Q1 (5 Marks) Determine if the columns of matrix $A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$ are linearly independent.

Sol: We only need to show that $A\vec{x} = \vec{0}$ has only trivial solution. ①

The augmented matrix $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \end{array}\right]$ ②

$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$ $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ only trivial solution.

Hence. the columns of matrix A are linearly independent. ②

Q2 Determine whether the vectors are linearly independent. Give the reason. (Hint: No need to solve the linear system.)

[2 marks for dependent. 1 mark for reason.]

(a) (3 Marks) $\begin{bmatrix} -8 \\ 12 \end{bmatrix}$ $\begin{bmatrix} 2 \\ -3 \end{bmatrix}$

$\begin{bmatrix} -8 \\ 12 \end{bmatrix} = (-4)\begin{bmatrix} 2 \\ -3 \end{bmatrix}$ Then. dependent.

(b) (3 Marks) $\begin{bmatrix} 5 \\ -3 \\ -1 \end{bmatrix}$ $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ $\begin{bmatrix} -7 \\ 2 \\ 4 \end{bmatrix}$

Dependent. since there is vector $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

(c) (3 Marks) $\begin{bmatrix} 5 \\ 1 \end{bmatrix}$ $\begin{bmatrix} 2 \\ 8 \end{bmatrix}$ $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ $\begin{bmatrix} -1 \\ 7 \end{bmatrix}$

Dependent. since the number of vectors is greater than the entries of each vector.

Q3 Define a transformation $T: \mathfrak{R}^n \to \mathfrak{R}^m$ by $T(\vec{x}) = A\vec{x}$.

(a) (9 Marks) If T maps $\vec{u} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ into $\begin{bmatrix} 4 \\ 1 \end{bmatrix}$ and maps $\vec{v} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$ into $\begin{bmatrix} -1 \\ 3 \end{bmatrix}$. Find the images under T of $2\vec{u}$, $3\vec{v}$ and $2\vec{u} + 3\vec{v}$. (Hint: Matrix transformation is linear transformation.)

$T(2\vec{u}) = 2T(\vec{u}) = 2\begin{bmatrix} 4 \\ 1 \end{bmatrix} = \cancel{\begin{bmatrix} 6 \\ 8 \end{bmatrix}} \begin{bmatrix} 8 \\ 2 \end{bmatrix}$ ③

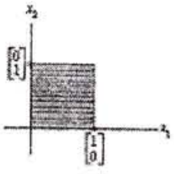
$T(3\vec{v}) = 3T(\vec{v}) = 3\begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} -3 \\ 9 \end{bmatrix}$ ③

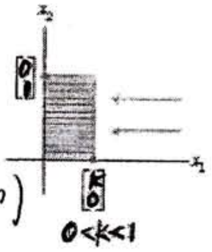
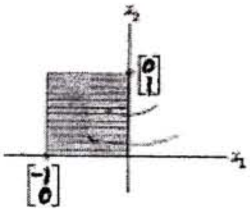
$T(2\vec{u} + 3\vec{v}) = T(2\vec{u}) + T(3\vec{v}) = \begin{bmatrix} 8 \\ 2 \end{bmatrix} + \begin{bmatrix} -3 \\ 9 \end{bmatrix} = \begin{bmatrix} 5 \\ 11 \end{bmatrix}$ ③

(b) (3 Marks) Let A be a 6x5 matrix. What must n and m be in order to define T?

$m = 6 \qquad n = 5. \qquad$ i.e. $\mathbb{R}^5 \longrightarrow \mathbb{R}^6.$

Q4 (4 Marks) Find the standard matrix for the following transformation.



Transformation	Image of the Unit Square	Standard Matrix
Horizontal contraction and expansion (Horizontal contraction and expansion)		$\begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix} = ?$ ②
Reflection through the x_2-axis (Reflection through the x_2-axis)		$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = ?$ ②

Q5 Let $T(x_1, x_2) = (3x_1 + x_2,\ 5x_1, +7x_2,\ x_1, +3x_2)$.

(a) (4 Marks) What's the standard matrix for the linear transformation T. That is $T(\vec{x}) = A\vec{x}$, find A.

$$A = \begin{bmatrix} 3 & 1 \\ 5 & 7 \\ 1 & 3 \end{bmatrix}$$

(b) (3 Marks) Show that T is a one-to-one linear transformation.

Need to show that $A\vec{x} = \vec{0}$ has only trivial solution.

$$\left[\begin{array}{cc|c} 3 & 1 & 0 \\ 5 & 7 & 0 \\ 1 & 3 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 3 & 0 \\ 5 & 7 & 0 \\ 3 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 3 & 0 \\ 0 & -8 & 0 \\ 0 & -8 & 0 \end{array}\right] \quad ②$$

$$\sim \left[\begin{array}{cc|c} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & -8 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \qquad \vec{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Therefore. T is one-to one ①

(c) (3 Marks) Is T onto transformation?

From part (b), $A \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$.

A doesn't have pivot position in each row. ①

Therefore. T is not onto transformation. ②