0$

L.T.

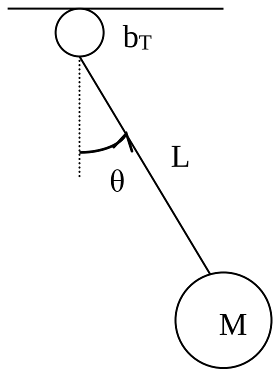
$$m\left[s^2X(s) - sx(0) - \dot{x}(0)\right] + kX(s) = F(s)$$

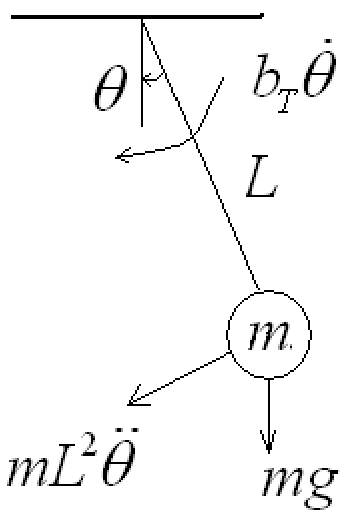
$$\left[ms^2 + k\right]X(s) = F(s) + 0.01ms + 0.1m$$

Since it is a transfer function, all initial conditions are zero

----

7. Simple pendulum shown has the length L, mass M, and the viscous friction coefficient b_T [N/ms^{-1}] of the pendulum bearing. Obtain the linearized model for the case of motion with small angle θ displacement with regard to the vertical.



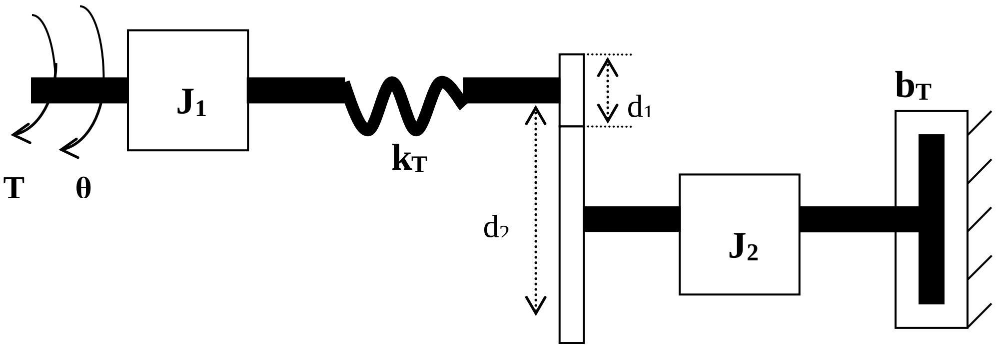


$$\Sigma M = mL^2\ddot{\theta} + b_T\theta + mgL\sin\theta = 0$$

The linearized model for θ is:

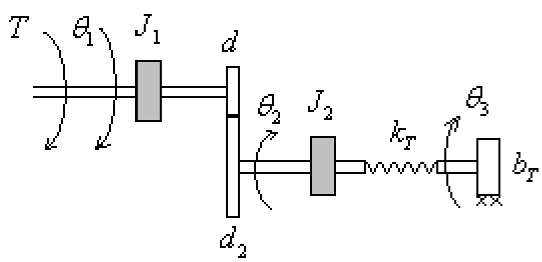
$$mL^2\ddot{\theta} + b_T\theta + mgL\theta = 0$$

8. For the geared system shown, with torsional spring coefficient k_T and rotational damper coefficient b_T obtain



a) the non-geared equivalent
b) the transfer function θ(s)/T(s)

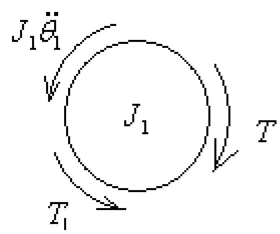
(a)



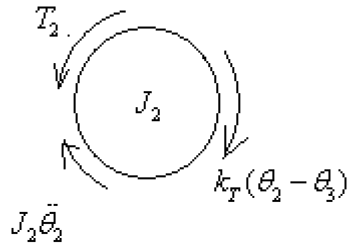
$$\frac{d_1}{2}\theta = \frac{d_2}{2}\theta_2$$

$$\theta_2 = \frac{d_1}{d_2}\theta_1$$

$$\begin{cases} T_1 = F_1 \dfrac{d_1}{2} \\ T_2 = F_2 \dfrac{d_2}{2} \\ F_1 = F_2 \end{cases} \Rightarrow \frac{T_1}{d_1} = \frac{T_2}{d_2} \Rightarrow T_2 = \frac{d_2}{d_1} T_1$$



$$J_1\ddot{\theta}_1 + T_1 = T \qquad (1)$$



$$J_2\ddot{\theta}_2+k_T\left(\theta_2-\theta_3\right)-T_2=0$$

$$\Rightarrow T_2=\frac{d_2}{d_1}T_1=J_2\ddot{\theta}_2+k_T\left(\theta_2-\theta_3\right)$$

$$T_1=\frac{d_1}{d_2}\left[J_2\ddot{\theta}_2+k_T(\theta_2-\theta_3)\right] \qquad (2)$$

Sub. (2) into (1):

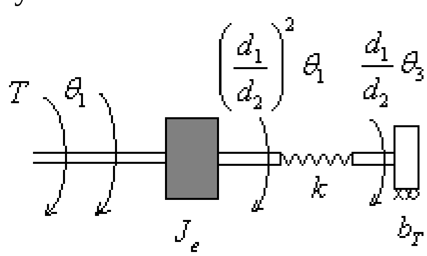
$$J_1\ddot{\theta}_1+\frac{d_1}{d_2}\left[J_2\ddot{\theta}_2+k_T\left(\theta_2-\theta_3\right)\right]=\left[J_1+\left(\frac{d_1}{d_2}\right)^2 J_2\right]\ddot{\theta}_1+\frac{d_1}{d_2}k_T\left(\theta_2-\theta_3\right)=T$$

Let $J_e=J_1+\left(\frac{d_1}{d_2}\right)^2 J_2$

We obtain:

$$J_e\ddot{\theta}_1+k_T\left(\frac{d_1}{d_2}\right)^2\theta_1-k_T\frac{d_1}{d_2}\theta_3=T$$

The non-geared equivalent system is shown as:



(b)

From the non-geared equivalent system:

$$J_e\ddot{\theta}_1+k_T\left[\left(\frac{d_1}{d_2}\right)^2\theta_1-\frac{d_1}{d_2}\theta_3\right]=T$$

Laplace Transform

$$J_e s^2\Theta(s)+k_T\left(\frac{d_1}{d_2}\right)^2\Theta(s)-k_T\left(\frac{d_1}{d_2}\right)\Theta(s)=T(s) \qquad (3)$$

$$b_T \frac{d_1}{d_2}\dot{\theta}_3 - k_T\left[\left(\frac{d_1}{d_2}\right)^2 \theta_1 - \frac{d_1}{d_2}\theta_3\right] = 0$$

Laplace Transform

$$b_T \frac{d_1}{d_2} s\Theta_3(s) - k_T\left(\frac{d_1}{d_2}\right)^2 \Theta_1(s) + k_T \frac{d_1}{d_2}\Theta_3(s) = 0$$

$$\Rightarrow (b_T s + k_T)\Theta_3(s) = k_T\left(\frac{d_1}{d_2}\right)\Theta_1(s)$$

$$\Rightarrow \Theta_3(s) = \frac{d_1}{d_2}\frac{k_T}{b_T s + k_T}\Theta_1(s) = T(s)$$

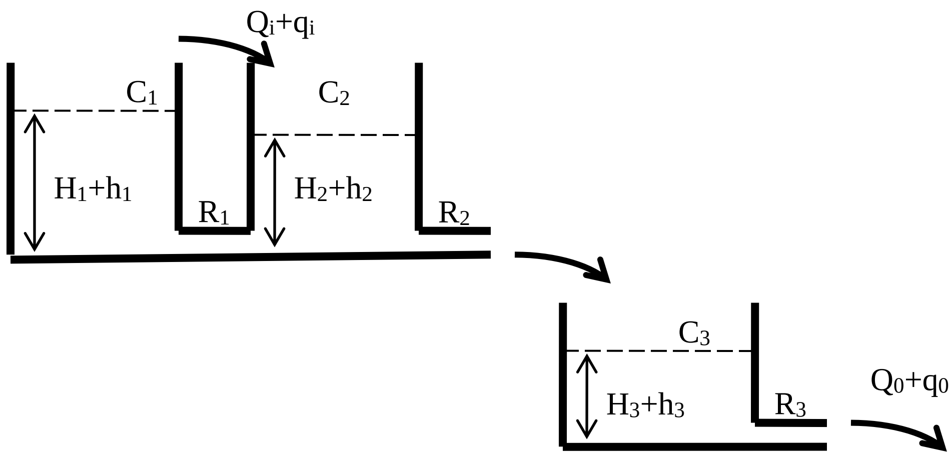
$$= \frac{b_T s + k_T}{\left(b_T s + k_T\right)\left[J_1 + \left(\frac{d_1}{d_2}\right)^2 J_2\right]s^2 + \left(\frac{d_1}{d_2}\right)^2 b_T k_T s}$$

$$\frac{\Theta_1(s)}{T(s)} = \frac{1}{J_e s^2 + \left(\frac{d_1}{d_2}\right)^2 k_T - \left(\frac{d_1}{d_2}\right)^2 \frac{k_T^2}{b_T s + k_T}}$$

$$= \frac{b_T s + k_T}{\left(b_T s + k_T\right)\left[J_1 + \left(\frac{d_1}{d_2}\right)^2 J_2\right]s^2 + \left(\frac{d_1}{d_2}\right)^2 b_T k_T s}$$

---

9. For the liquid level system shown below obtain the transfer function $q_i(s)/q_0(s)$.



$$R_1 q_1 = h_1 - h_2 \Rightarrow R_1 Q_1(s) = H_1(s) - H_2(s) \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

$$R_2 q_2 = h_2 \Rightarrow R_2 Q_2(s) = H_2(s) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

$$R_3 Q_0(s) = H_3(s) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

$$C_1 \frac{dh_1}{dt} = q_i - q_1 \Rightarrow C_1 s H_1(s) = Q_i(s) - Q_1(s) \ldots\ldots\ldots\ldots\ldots\ldots(4)$$

$$C_2 \frac{dh_2}{dt} = q_1 - q_2 \Rightarrow C_2 s H_2(s) = Q_1(s) - Q_2(s) \ldots\ldots\ldots\ldots\ldots\ldots(5)$$

$$C_3 \frac{dh_3}{dt} = q_2 - q_0 \Rightarrow C_3 s H_3(s) = Q_2(s) - Q_0(s) \ldots\ldots\ldots\ldots\ldots\ldots(6)$$

From (1) (2) (4) (5):

$$Q_i(s) = \left[R_1 C_1 R_2 C_2 s^2 + \left(R_1 C_1 + R_2 C_2 + R_2 C_1\right) s + 1\right] Q_2(s) \ldots\ldots\ldots(7)$$

From (3) and (6)

$$R_3 C_3 s Q_0(s) = Q_2(s) - Q_0(s) \ldots\ldots\ldots\ldots\ldots\ldots\ldots(8)$$

Combine (7) and (8),

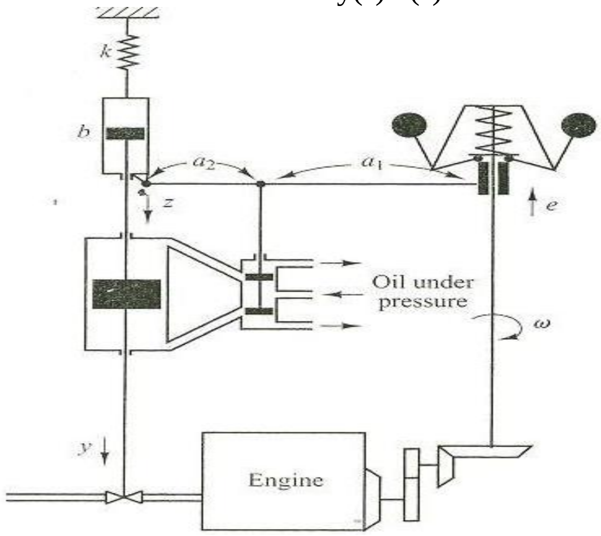
$$\frac{Q_i(s)}{Q_0(s)} = \left(R_3 C_3 s + 1\right)\left(R_1 C_1 R_2 C_2 s^2 + \left(R_1 C_1 + R_2 C_2 + R_2 C_1\right) s + 1\right) \ldots\ldots(9)$$

----

10. For the hydraulic servo system for speed control, obtain

a) the block diagram for the part of the system between the input e(s) and output y(s)

b) find the transfer function y(s)/e(s)



example A-4-11 in the text book.

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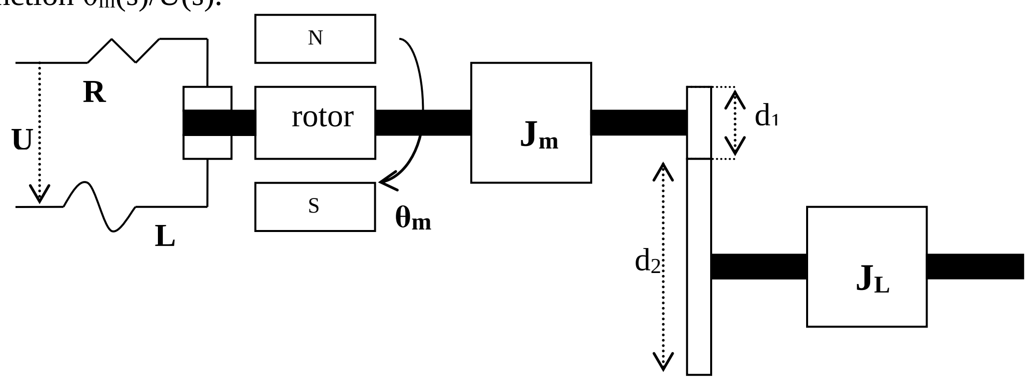
11. For the system shown, where U is the DC voltage supplied to the DC rotor, R is rotor resistance , L is rotor inductance, N and S are the permanent magnets of the stator (rotor inductance is negligible). J_m and J_L are moments of inertias. Obtain the transfer function $\theta_m(s)/U(s)$.



$$J_e = J_m + \left(\frac{d_1}{d_2}\right)^2 J_2$$

$$U - Ri - e_b = 0 \Rightarrow U - Ri - k_l\dot{\theta}_m = 0$$

$$k_t i - J_e\ddot{\theta}_m = 0$$

Take Laplace Transform, we have

$$U(s) - RI(s) - k_l s\Theta_m(s) = 0$$

$$k_t I(s) - J_e s^2\Theta_m(s) = 0$$

Combine the two equations,

$$U(s) - R\frac{J_e s^2}{k_t}\Theta_m(s) - k_l s\Theta_m(s) = U(s) - \left(R\frac{J_e}{k_t}s^2 + k_l s\right)\Theta_m(s) = 0$$

$$\Rightarrow \frac{\Theta_m(s)}{U(s)} = \frac{1}{R\frac{J_e}{k_t}s^2 + k_l s} = \frac{k_t}{RJ_e s^2 + k_l k_t s}$$

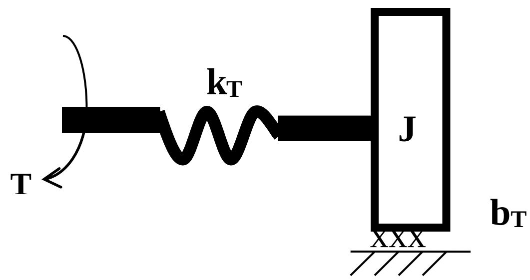
If $k_t = k_l = k$

We have

$$\frac{\Theta_m(s)}{U(s)} = \frac{k}{R\left[J_m + \left(\frac{d_1}{d_2}\right)^2 J_2\right]s^2 + k^2 s}$$

----

12. For the underdamped system shown below, find k_T, b_T and J to yield maximum overshoot of 0.4 and a settling time for 2% criterium of 4 seconds for a unit step input torque T(t). The steady state position of the disk is 0.1 [rad].



$(Js^2 + bs + k)a = T$

$T/a = 1/(Js^2 + bs + k) = (1/J)(1/(s^2 + (b/J)s + k/J)$

$T(s) = (1/J)(1/(s^2 + (b/J)s + k/J)(1/s)$

Steady state error to T(s)=1/s

$a(inf) = \lim_{s\text{-}0} s\,(1/J)(1/(s^2 + (b/J)s + k/J)(1/s) = \lim_{s\text{-}0}(1/J)(1/(s^2 + (b/J)s + k/J) = 1/k = 0.1$

k=10 [N/m]

$2\zeta\omega_n = b/J$ $\quad\omega_n = \sqrt{k/J}$ $\quad\zeta = b/J2\omega_n = b/(J2\sqrt{k/J}) = b/(2\sqrt{kJ})$

$\exp\{-\pi\zeta/\sqrt{(1-\zeta^2)}\} = 0.4$ $\quad -\pi\zeta/\sqrt{(1-\zeta^2)} = \ln 0.4 = -0.92$

$\zeta/\sqrt{(1-\zeta^2)} = -0.92/3.14 = -0.29$ $\quad \zeta = -0.29\sqrt{(1-\zeta^2)}$

$\zeta^2 = 0.29^2\,(1-\zeta^2) = 0.085 - 0.085\,\zeta^2$ $\quad 1.085\,\zeta^2 = 0.085$

$\zeta = 0.28 = b/(2\sqrt{kJ})$

$4/(\zeta\,\omega_n) = 4$ $\quad \zeta\,\omega_n = 1$ $\quad \omega_n = 1/\zeta = 1/0.28 = 3.57$

$\omega_n = 3.57 = \sqrt{k/J} = \sqrt{1/J}$ $\quad J = 3.57^2$

J= **12.7 [Nm^2]**

$\zeta = 0.28 = b/(2\sqrt{kJ}) = b/(2\sqrt{10 * 12.7}) = b/(2*22.76) = b/44.51$ $\quad b = 0.28*44.51$

b=12.74 [Nm s /rad]

HINTS $\exp\{-\pi\zeta/\sqrt{(1-\zeta^2)}\}$, $\pi/[\omega_n\sqrt{(1-\zeta^2)}$, $4/(\zeta\,\omega_n)$, $s^2 + 2\zeta\omega_n s + \omega_n^2$