6. For which real number c does the limit

$$\lim_{x\to\infty} \sqrt{4x^2+1} - cx$$

exist?

$= \infty - \infty$ Indeterminant form, so factor:

$\lim_{x\to\infty} \sqrt{4x^2+1} - cx = \lim_{x\to\infty} \sqrt{x^2}\sqrt{4+\frac{1}{x^2}} - cx = \lim_{x\to\infty} |x|\sqrt{4+\frac{1}{x}} - cx$ (since $x \to$, $|x| = x$)

$= \lim_{x\to\infty} x\left(\sqrt{4+\frac{1}{x}} - c\right)$. If the limit is to have any chance of not being ∞, then $\lim_{x\to\infty} \sqrt{4+\frac{1}{x}} - c = 0$, so $c = 2$. Now,

$\lim_{x\to\infty} \frac{\sqrt{4+\frac{1}{x}} - 2}{\frac{1}{x}} = \frac{0}{0}$ L'Hôp

$= \lim_{x\to\infty} \frac{\frac{1}{2}\left(4+\frac{1}{x}\right)^{-\frac{1}{2}}\left(-\frac{1}{x^2}\right)}{\left(-\frac{1}{x^2}\right)}$

$= \frac{1}{2\sqrt{4}} = \frac{1}{4}$

$\therefore c = 2$ works

A. $\ln(x)$ B. $\frac{\sqrt{x^2+1}}{x}$ C. 0 D. $\frac{1}{2}$ E. 1

F. $\frac{3}{2}$ G. 2 H. e I. π J. Such number does not exist

Correct answer: G

7. Consider the equation $x^2 - 5 = 0$. If the initial guess is $x_0 = 3$, the result of applying the Newton's method once produces numerical solution equal to x_1.

Choose the nearest answer.

$f'(x) = 2x$

$f'(3) = 6$

$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$

$= 3 - \frac{4}{6}$

$= 2\frac{1}{3} \cong 2.33$

A. $x_1 = 2.25$ B. $x_1 = 2.27$ C. $x_1 = 2.29$ D. $x_1 = 2.31$ E. $x_1 = 2.33$

F. $x_1 = 2.35$ G. $x_1 = 2.37$ H. $x_1 = 2.39$ I. $x_1 = 2.41$ J. $x_1 = 2.43$

Correct answer: E

8. Find the slope of the line tangent to the curve $f(x) = \ln\frac{\sin x}{x}$ at the point $x = 1$.

Choose the nearest answer.

$f(x) = \ln\sin x - \ln x$

$f'(x) = \frac{\cos x}{\sin x} - \frac{1}{x} \Rightarrow f'(1) = \frac{\cos(1)}{\sin(1)} - 1 = -0.36$

A. -2.0 B. -1.2 C. -0.8 D. -0.4 E. 0.0

F. 0.5 G. 1.0 H. 1.2 I. 2.0 J. Does not exist

Correct Answer: D

25. Find the average (mean) value of the function $f(x) = \frac{6x^2 + 4x}{x^3 + x^2}$ on $[1, 3]$.

The average value of a function, f, on $[a, b]$ is

$$\frac{1}{b-a}\int_a^b f(x)\,dx.$$

Thus, the average of $\frac{6x^2+4x}{x^3+x^2}$ on $[1,3]$ is

$$\frac{1}{3-1}\int_1^3 \frac{6x^2+4x}{x^3+x^2}\,dx \qquad \text{let } u = x^3+x^2$$

$$du = (3x^2+2x)\,dx$$

$$= \frac{1}{2}\int_1^3 \frac{2(3x^2+2x)\,dx}{x^3+x^2}$$

$$= \int_*^{**} \frac{1}{u}\,du = \ln|u|\Big|_*^{**} = \ln|x^3+x^2|\Big|_1^3$$

$$= \ln|27+9| - \ln|2| = \ln 18 \cong 2.89$$

UNIVERSITY OF VICTORIA
EXAMINATIONS August 2013
MATHEMATICS 100, SECTIONS [A01]-[A02]

Name: Solved by PW Student ID: ______ Section: ______

TO BE ANSWERED ON THE PAPER AND ON BUBBLE SHEETS Duration: 3 hours

Instructor: Peter Williamson

Course: Math 100
Section [A01], CRN: 30458
Section [A02], CRN: 30459

Question	Value	Marks
1–20	40	
21	5	
22	5	
23	5	
24	5	
25	5	
26	5	
27	5	
Total	75	

STUDENTS MUST COUNT THE NUMBER OF PAGES IN THIS EXAMINATION PAPER BEFORE BEGINNING TO WRITE AND REPORT ANY DISCREPANCY IMMEDIATELY TO THE INVIGILATOR.

THIS QUESTION PAPER HAS 17 PAGES PLUS COVER SHEET.

INSTRUCTIONS:

1. Make sure your name is in **3** places (**NOW**)
   (a) on the top of this page
   (b) back of the last page of this exam
   (c) on your green multiple choice sheet
   (d) **ALSO: make sure your name (last name first) and student number excluding V00 are coded with dots on the multiple choice sheet**
2. Use an HB pencil or softer to enter your answer on the computer sheet.
3. You may only use a Sharp EL-510R or EL-510-RNB calculator - no other calculator is acceptable on this examination. **A calculator that is not acceptable will be withheld by the invigilator until the end of the examination.**
4. The examination consists of **20 multiple choice questions** worth each 2 points for a total of 40 marks and **7 full answer questions** worth each 5 points for a total of 35 marks. For verification purposes, show all calculations on your question paper for each multiple choice and full answer question. Do all work on the test pages using the backs if necessary. Use no extra paper. Unverified answers may be disallowed.
5. For questions asking to **Choose the nearest answer**, choose the value which is nearest to your answer. If your answer is equidistant from the two nearest choices, choose the larger of these two choices. No marks will be deducted for wrong answers.
6. Submit BOTH your examination paper and your answer sheet.
7. **Cell phones** should be turned off during the final examination and **are not allowed** to be used in any manner until all papers are returned to the invigilator. **No head phones** are allowed during the final examination.

9. If $f(x) = \sin(\cos(x))$, find $f'\left(\frac{\pi}{2}\right)$.

Choose the nearest answer.

$$f'(x) = \cos(\cos(x)) \cdot (-\sin x)$$
$$f'\left(\tfrac{\pi}{2}\right) = \cos\left(\cos\left(\tfrac{\pi}{2}\right)\right) \cdot \left(-\sin\tfrac{\pi}{2}\right)$$
$$= \cos(0) \cdot (-1)$$
$$= -1$$

A. -2.0 B. -1.5 C. -1.0 D. -0.5 E. 0.0

F. 0.5 G. 1.0 H. 1.5 I. 2.0 J. 2.5

Correct Answer: C

10. For which value(s) of a does the following limit NOT exist?

$$\lim_{x \to a} \frac{x^2 - 3x}{3x^3 - 27x}$$

$$= \lim_{x \to a} \frac{x(x-3)}{3x(x-3)(x+3)}$$

$$= \lim_{x \to a} \frac{1}{3(x+3)}$$

The limit doesn't exist when $a = -3$ only.

A. only 0 B. only -3 C. only 0 and -3 D. only 3 E. only 0 and 3

F. 0, 3, and -3 G. only -3 and 3 H. only 9 I. only -9 J. exists for all a

Correct answer: [B. -3]

23. Use the limit definition of the derivative to find the slope of the line tangent to the curve

$$y = x^2 - 3x$$

at the point ~~(0,0)~~ $x=0$.

Do not use other differentiation rules to calculate the derivative.

$$\frac{dy}{dx} = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \left[\frac{\cancel{x^2}+2xh+h^2-3x-3h-\cancel{x^2}+\cancel{3x}}{h} \right.$$

$$= \lim_{h\to 0} \frac{\cancel{h}(2x+h-3)}{\cancel{h}} = \lim_{h\to 0} 2x+h-3 = 2x-3.$$

Thus $\left.\frac{dy}{dx}\right|_{x=0} = -3.$

1. Compute $f'(1)$ for

$$f(x) = \frac{x^2}{\sqrt{x} - 3}$$

Choose the nearest answer.

$$f'(x) = \frac{2x(\sqrt{x}-3) - (\frac{1}{2}x^{-\frac{1}{2}})x^2}{(\sqrt{x}-3)^2}$$

$$f'(1) = \frac{2 \cdot 1(\sqrt{1}-3) - (\frac{1}{2} \cdot \frac{1}{\sqrt{1}})1^2}{(\sqrt{1}-3)^2}$$

$$= \frac{-4 - \frac{1}{2}}{4} = -1.125$$

A. -2.6 B. -1.1 C. -0.75 D. -0.1 E. 0.5

F. 2.0 G. 2.7 H. 3.1 I. 3.8 J. Does not exist

Correct answer: [-1.125: answer B]

2. Compute

$$D_x[\sin^2(x^2+3)]$$

$$= 2\sin(x^2+3) \cdot \cos(x^2+3) \cdot 2x$$

$$= 4x \sin(x^2+3)\cos(x^2+3)$$

A. $\cos^2(x^2+3)$ B. $2\cos(x^2+3)\sin(x^2+3)$ C. $4x\cos(x^2+3)$

D. $4x\sin(x^2+3)$ E. $4x\cos(x^2+3)\sin(x^2+3)$ F. $2x\cos^2(x^2+3)$

G. $2x\sin^2(x^2+3)$ H. $2(x^2+3)\cos(x^2+3)\sin(x^2+3)$

I. $2\sin(x^2+3)$ J. None of the above

Correct Answer: [answer E]

17. Use Simpson's Approximation, with n=4 to approximate the following integral:

$$\int_0^4 \sin(x^2)dx$$

Choose the nearest answer.

$$S_4 = \frac{\Delta x}{3}\left(y_0 + 4y_1 + 2y_2 + 4y_3 + y_4\right)$$

$$\Delta x = \frac{4-0}{4} = 1$$

$$S_4 = \frac{1}{3}\left(\sin(0^2) + 4\sin(1^2) + 2\sin(2^2) + 4\sin(3^2) + \sin(4^2)\right)$$

$$= 1.071$$

A. 0.07	B. 0.36	C. 0.79	D. 1.07	E. 1.55
F. 1.82	G. 2.11	H. 2.67	I. 2.98	J. 3.16

Correct answers: 1.0709.. (D)

18. Let $F(x) = \int_1^{3x} e^{t^2}dt$; find $F'(0)$.

Choose the nearest answer.

Suppose $H(t)$ is the antiderivative of e^{t^2}.

Then $\int_1^{3x} e^{t^2}dt = H(3x) - H(1) = F(x)$.

Then $F'(x) = H'(3x)\cdot 3 = e^{(3x)^2}\cdot 3$

$F'(0) = e^0 \cdot 3 = 3$

A. 1	B. 2	C. 3	D. 4	E. 5
F. 6	G. 7	H. 8	I. 9	J. 10

Correct answer: 3 (C)

3. What is the global maximum value of the following function on the closed interval $[-2.5, 0.5]$?

$$f(x) = x^2 e^{2x}$$

Choose the nearest answer.

$$f'(x) = 2xe^{2x} + 2x^2e^{2x} = 0$$
$$\Rightarrow 2e^{2x}(x + x^2) = 0$$
$$\Rightarrow x(x+1) = 0 \quad \text{since } e^{2x} > 0 \text{, i.e. never equal to zero.}$$
$$\Rightarrow x = 0 \text{ or } -1 \text{ are the critical points,}$$

$f(-2.5) = 0.042$ $f(-1) = 0.135$ $f(0) = 0$ $\boxed{f(0.5) = 0.680}$ max

A. 0.02 B. 0.13 C. 0.24 D. 0.35 E. 0.46

F. 0.57 G. 0.68 H. 0.79 I. 0.90 J. Does not exist

Correct answer: [0.68 : answer G]

4. Compute the limit

$$\lim_{x\to 0} \frac{\tan(x)}{x^2} = \frac{0}{0}$$ indeterminant form, so apply L'Hôpital's rule.

$$= \lim_{x\to 0} \frac{\sec^2(x)}{2x}$$

$$\left.\begin{aligned} \lim_{x\to 0^+} \frac{\sec^2(x)}{2x} &= +\infty \\ \lim_{x\to 0^-} \frac{\sec^2(x)}{2x} &= -\infty \end{aligned}\right\} \therefore \lim_{x\to 0} \frac{\sec^2(x)}{2x} \text{ D.N.E.}$$

A. $-\infty$ B. -1 C. $-\frac{1}{2}$ D. 0 E. $\frac{1}{2}$

F. 1 G. π H. π^2 I. ∞ J. Neither $+\infty$ nor $-\infty$ and does not exist

Correct answer: J

5. Compute the limit

$$\lim_{x\to 2} \frac{x-2}{x^2-4}$$

$$= \lim_{x\to 2} \frac{x-2}{(x-2)(x+2)} = \lim_{x\to 2} \frac{1}{x+2} = \frac{1}{4}$$

A. 2 B. $\frac{1}{2}$ C. $\frac{1}{4}$ D. 0 E. Does not exist

F. $-\frac{1}{4}$ G. $-\frac{1}{2}$ H. -2 I. $(x+2)$ J. None of the above

Correct answer: C

24. Show by the method of Riemann sums that:

$$\int_1^3 (x-1)dx = 2$$

That is, subdivide the interval $[1,3]$ into n equal subdivisions and using right-hand end-points to evaluate your function, calculate the corresponding Riemann sum. Then take the limit as $n \to \infty$ to obtain the exact value of the integral.

You may need the following formulas:

$$\sum_{k=1}^{n} c = nc \quad \text{where c is a constant;} \qquad \sum_{k=1}^{n} k = \frac{n^2}{2} + \frac{n}{2}; \qquad \sum_{k=1}^{n} k^2 = \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}.$$

$\Delta x_i = \frac{3-1}{n} = \frac{2}{n}$ for all i (subintervals are all the same size).

$$R = \sum_{i=1}^{n} f(x_i^*) \Delta x_i$$

$$x_i = 1 + i\Delta x$$
$$= 1 + \frac{2i}{n}$$

$$= \sum_{i=1}^{n} \left(1 + \frac{2i}{n} - 1\right)\frac{2}{n}$$

$$= \frac{4}{n^2} \sum_{i=1}^{n} i = \frac{4}{n^2}\left(\frac{n^2}{2} + \frac{n}{2}\right) = 2 + \frac{2}{n}$$

Thus $\int_1^3 (x-1)dx = \lim_{n\to\infty} \left(2 + \frac{2}{n}\right) = 2.$

22. The base of a rectangle is increasing at the rate of 4 cm/s while its height is decreasing at the rate of 3 cm/s. At what rate is its area changing when its base is 20 cm and its height is 12 cm?

$A(t) = x(t)\,y(t)$

$$\frac{dA}{dt} = \frac{dx}{dt}y + x\frac{dy}{dt}$$

$$= (4 \cdot 12 - 3 \cdot 20)\ \mathrm{cm^2/s}$$

$$= -12\ \mathrm{cm^2/s}$$

The area is decreasing at $12\ \mathrm{cm^2/s}$

15. Solve the initial value problem:

$$F'(x) = \cos(x) + e^{2x} - 1 \text{ and } F(0) = 0$$

$$F(x) = \sin x + \frac{1}{2}e^{2x} - x + C \leftarrow \text{antiderivative of } F'(x).$$

$$F(0) = 0 + \frac{1}{2} + C = 0$$

$$\text{so } C = -\frac{1}{2}$$

$$\therefore F(x) = \sin x + \frac{1}{2}e^{2x} - x - \frac{1}{2}$$

A. $F(x) = \sin(x) + \frac{1}{2}e^{2x}$

B. $F(x) = -\sin(x) + \frac{1}{2}e^{x} - x$

C. $F(x) = \sin(x) + \frac{1}{2}e^{2x} - x - 1$

D. $F(x) = -\sin(x) + 2e^{2x} - x$

E. $F(x) = \sin(x) + \frac{1}{2}e^{2x} - x - \frac{1}{2}$

F. $F(x) = -\sin(x) + 2e^{2x} - x - \frac{1}{2}$

G. $F(x) = \sin(x) + 2e^{2x} - x - 2$

H. $F(x) = -\sin(x) + 2e^{x} - x - 1$

I. $F(x) = \sin(x) + \frac{1}{2}e^{x} - x + 1$

J. $F(x) = \sin(x) + 2e^{2x} - x - 2$

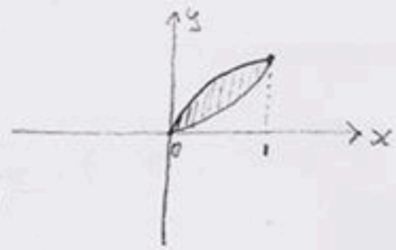
Correct Answer: (E)

16. Find the area bounded by $y = \sqrt{x}$ and $y = x^2$.

Choose the nearest answer.

Intersection points

$$\sqrt{x} = x^2$$
$$\Rightarrow x = x^4$$
$$\Rightarrow x(x^3 - 1) = 0$$
$$\Rightarrow x = 0 \text{ or } +1$$



$$\therefore A = \int_0^1 \sqrt{x} - x^2\,dx = \frac{2}{3}x^{\frac{3}{2}} - \frac{x^3}{3}\Big|_0^1 = \left(\frac{2}{3} - \frac{1}{3}\right) - (0 - 0$$

$$= \frac{1}{3}$$

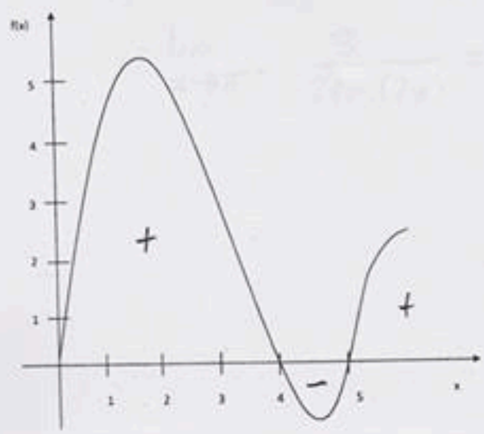
A. 0.0 B. 0.1 C. 0.2 D. 0.3 E. 0.4

F. 0.5 G. 0.6 H. 0.7 I. 0.8 J. 0.9

Correct answer: 0.3333 (D)

11. For which values of a and b below does the following definite integral have the minimum value?

$$\int_a^b f(x)dx$$



A. $a=0, b=0$ B. $a=0, b=1$ C. $a=0, b=5$ D. $a=1, b=4$ E. $a=1, b=5$

F. $a=2, b=4$ G. $a=3, b=4$ H. $a=2, b=5$ I. $a=3, b=5$ J. $a=4, b=5$

Correct answer: [J. $a=4, b=5$]

12. Find the interval containing a solution to the equation $x^5+\frac{1}{4}x^3=1$

$f(x)=x^5+\frac{1}{4}x^3-1=0$

$f(0)=-1$
$f(1)=\frac{1}{4}$

$\therefore$ By I.V.P. $[0,1]$ contains a solution.

A. $[-1,0]$ B. $[0,1]$ C. $[1,2]$ D. $[2,3]$ E. $[3,4]$

F. $[4,5]$ G. $[5,6]$ H. $[6,7]$ I. $[7,8]$ J. None of the above

Correct answer: [B. [0,1]]

FULL ANSWER

21. Sketch the graph of the function $y(x) = \frac{x^2-9}{x+2}$, by completing a full analysis as outlined in class, labelling all important points.

y-int: $y(0) = -\frac{9}{2}$

x-int: $\frac{x^2-9}{x+2} = 0 \Rightarrow x = \pm 3$ $\qquad x \neq 2$

$\lim_{x\to 2^-} \frac{x^2-9}{x+2} = +\infty$; $\lim_{x\to 2^+} \frac{x^2-9}{x+2} = -\infty$ $\qquad x=2$ is a vertical asymptote.

$y'(x) = \frac{2x(x+2)-(x^2-9)}{(x+2)^2} = 0 \qquad x \neq -2$

$\Rightarrow x^2+4x+9=0$ $\Rightarrow$ Quadratic formula gives $\frac{-4 \pm \sqrt{4^2-4\cdot(1)\cdot 9}}{2}$

& $16-36<0$ so there are no real roots.

$\therefore$ The only critical point is $x=-2$.

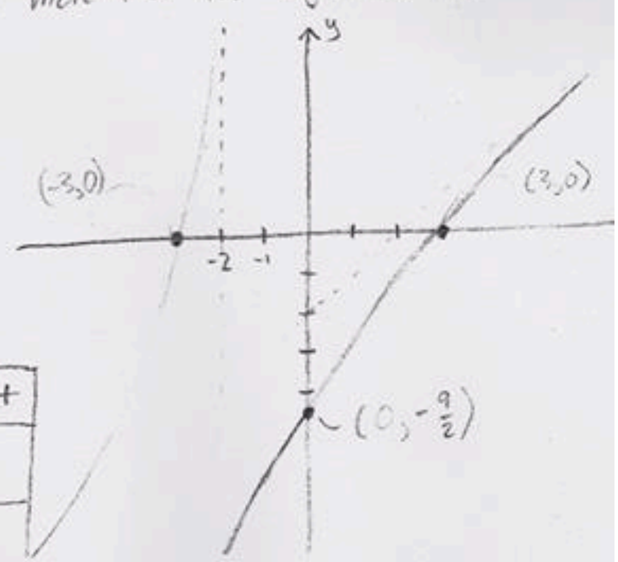
$y''(x) = \frac{2x+4\cdot(x+2)^2 - 2(x+2)(x^2+4x+9)}{(x+2)^{\cancel{4}\,3}} = \frac{-10}{(x+2)^3}$

The only potential inflection point is at $x=-2$.

Since the degree of the numerator is one more than the degree of the denominator, we have a slant asymptote:

$$\begin{array}{r} x-2 \\ x+2 \overline{)\, x^2-9} \\ x^2+2x \\ -2x-9 \\ -2x-4 \\ -5 \end{array} \quad \therefore \frac{x^2-9}{x+2} = \boxed{x-2} - \frac{5}{x+2}$$

slant asymptote.



	-3	-2	3	
$f(x)$	$-$	$+$	$-$	$+$
$f'(x)$	$+$		$+$	
$f''(x)$	$+$		$-$	

13. Find the limit:

$$\lim_{x\to\pi^-}(3x-3\pi)\csc(2x).$$

$= 0\cdot\infty$

$= \lim_{x\to\pi^-} \frac{3x-3\pi}{\sin(2x)} = \frac{0}{0}$ Apply L'Hôpital:

$= \lim_{x\to\pi^-} \frac{3}{2\cos(2x)} = \frac{3}{2\cdot 1} = \frac{3}{2}$

A. -2.0 B. -1.5 C. -1.0 D. -0.5 E. 0.0

F. 0.5 G. 1.0 H. 1.5 I. 2.0 J. 2.5

Correct answer: 1.5 (H)

14. Find the limit:

$$\lim_{x\to\infty}\ln\left(1+\frac{2}{x}\right)^x$$

$= \lim_{x\to\infty} x\ln\left(1+\frac{2}{x}\right) = \lim_{x\to\infty}\frac{\ln\left(1+\frac{2}{x}\right)}{\frac{1}{x}} = \frac{0}{0}$ Apply L'Hôpital

$= \lim_{x\to\infty}\frac{\frac{1}{1+\frac{2}{x}}\cdot\frac{-2}{x^2}}{\frac{-1}{x^2}} = \lim_{x\to\infty}\frac{+2}{1+\frac{2}{x}} = 2$

A. $-\infty$ B. -2 C. -1 D. 0 E. e^{-1}

F. 1 G. 2 H. e I. ∞ J. None of the above

Correct answer: 2 (G)

19. What value of c would make the function $f(x)$ continuous at the point $x = 0$, where

$$f(x) = \begin{cases} \frac{\sin(3x)}{2x} & ,\text{if } x \neq 0 \\ c & ,\text{if } x = 0 \end{cases}$$

$$\lim_{x \to 0} \frac{\sin(3x)}{2x} = \lim_{x \to 0} \frac{3\sin 3x}{2 \cdot (3x)} = \frac{3}{2}$$

$\therefore c = \frac{3}{2}$ for $f(x)$ to be continuous at 0.

Recall $f(x)$ is continuous at $x = a$ if

$\lim_{x \to a} f(x) = f(a)$.

A. 0.0 B. 0.5 C. 1.0 D. 1.5 E. 2.0

F. 2.5 G. 3.0 H. 3.5 I. 4.0 J. Such value does not exist

Correct answer: (D)

20. Find $L(1)$, where $L(x)$ is the linear approximation of $f(x) = \tan(e^x)$ about the point $x = 0$.

Choose the nearest answer.

$f'(x) = \sec^2(e^x) \cdot e^x$

$f'(0) = \sec^2(1) \cdot 1 = \sec^2(1)$

$L_0(x) = f(0) + f'(0) \cdot (x - 0)$

$= \tan(1) + \sec^2(1)\, x$

$L_0(1) = \tan(1) + \sec^2(1) = 4.98$

A. 0 B. 1 C. 2 D. 3 E. 4

F. 5 G. 6 H. 7 I. 8 J. 9

Correct answer: 4.98 (F)