MAT1330 - Calculus for the Life Sciences

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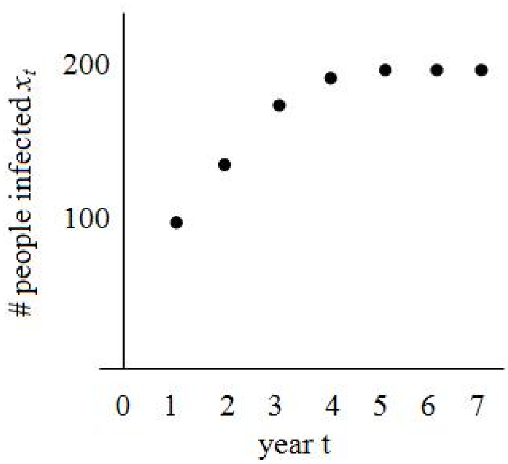
1 Discrete-Time Dynamical Systems

A basic rule of life: everything changes. This is especially true in biology.
Question: What are some biological examples of things that change?
Eg) The number of people who catch the flu on campus is equal to half those who had it last year plus 100.

$$x_t \xrightarrow{\text{Some people get infected again}} \frac{1}{2}x_t \xrightarrow{\text{Influx of new people with flu}} x_{t+1} = \frac{1}{2}x_t + 100$$

This is called a discrete-time dynamical system (dynamical because it changes, discrete because it only does so in fixed chunks)
In general, we can write $x_{t+1} = f(x_t)$ where f is the updating function.
Eg) If $x_0 = 0$, ie initially there is no flu, what happens eventually?

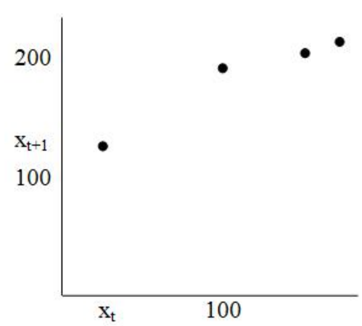
$x_0 = 0$
$x_1 = 100$
$x_2 = 150$
$x_3 = 175$
$x_4 = 187.5$
$x_5 = 193.75$
$x_6 = 196.875$
$x_7 = 198.4375$
$x_8 = 199.21875$
$x_9 = 199.609375$



The curve through these points is complicated.
Therefore, we suspect that eventually 200 people will be infected.

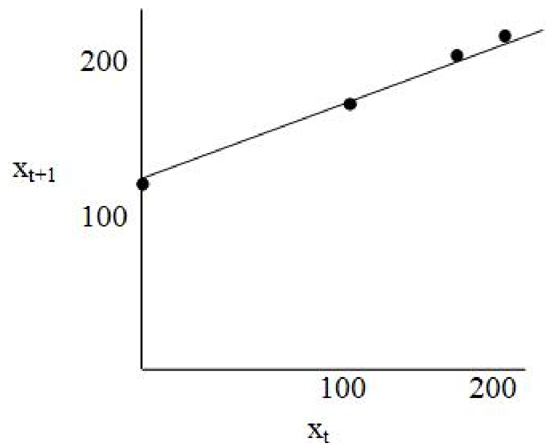
Homework: What happens if 400 people were initially infected?
We could think of these data as the initial and final size of the endemic period.

Year	Initial x_t	Final x_{t+1}
1	0	100
2	100	150
3	150	175
4	175	187.5



Question: What does the curve through these points look like?

Answer: A straight line.



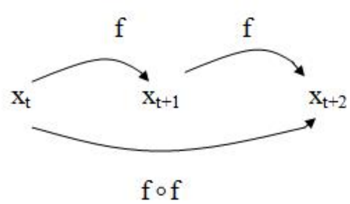
This line has intercepted 100 and slope $\frac{1}{2}$: $x_{t+1} = \frac{1}{2}x_t + 100$.
But this is the original formula.
Therefore plotting x_t vs t is complicated, but plotting x_t vs x_{t+1} is easy.
How can we use this to make predictions?

Eg) We can make a flu vaccine, but we need two years usually to know how much to make. How can we find out from the number of people infected now how many will be infected in two years' time?

$$\begin{aligned}
x_{t+1} &= \frac{1}{2}x_t + 100 \\
x_{t+2} &= \frac{1}{2}x_{t+1} + 100 \\
&= \frac{1}{2}\left[\frac{1}{2}x_t + 100\right] + 100 \\
&= \frac{1}{4}x_t + 50 + 100 \\
&= \frac{1}{4}x_t + 150
\end{aligned}$$

This is called <u>composition</u> of functions:

$$\begin{aligned}
x_{t+1} &= f(x_t) \\
x_{t+2} &= f(x_{t+1}) \\
&= f(f(x_t)) \\
&= (f_\circ f)x_t
\end{aligned}$$



Eg) $y_{t+1} = 3y_t - 2$. Find $f_\circ f$.

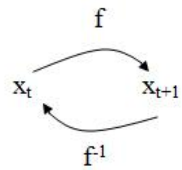
$$\begin{aligned}
f(y_t) &= 3y_t - 2 \\
f(f(y_t)) &= 3(3y_t - 2) - 2 \\
&= 9y_t - 6 - 2 \\
&= 9y_t - 8
\end{aligned}$$

2 Inverses

Suppose we know how many people are infected now, but we want to find out how many were infected last year.

The inverse of f is f^{-1} such that $f_\circ f^{-1}(x_t) = x_t$ or $f_\circ^{-1} f(x_t) = x_t$. In particular, if $x_{t+1} = f(x_t)$ then $x_t = f^{-1}(x_{t+1})$.
Why? Apply f^{-1} to both sides:

$$\begin{aligned} x_{t+1} &= f(x_t) \\ f^{-1}(x_{t+1}) &= f^{-1}(f(x_t)) \\ f^{-1}(x_{t+1}) &= x_t \end{aligned}$$



Thus, the inverse of a function corresponds to a function that goes backwards in time.

Eg) $x_{t+1} = \frac{1}{2}x_t + 100$. Find the inverse.

$$\begin{aligned} x_{t+1} &= \frac{1}{2}x_t + 100 \\ x_{t+1} - 100 &= \frac{1}{2}x_t \\ 2(x_{t+1} - 100) &= x_t \\ x_t &= 2x_{t+1} - 200 \end{aligned}$$

Eg) The rule for dating younger people says that the youngest person you can date is half your age plus 7. For example,
If you're 20, you shouldn't date younger than 17.
If you're 30, you shouldn't date younger than 22.
If you're 40, you shouldn't date younger than 27.
What is the oldest you can date?

$$\begin{aligned} y &= \frac{x}{2} + 7 \\ y - 7 &= \frac{x}{2} \\ x &= 2(y - 7) \\ x &= 2y - 14 \end{aligned}$$

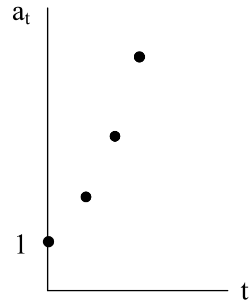
Therefore, the oldest you should date is twice your age minus 14. Then,
If you're 20, you shouldn't date older than 26.
If you're 30, you shouldn't date older than 46.
If you're 40, you shouldn't date older than 66.

Solutions: We've graphed the function, but we would like to solve for x_t in terms of t if possible. However, we need to know where we started. The starting values are called the initial conditions.

Definition 2.1. *The sequence $x_0, x_1, x_2, ...$ is the solution of the discrete-time dynamical system $x_{t+1} = f(x_t)$ starting from the initial condition x_0.*

Eg) $a_{t+1} = 3a_t,\ a_0 = 1$ (bacteria triples every hour). Determine how many bacteria there are after 8 hours without knowing how many there were after 7 hours.

$$
\begin{aligned}
a_1 &= 3a_0 = 3\cdot 1\\
a_2 &= 3a_1 = 3\cdot 3\cdot 1 = 3^2\cdot 1\\
a_3 &= 3a_2 = 3\cdot 3\cdot 3\cdot 1 = 3^3\cdot 1\\
&\ \vdots\\
a_n &= 3^n\cdot 1
\end{aligned}
$$



Therefore, we can define $a_8 = 3^8 = 6561$ without having to find $a_1, ...a_7$.

Homework: What is the solution if $a_0 = 2$?
Eg) $x_{t+1} = \frac{1}{2}x_t + 100$

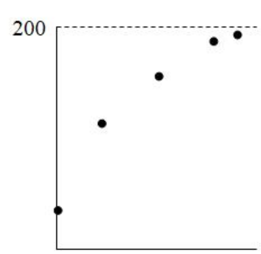
$$
\begin{aligned}
x_1 &= \frac{1}{2}x_0 + 100\\
x_2 &= \frac{1}{2}\left(\frac{1}{2}x_0 + 100\right) + 100 = \frac{1}{4}x_0 + \frac{1}{2}100 + 100\\
x_3 &= \frac{1}{2}\left(\frac{1}{4}x_0 + \frac{1}{2}100 + 100\right) + 100 = \frac{1}{8}x_0 + \frac{1}{4}100 + \frac{1}{2}100 + 100
\end{aligned}
$$

But it's not easy to guess the general formula from this.
However, earlier we saw that the values get closer and closer to 200. Let's try subtracting 200 from each value.

$$
\begin{aligned}
x_1 - 200 &= \frac{1}{2}x_0 - 100 = \frac{1}{2}(x_0 - 200)\\
x_2 - 200 &= \frac{1}{2}x_1 - 100 = \frac{1}{2}(x_1 - 200) = \frac{1}{2^2}(x_0 - 200)\\
x_3 - 200 &= \frac{1}{2}x_2 - 100 = \frac{1}{2}(x_2 - 200) = \frac{1}{2^3}(x_0 - 200)\\
x_4 - 200 &= \frac{1}{2}x_3 - 100 = \frac{1}{2}(x_3 - 200) = \frac{1}{2^4}(x_0 - 200)\\
&\ \vdots\\
x_n - 200 &= \frac{1}{2}x_{n-1} - 100 = \frac{1}{2}(x_{n-1} - 200) = \frac{1}{2^n}(x_0 - 200)\\
\therefore x_n &= \frac{1}{2^n}(x_0 - 200) + 200
\end{aligned}
$$

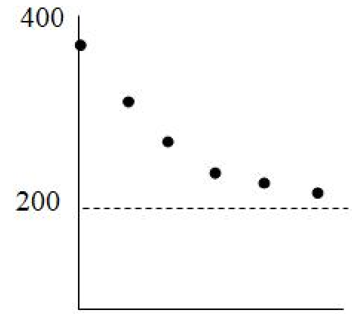
Therefore, the solution for $x_0 = 1$ is

$$
\begin{aligned}
x_0 &= 1\\
x_1 &= 100.5\\
x_2 &= 150.25\\
x_3 &= 175.125\\
&\ \vdots\\
x_{10} &= \frac{1}{2^{10}}(1 - 200) + 200\\
&= 199.805664
\end{aligned}
$$



The solution for $x_0 = 400$ is

$$
\begin{aligned}
x_0 &= 400 \\
x_1 &= 300 \\
x_2 &= 250 \\
x_3 &= 225 \\
x_4 &= 212.5 \\
x_5 &= 206.25 \\
x_6 &= 203.125 \\
&\vdots \\
x_{10} &= \frac{1}{2^{10}}(400-200)+200 \\
&= 200.3906247
\end{aligned}
$$



Question: Can we always guess the formula?
Answer: No. As we'll see later, formulas get very complex and sometimes lead to chaos. In this case, not only can we not guess, it's totally impossible to write down a formula.

3 Cobwebbing

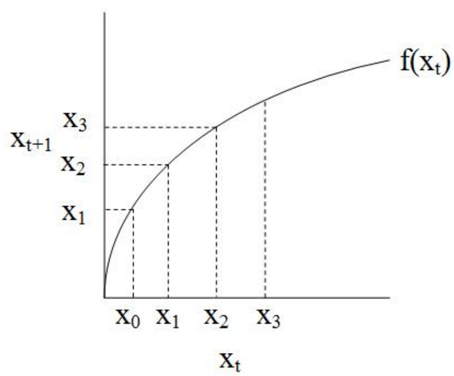
Suppose we have a discrete-time dynamical system $x_{t+1} = f(x_t)$.

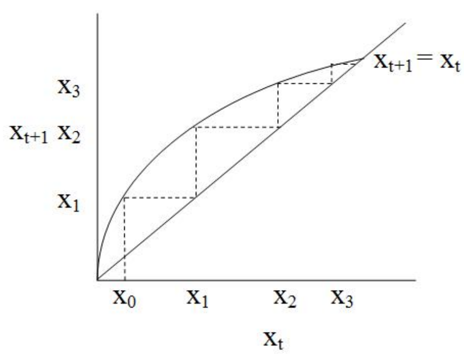
Graphically, we'd like to find out what happens if we start at x_0. This involves measuring each x_{t+1} and then plugging that back in for the next x_t.

Better way: Draw the line $x_{t+1} = x_t$ and then go off that. This line restarts each step, turning x_{t+1} into the next x_t (since $x_{t+1} = x_t$ on this line).

Using this method, we get
a) the steps and
b) a sense of the long-term behaviour.

This method is called cobwebbing.

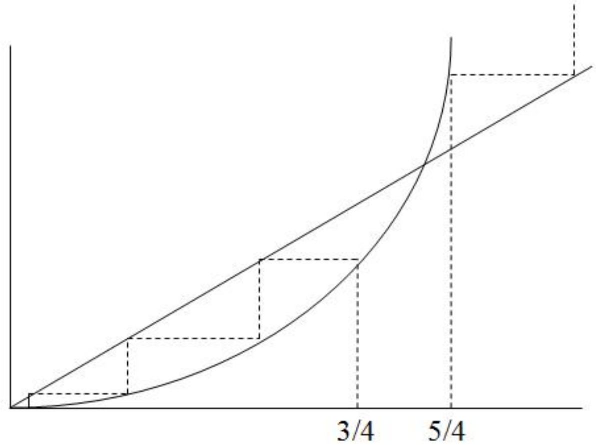




Eg) $x_{t+1} = x_t^2, \quad x_0 = \frac{3}{4}, \quad x_0 = \frac{5}{4}$

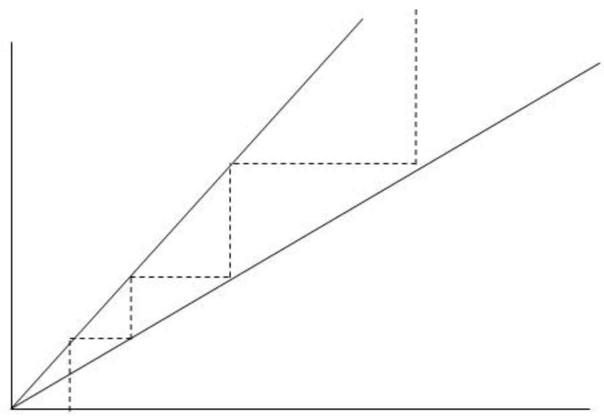
For $x_0 = \dfrac{3}{4}$, solutions approach 0.

For $x_0 = \dfrac{5}{4}$, solutions approach ∞.



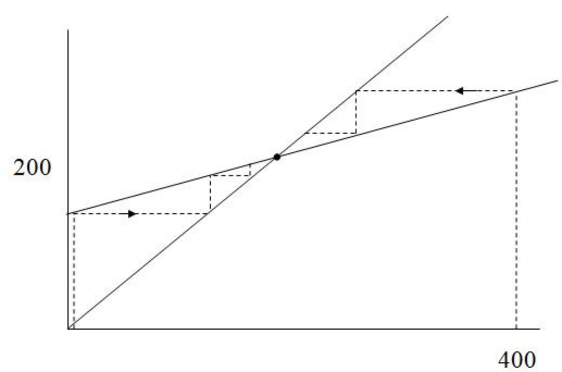
Eg) $a_{t+1} = 3a_t$, $x_0 = 1$

Solutions move away from 0 and approach ∞.



Eg) $x_{t+1} = \frac{1}{2}x_t + 100,\ x_0 = 0,\ x_0 = 400$

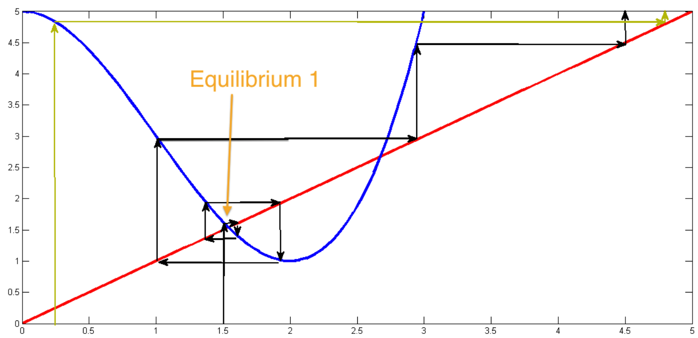
Solutions approach 200.



Eg) $y_{t+1} = y_t^3 - 3y_t^2 + 5$

$x_0 = \frac{1}{4}$
$x_1 = 4.828125$
$x_2 = 47.615$
$x_3 = 101155.86$

$x_0 = \frac{3}{2}$
$x_1 = 1.625$
$x_2 = 1.369$
$x_3 = 1.943$
$x_4 = 1.0096$
$x_5 = 2.97119$



Very complex behaviour can be understood fairly easily.

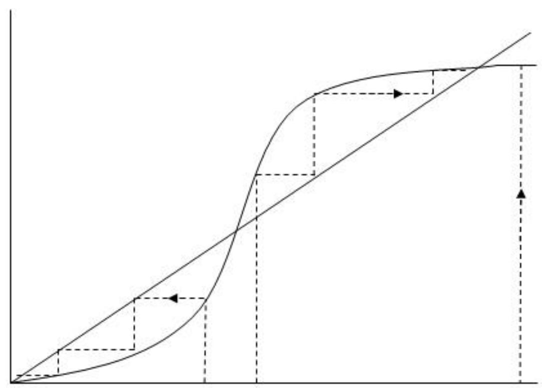
3.1 Equilibria

When the function crosses the diagonal, what happens?

If we're nearby, we may move closer or further away.

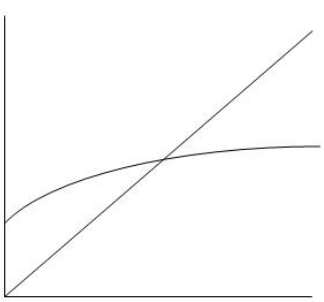
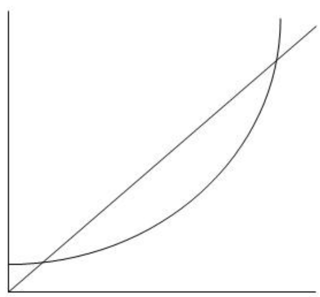
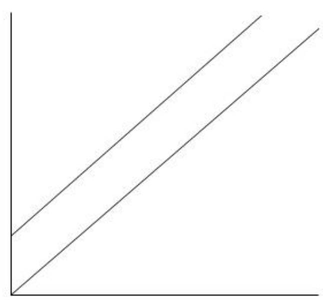
If we are exactly on the point, then there is no change.

These points are called equilibrium points.



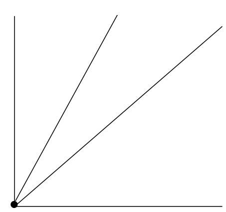
Definition 3.1. *A point x^* is an equilibrium of $x_{t+1} = f(x_t)$ if $f(x^*) = x^*$.*

Eg) In the figures below, we have 1 equilibrium, 2 equilibria and no equilibria.

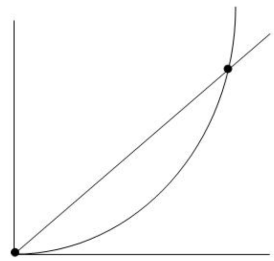
Since $f(x^*) = x^*$, we can use this to solve for equilibria.
Eg)

$$\begin{aligned} a_{t+1} &= 3a_t \\ a^* &= 3a^* \\ 0 &= 2a^* \\ a^* &= 0 \text{ is the only equilibrium.} \end{aligned}$$



Eg) $x_{t+1} = x_t^2$

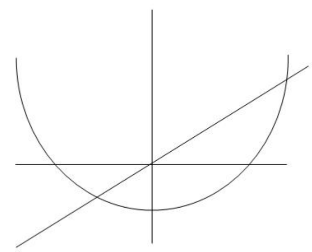
$$\begin{aligned} x^* &= x^{*2} \\ 0 &= x^{*2} - x^* \\ 0 &= x^*(x^* - 1) \\ x^* &= 0 \text{ or } 1 \end{aligned}$$



Homework: Find the equilibria for $x_{t+1} = \frac{1}{2}x_t + 100$.

Eg) Bacteria grow according to the equation $x_{t+1} = x_t^2 - 6$.

$$\begin{aligned} x_{t+1} &= x_t^2 - 6 \\ x^* &= x^{*2} - 6 \\ x^{*2} - x^* - 6 &= 0 \\ (x^* - 3)(x^* + 2) &= 0 \\ x^* &= 3, \ -2 \end{aligned}$$

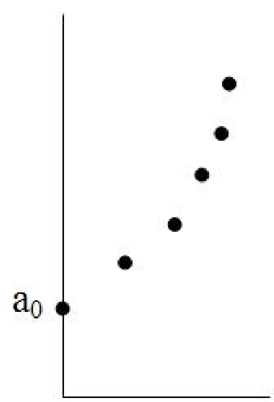


However, one of these is not biologically meaningful. Be careful: if you write down answers that are biologically meaningless when the question refers to biology, you'll lose marks.

3.2 Exponential Functions

Consider

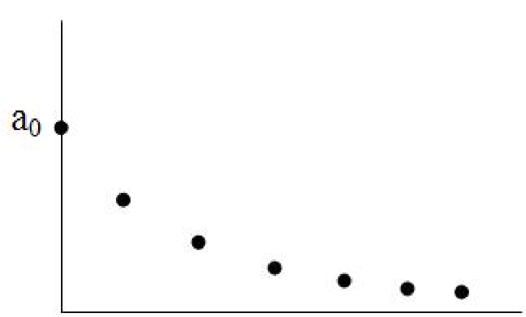
$$\begin{aligned} a_{t+1} &= 3a_t \\ a_1 &= 3a_0 \\ a_2 &= 3^2 a_0 \\ a_3 &= 3^3 a_0 \\ &\vdots \\ a_t &= 3^t a_0 \end{aligned}$$



The population triples every hour. This bacteria will grow without bound.

Eg) $a_{t+1} = \frac{1}{3}a^t$

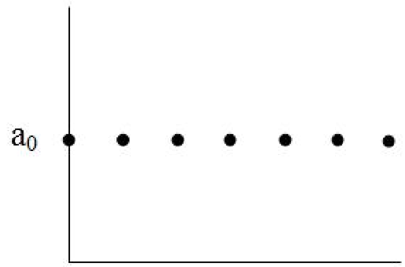
$$\begin{aligned} a_1 &= \frac{1}{3}a_0 \\ a_2 &= \frac{1}{9}a_0 \\ a_3 &= \frac{1}{27}a_0 \\ &\vdots \\ a_t &= \frac{1}{3^t}a_0 = 3^{-t}a_0 \end{aligned}$$



The population reduces by a third every hour and eventually the population dies out.

Eg) $a_{t+1} = a_t$

$$\begin{aligned} a_1 &= a_0 \\ a_2 &= a_0 \\ a_3 &= a_0 \\ &\vdots \\ a_t &= a_0 \end{aligned}$$



The population remains constant and never changes.

General rule: Consider $a_{t+1} = ra_t$. The solution is $a_t = r^t a_0$ and

- If $r > 1$, then the population increases.
- If $r = 1$, then the population remains constant.
- If $r < 1$, then the population decreases.

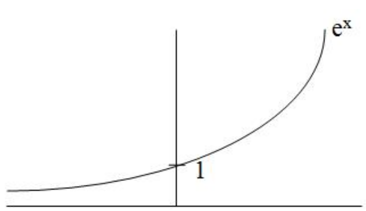
Exponents: How does a function $f(t) = r^t$ behave? If r is positive, this function is called the exponential function with base r.

Laws of exponents:

$$\begin{aligned} r^x r^y &= r^{x+y} & 3^2 \cdot 3^3 &= 3^5 = 243 \\ (r^x)^y &= r^{xy} & (3^2)^3 &= 3^6 = 729 \\ r^{-x} &= \frac{1}{r^x} & 3^{-2} &= \frac{1}{3^2} = \frac{1}{9} \\ \frac{r^x}{r^y} &= r^{x-y} & \frac{3^2}{3^3} &= 3^{2-3} = \frac{1}{3} \\ r^1 &= r & 3^1 &= 3 \\ r^0 &= 1 & 3^0 &= 1 \end{aligned}$$

Eg) $(r^{\frac{1}{2}})^2 = r^1$.

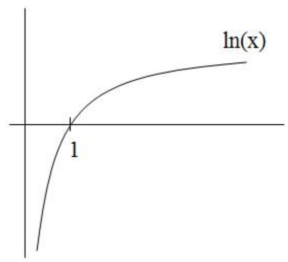
If $f(r) = r^{\frac{1}{2}}$ then $f^{-1}(r) = r^2$, ie, $r^{\frac{1}{2}}$ “undoes” r^2.
But we know this: it’s the square root.
Therefore $r^{\frac{1}{2}} = \sqrt{r}$
In general $r^{\frac{1}{n}} = \sqrt[n]{r}$ since $(r^{\frac{1}{n}})^n = r$.
Special number: e=2.718281828454... (irrational, like π)
$f(x) = e^x$ is called the exponential function.



Laws of exponentials:

$$
\begin{aligned}
e^5e^6 &= e^{5+6} = e^{11} \\
e^5 + e^6 & \quad \text{cannot be simplified} \\
(e^5)^6 &= e^{30} \\
e^{-5} &= \frac{1}{e^5} \\
\frac{e^5}{e^6} &= e^{5-6} = e^{-1} \\
e^0 &= 1
\end{aligned}
$$

To "undo" e^x, we define a new function $\ln x$ ("lon" x aka "logarithm naturelle").
Thus, $\ln(e^x) = x$ and $e^{\ln x} = x$.



Log laws:

$$
\begin{aligned}
\ln(xy) &= \ln x + \ln y \\
\ln(x^p) &= p \ln x \\
\ln\left(\frac{1}{x}\right) &= -\ln(x) \\
\ln\left(\frac{x}{y}\right) &= \ln x - \ln y \\
\ln(e) &= 1 \\
\ln(1) &= 0
\end{aligned}
$$

$$
\begin{aligned}
\ln(5) + \ln(6) &= \ln(5 \cdot 6) \\
&= \ln(30) \\
\ln(5)\ln(6) & \quad \text{cannot be simplified} \\
\ln(5^6) &= 6\ln(5) \\
\ln\left(\frac{5}{6}\right) &= \ln(5) - \ln(6)
\end{aligned}
$$

Question: Why would we want this sort of function?
Answer 1: To solve exponentials, since we need the inverse of e^x.
Question 2: No really, why?
Answer 2: To deal with powers.
Eg) $3^x = 4$ Find x.

We don't have a way of extracting x.
Trick: Take ln of both sides (we do this whenever there's a variable in the power)

$$
\begin{aligned}
\ln(3^x) &= \ln(4) \\
x\ln(3) &= \ln(4) \qquad\qquad (\ln(a^b) = b\ln(a)) \\
x &= \frac{\ln(4)}{\ln(3)} \\
&= 1.26
\end{aligned}
$$

Eg) A bacterial population grows at rate $f(t) = 30e^{0.08t}$ where t is in hours.
a) How many bacteria are there originally?
b) How long does it take for the bacteria to double in size?

c) How long does it take until there are 100 bacteria?

$$
\begin{aligned}
&a)\ f(0) = 30e^0 = 30(1) = 30\\
&b)\ 60 = 30e^{0.08t}\\
&\quad 2 = e^{0.08t}\\
&\ln(2) = \ln e^{0.08t}\\
&\ln(2) = 0.08t \qquad\qquad \text{ln and } e \text{ are inverses.}\\
&\quad t = \frac{\ln(2)}{0.08}\\
&\quad = 8.66 \text{ hours}\\
&c)\ 100 = 30e^{0.08t}\\
&\frac{10}{3} = e^{0.08t}\\
&\ln\left(\frac{10}{3}\right) = \ln e^{0.08t}\\
&\ln\left(\frac{10}{3}\right) = 0.08t\\
&t = \frac{\ln(\frac{10}{3})}{0.08}\\
&= 15 \text{ hours}
\end{aligned}
$$

General rule for variables in the power: you probably need a logarithm.

Eg) A population has initial size 50 and doubling time $t_2 = 9$ hours. Find an expression for the size of the population as a function of time.

$$
\begin{aligned}
f(t) &= 50e^{\alpha t} \qquad\qquad \text{(Note that } f(0) = 50)\\
f(9) &= 100\\
f(9) &= 50e^{\alpha 9}\\
50e^{9\alpha} &= 100\\
e^{9\alpha} &= 2\\
\ln e^{9\alpha} &= \ln(2)\\
9\alpha &= \ln(2)\\
\alpha &= \frac{\ln(2)}{9}\\
\therefore f(t) &= 50e^{\frac{\ln(2)}{9}t}
\end{aligned}
$$

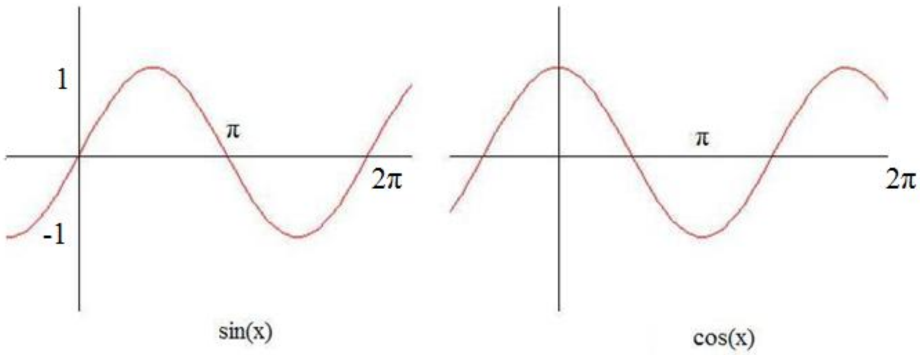
3.3 Trigonometry

In applied math, we use <u>radians</u> rather than degrees, to measure angles.
Since a circle with radius 1 has circumference 2π, we say there are 2π radians in a circle. Thus

$$
\begin{array}{ccc}
2\pi \leftrightarrow 360^\circ & \frac{\pi}{2} \leftrightarrow 90^\circ & \frac{\pi}{6} \leftrightarrow 30^\circ \\
\pi \leftrightarrow 180^\circ & \frac{\pi}{3} \leftrightarrow 60^\circ & \frac{\pi}{4} \leftrightarrow 45^\circ
\end{array}
$$

(Start thinking in radians now. We'll almost never use degrees again.)

The trigonometric functions <u>sine</u> and <u>cosine</u> take angles as inputs and produce numbers between -1 and 1.



Both sine and cosine repeat every 2π radians. This value 2π is called the period of the oscillation.

$$\sin\theta = \sin(\theta + 2\pi) = \sin(\theta + 4\pi) = \sin(\theta + 36\pi)$$
$$\cos\theta = \cos(\theta - 2\pi) = \cos(\theta - 4\pi) = \cos(\theta - 54\pi)$$

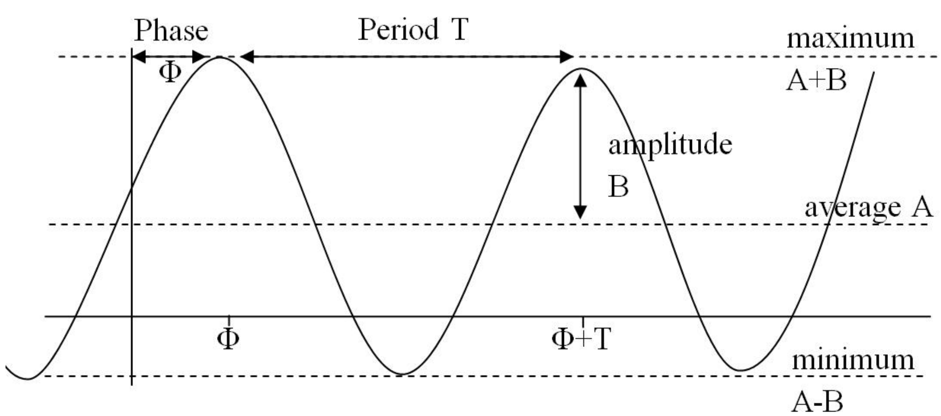
The graphs of sin and cosine are similar except they are shifted from each other by $\frac{\pi}{2}$. Therefore $\cos(\theta) = \sin(\theta + \frac{\pi}{2})$.

Eg)

$$\cos\pi = \sin(\pi + \frac{\pi}{2}) = \sin\frac{3\pi}{2} = -1$$

Since sin and cosine are basically the same except for this shift, we'll use cosine to describe oscillations. A measurement that varies regularly between high and low values is said to oscillate. To describe the oscillation with a cosine function, we need

- the amplitude - the difference between the maximum and the average
- the average - halfway between the maximum and minimum values
- the period - the time between successive peaks
- the phase - the time of the first peak

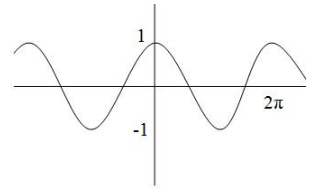


For any curve that oscillates, if we know the amplitude, average, period, and phase, then we can write it as a cosine curve.

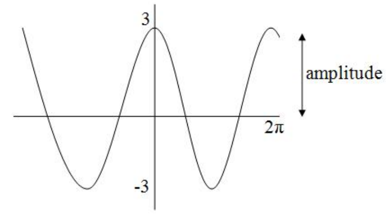
Eg)

Start with $f(t) = \cos t$.

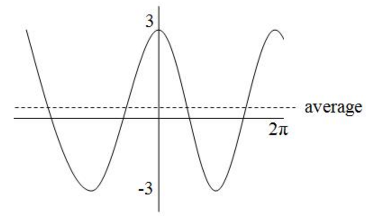
Triple the amplitude: $f(t) = 3\cos t$.



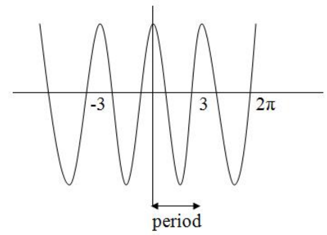
Move the average by adding 1: $f(t) = 1 + 3\cos t$.

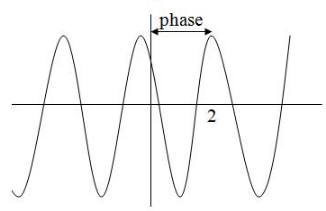


Change the period from 2π to 3 by shifting horizontally (ie time) by a factor of $\frac{2\pi}{3}$: $f(t) = 1 + 3\cos\left(\frac{2\pi t}{3}\right)$



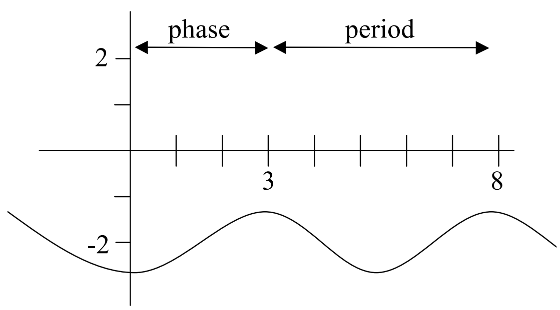
Shift the curve horizontally so the first peak is at 2 (that is, shift t by 2) by subtracting 2 from t: $f(t) = 1 + 3\cos\left(\frac{2\pi}{3}(t-2)\right)$





In general, the function $f(t) = A + B\cos(\frac{2t}{T}(t-\phi))$ has amplitude B, average A, period T and phase ϕ.

Eg) Plot $f(t) = -2 + 0.5\cos\left(\frac{2\pi}{5}(t-3)\right)$



4 Derivatives

A basic rule of life: everything changes. This is especially true in biology.

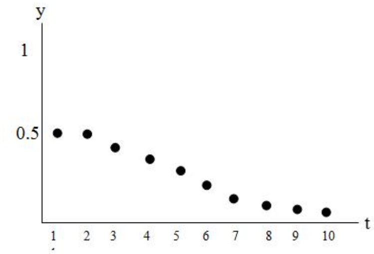
Eg) The time-course of a pandemic is described by $y(t) = t2^{-t}$ where t is the time in days and y is the proportion of the population infected. How can we fully understand the dynamics of the outbreak?

1) Plot points.

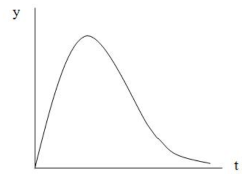
t	1	2	3	4	5	6	7	8	9	10
y	0.5	0.5	0.375	0.25	0.15625	0.09375	0.0546875	0.03125	0.017578	0.009765

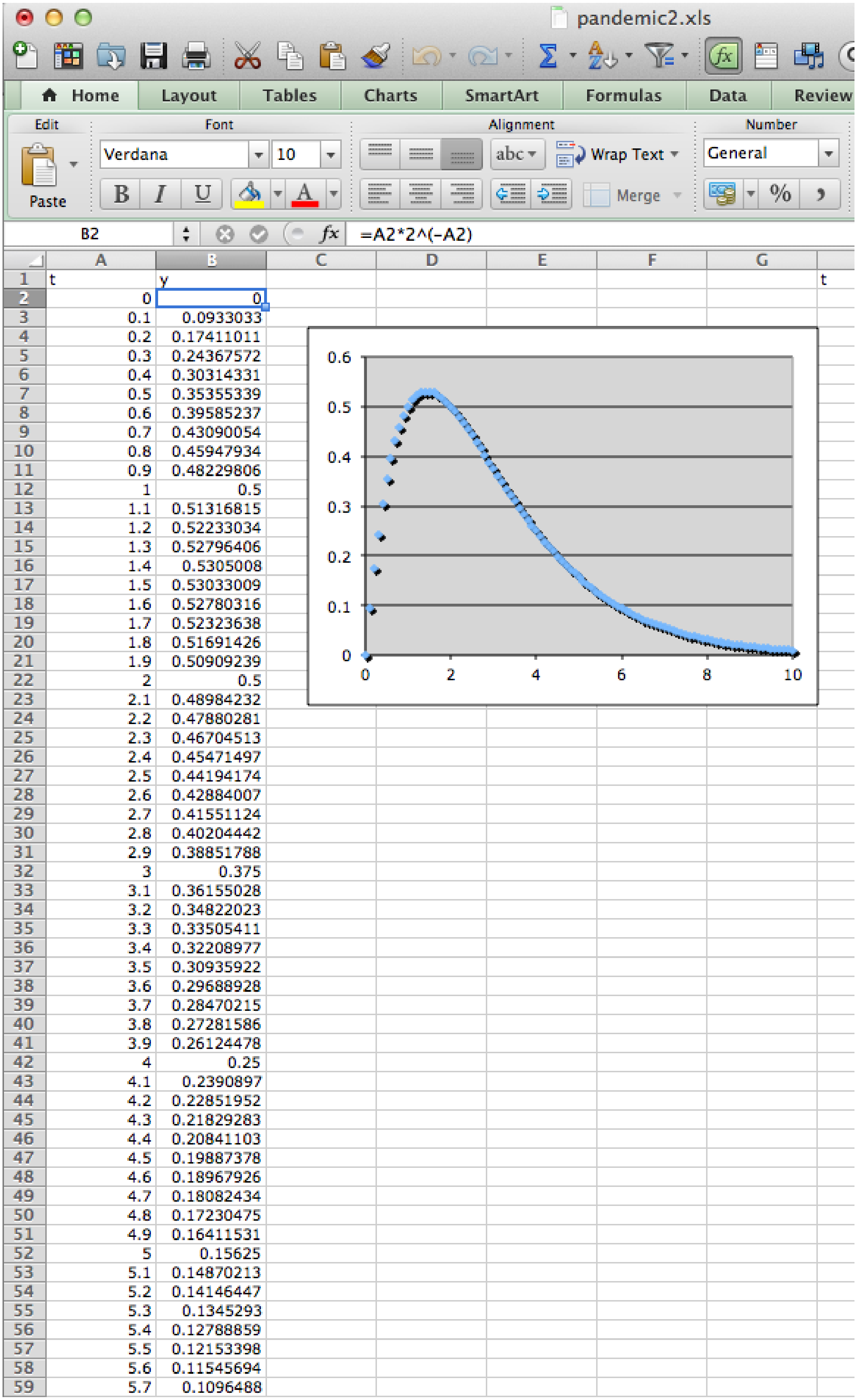
Questions:

1) What happens eventually?
2) When is the disease at its worst?
3) How bad is the disease at its worst?
4) What happens at the beginning of the epidemic?



We can answer the first question immediately:
The disease dies out.
How can we answer the other questions?
One possibility is to plot more points.





	A	B
1	t	y
2	0	0
3	0.1	0.0933033
4	0.2	0.17411011
5	0.3	0.24367572
6	0.4	0.30314331
7	0.5	0.35355339
8	0.6	0.39585237
9	0.7	0.43090054
10	0.8	0.45947934
11	0.9	0.48229806
12	1	0.5
13	1.1	0.51316815
14	1.2	0.52233034
15	1.3	0.52796406
16	1.4	0.5305008
17	1.5	0.53033009
18	1.6	0.52780316
19	1.7	0.52323638
20	1.8	0.51691426
21	1.9	0.50909239
22	2	0.5
23	2.1	0.48984232
24	2.2	0.47880281
25	2.3	0.46704513
26	2.4	0.45471497
27	2.5	0.44194174
28	2.6	0.42884007
29	2.7	0.41551124
30	2.8	0.40204442
31	2.9	0.38851788
32	3	0.375
33	3.1	0.36155028
34	3.2	0.34822023
35	3.3	0.33505411
36	3.4	0.32208977
37	3.5	0.30935922
38	3.6	0.29688928
39	3.7	0.28470215
40	3.8	0.27281586
41	3.9	0.26124478
42	4	0.25
43	4.1	0.2390897
44	4.2	0.22851952
45	4.3	0.21829283
46	4.4	0.20841103
47	4.5	0.19887378
48	4.6	0.18967926
49	4.7	0.18082434
50	4.8	0.17230475
51	4.9	0.16411531
52	5	0.15625
53	5.1	0.14870213
54	5.2	0.14146447
55	5.3	0.1345293
56	5.4	0.12788859
57	5.5	0.12153398
58	5.6	0.11545694
59	5.7	0.1096488

This mostly give us our answers. The disease is worst after around 1.4 days, infecting 53% of the population and at the beginning there is an initial rise.
None of these were obvious from the plot that we did by hand. And plotting 101 points is very labour intensive. Also, we didn't have an exact answers for questions 2 or 3.
(Secret knowledge: the answer to question 2 is $t = \frac{1}{\ln 2} = 1.442695$ and the answer to question 3 is $y(1.442695) = 0.53073785$. Now how did we get that?)
Knowing the exact nature of the outbreak, including when it peaks, has implications for

- quarantine
- public information
- hospitals
- business
- researchers

Only mathematics can tell us precisely.
Question: What are some examples of systems in biology where change is important?

Let's look at the way our disease changes every half day.

$$\Delta y = y(t) - y(t - 0.5)$$
$$\frac{\Delta y}{\Delta t} = \frac{y(t) - y(t - 0.5)}{0.5}$$

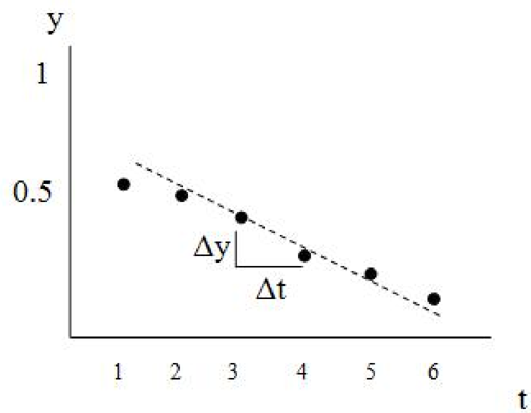
(Remember: "Δ" means "change")

H	I	J	K	L
t	y	Change in y	Av change in y	
0	0			
0.5	0.35355339	0.35355339	0.70710678	
1	0.5	0.14644661	0.29289322	
1.5	0.53033009	0.03033009	0.06066017	
2	0.5	-0.0303301	-0.0606602	
2.5	0.44194174	-0.0580583	-0.1161165	
3	0.375	-0.0669417	-0.1338835	
3.5	0.30935922	-0.0656408	-0.1312816	
4	0.25	-0.0593592	-0.1187184	
4.5	0.19887378	-0.0511262	-0.1022524	
5	0.15625	-0.0426238	-0.0852476	
5.5	0.12153398	-0.034716	-0.069432	
6	0.09375	-0.027784	-0.055568	
6.5	0.07181553	-0.0219345	-0.0438689	
7	0.0546875	-0.017128	-0.0342561	
7.5	0.04143204	-0.0132555	-0.0265109	
8	0.03125	-0.010182	-0.0203641	
8.5	0.02347815	-0.0077718	-0.0155437	
9	0.01757813	-0.0059	-0.0118001	
9.5	0.01312015	-0.004458	-0.008916	
10	0.00976563	-0.0033545	-0.006709	

This tells us that

- The most change happens at the beginning (0.707)
- The outbreak starts off increasing, but then decreases (+ vs −)
- The average rate of change decreases, then increases, then decreases in magnitude.
- This change occurs around $t = 3$, that is, the disease eradication starts slowing down.

Definition 4.1. *A line connecting two points on a graph is called a* <u>*secant line*</u>*. We often pick one of the points for further study, called a* <u>*base point*</u>*.*



Eg) If $(4, 0.25)$ is our base point, then the average rate of change between $t = 4$ and $t = 5$ is

$$\frac{\Delta y}{\Delta t} = \frac{0.15625 - 0.25}{5 - 4} = -0.09375$$

This is the slope of the secant line connecting these data points.
In general, if $(x_0, f(x_0))$ is a base point of a function $f(x)$ and a second point is $(x_1, f(x_1))$ then the slope is

$$m = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

The equation of the secant line is thus $y = f(x_0) + m(x - x_0)$.

Eg) The equation of the secant line above is

$$\begin{aligned} y &= 0.25 + (-0.09375)(x - 4) \\ &= 0.25 - 0.09375x + 0.375 \\ &= -0.09375x + 0.625 \end{aligned}$$

Eg) Find the equation of the secant line at $t = 2$.
The base point is $(2, 0.5)$ and the second point is $(3, 0.375)$.

$$\begin{aligned} m &= \frac{0.375 - 0.5}{3 - 2} = -0.125 \\ y &= 0.5 - 0.125(x - 2) \\ &= 0.5 - 0.125x + 0.25 \\ &= -0.125x + 0.75 \end{aligned}$$

Therefore different secant lines generally have different equations.
How can we find the exact rate of change?

Eg) Find the slope at $t = 1$ using second points of $t = 2, 1.5, 1.1$. What value does the slope approach?

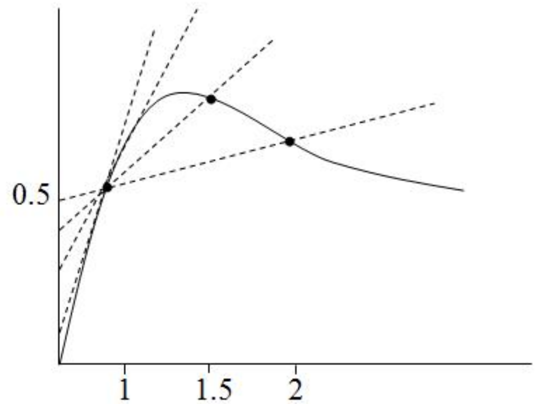
$$\begin{aligned} y(1) &= 0.5 \qquad y(2) = 0.5 \\ m &= \frac{0.5 - 0.5}{2 - 1} = 0 \\ y(1.5) &= 0.53033009 \\ m &= \frac{0.53033009 - 0.5}{1.5 - 1} = 0.06066018 \\ y(1.1) &= 0.51316815 \\ m &= \frac{0.51316815 - 0.5}{1.1 - 1} = 0.316815 \end{aligned}$$

Δt	$1+\Delta t$	$y(1+\Delta t)$	Δy	$\frac{\Delta y}{\Delta t}$
1	2	0.5	0	0
0.5	1.5	0.53033009	0.3033009	0.06066018
0.1	1.1	0.51316815	0.01316815	0.1316815
0.01	1.01	0.50151171	0.00151171	0.151171
0.001	1.001	0.5001532	0.0001532	0.1532004
0.0001	1.0001	0.50001534	0.00001534	0.15340376

As Δt becomes small, there is less and less time for anything to happen and the change Δy also becomes small. However, the average rate of change does not become tiny and approaches a value of about 0.153.

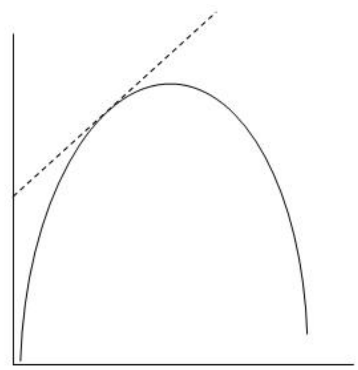
Question: What happens to the secant line as Δt becomes smaller?

Answer: Eventually it approaches the tangent line at $t = 1$.



The tangent line touches the curve at the single point $t = 1$ but doesn't cross the curve nearby, but just lies along it.

The closer that Δy gets to zero, the more accurate the secant line is...but of course Δt can't be zero, or else we'd be dividing by zero.



Question: How can we get a handle on this idea of getting closer and closer?
Answer: Limits: $\lim_{\Delta t \to 0} \frac{\Delta y}{\Delta t}$ is what we'd get if we made Δt as small as possible.

Eg)

$$\lim_{\Delta t \to 0} \frac{\Delta y}{\Delta t} = 0.153 \text{ at } t = 1$$

Definition 4.2. *The derivative of a function f is the instantaneous rate of change of a function $f(x)$ at $x = x_0$ and is computed as*

$$\lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$

The derivative measures how quickly a measurement is changing at a particular instant.
Notation: For historical reasons, there are two different notations for the derivative.

$$\text{Derivative of } f \text{ at } x_0 = \left.\frac{df}{dx}\right|_{x_0} = f'(x_0)$$

The "d"s represent limiting versions of the "Δ"s

Definition 4.3. *The slope of the graph of a function is equal to the slope of the tangent line to the graph, which is equal to the derivative of the function.*

The derivative is the single most important thing you'll learn in mathematics.

5 Limits

Basic rule: if nothing illegal occurs, then $\lim_{x\to a} f(x) = f(a)$.

Eg) $\lim_{x\to 2} x^2+1 = 2^2+1 = 5$

Eg) $\lim_{x\to -3} 2x+4 = 2(-3)+4 = -2$

Eg) $\lim_{x\to 5} 7 = 7$

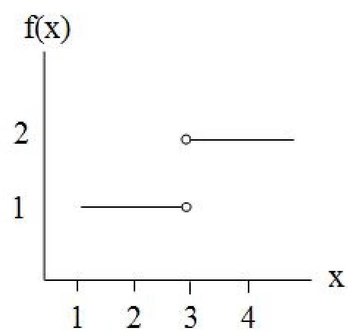
Eg) $\lim_{x\to 1} \frac{x+1}{x-2} = \frac{1+1}{1-2} = -2$

Question: What can go wrong?
Answer:

- Dividing by zero
- The function doesn't exist at a
- The function isn't consistant at a

Eg) $f(x) = \begin{cases} 2 & x>3 \\ 1 & x<3 \end{cases}$ This function doesn't exist at $x=3$.

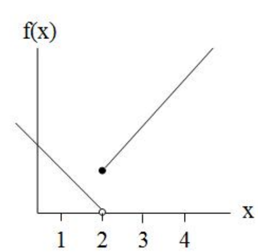
$$\begin{aligned} \lim_{x\to 2.5} f(x) &= 1 \\ \lim_{x\to 3.5} f(x) &= 2 \\ \lim_{x\to 3} f(x) &\text{ doesn't exist} \\ \text{But limit from the left: } \lim_{x\to 3^-} f(x) &= 1 \\ \text{limit from the right: } \lim_{x\to 3^+} f(x) &= 2 \end{aligned}$$



If the left and right limits don't match, the limit doesn't exist.

Eg) $g(x) = \begin{cases} 1-x & x<1 \\ x & x\geq 1 \end{cases}$

$$\begin{aligned} &\lim_{x\to 1^-} g(x) = 0 \\ &\lim_{x\to 1^+} g(x) = 1 \\ &\therefore \lim_{x\to 1} g(x) \text{ Does Not Exist (DNE)} \end{aligned}$$



But the limit exists at all other points.

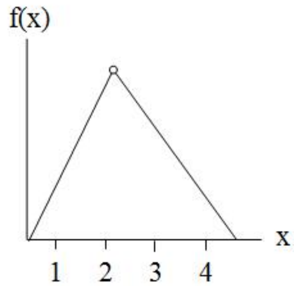
Eg) $h(x) = \begin{cases} 2x & x<2 \\ 6-x & x>2 \end{cases}$

$\lim_{x \to 2^-} h(x) = 4$

$\lim_{x \to 2^+} h(x) = 4$

$\therefore \lim_{x \to 2}$ exists and is equal to 4

even though the function wasn't defined at $x = 2$!



Eg) $j(x) = \frac{x-1}{x+2}$ isn't defined at $x = -2$ (we can't divide by zero)

$$\lim_{x \to -2} j(x) = \frac{-2-1}{-2+2} = \frac{-3}{0} = ?$$

Eg) $k(x) = \frac{x^2+x}{x}$ isn't defined at $x = 0$.

$$\lim_{x \to 0} k(x) = \frac{0^2+0}{0}$$

$$\text{But } k(x) = \frac{x(x+1)}{x} = x+1$$

$$\lim_{x \to 0} k(x) = 0+1 = 1$$

5.1 Rules for Limits

Suppose $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ exist. Then,

$$a) \ \lim_{x \to a}[f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$$

$$b) \ \lim_{x \to a}[f(x) \cdot g(x)] = \left[\lim_{x \to a} f(x)\right]\left[\lim_{x \to a} g(x)\right]$$

$$c) \ \lim_{x \to a}[cf(x)] = c \lim_{x \to a} f(x)$$

$$d) \ \lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} \quad \text{if } \lim_{x \to a} g(x) \neq 0$$

Eg)

$$\lim_{x \to 2} \frac{x^2+3x-4}{x-5} = \frac{2^2+3(2)-4}{2-5} = \frac{6}{-3} = -2$$

5.2 Infinite Limits

How can we find the limit $\lim_{x \to 0} \frac{1}{x^2}$?

Let's try some points:

x	1	0.1	0.01
$\frac{1}{x^2}$	1	100	10000

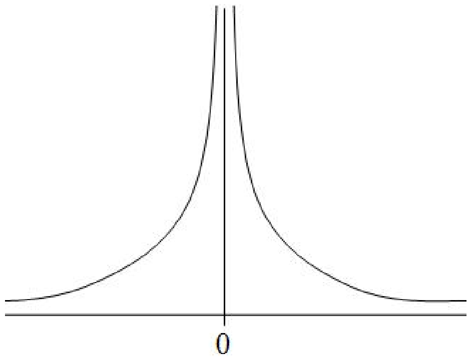
This limit doesn't quite "exist" but at least we have an idea of what it's doing.

So it seems as though $\lim_{x \to 0^+} \frac{1}{x^2} = \infty$.

What about $\lim_{x \to 0^-} \frac{1}{x^2}$?

x	-1	-0.1	-0.01
$\frac{1}{x^2}$	1	100	1000

Therefore $\lim_{x \to 0^-} = \infty$ and $\lim_{x \to 0^+} = \infty$.



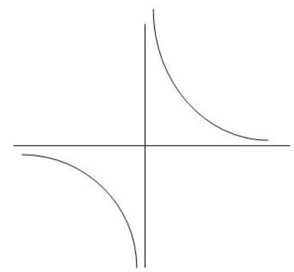
Eg) $\lim_{x \to 0} \frac{1}{x}$

x	$\frac{1}{x}$
1	1
0.1	10
0.01	100
0.001	1000

$\therefore \lim_{x \to 0^+} \frac{1}{x} = \infty$

x	$\frac{1}{x}$
-1	-1
-0.1	-10
-0.01	-100
-0.001	-1000

$\therefore \lim_{x \to 0^-} \frac{1}{x} = -\infty$

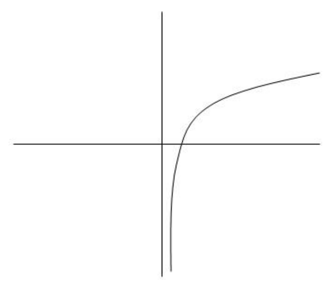


So $\lim_{x \to 0} \frac{1}{x}$ doesn't exist either, but we still know what it's doing.

Eg) Find $\lim_{x \to 0} \ln x$

x	$\ln x$
1	0
0.1	-2.3
0.01	-4.6
0.001	-6.9
0.000001	-13.8
0.000000001	-20.7

$\lim_{x \to 0^+} \ln x = -\infty$



This limit is only defined on one side.

5.3 Continuity

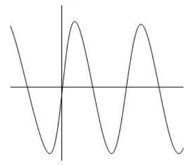
Definition 5.1. *A function is <u>continuous</u> at a point a if $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$. If not, we say the function is <u>discontinuous</u> at a.*

Therefore functions are continuous if nothing goes wrong.

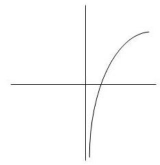
Continuity is defined at points. Functions may be continuous at some (or most) points and be discontinuous at others.

Definition 5.2. *A function is <u>continuous</u> if it is continuous at every point in its domain.*

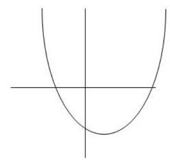
Eg) $\sin(x)$ is continuous for all x.

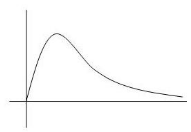


Eg) $\ln(x)$ is continuous for all $x > 0$.



Eg) $x^2 - 3x - 2$ is continuous for all x.





Eg) $\frac{3x}{e^x}$ is continuous for all x since $e^x \neq 0$.

5.3.1 Rules for Continuity

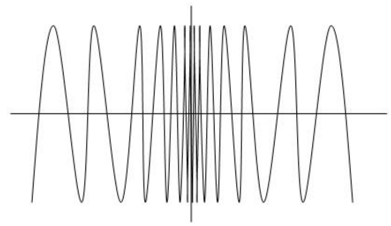
Suppose $f(x)$ and $g(x)$ are continuous at a.

$a)$ $f(x) \pm g(x)$ is continuous at a.
$b)$ $f(x) \cdot g(x)$ is continuous at a.
$c)$ $\dfrac{f(x)}{g(x)}$ is continuous at a unless $g(x) = 0$
$d)$ If $h(x)$ is continuous at $x = f(a)$ then $h(f(x))$ is continuous at a.

Eg) $1 - 2x + e^x$ is continuous everywhere.
Eg) $(1 - 2x)e^x$ is continuous everywhere.
Eg) $\frac{e^x}{1-2x}$ is continuous everywhere except when $1 - 2x = 0 \rightarrow x = \frac{1}{2}$.
Eg) e^{1-2x} is continuous everywhere (composition of functions).
Discontinuity trouble spots are

- Dividing by 0
- $\log(0)$
- Places where the definition of the function changes

Eg) $\sin\left(\frac{1}{x}\right)$



This function oscillates faster and faster as $x \to 0$.

The composition involves dividing by zero.

Eg) $\cot x = \frac{\cos x}{\sin x}$

This function divides by zero when $\sin x = 0 \to x = ..., -3\pi, -2\pi, -\pi, 0, \pi, 2\pi, 3\pi, ...$
Eg) $(2-x)^{-4} = \frac{1}{(2-x)^4}$

This function divides by zero if $(2-x)^4 = 0 \to 2 - x = 0 \to x = 2$. Therefore it's not continuous at $x = 2$.

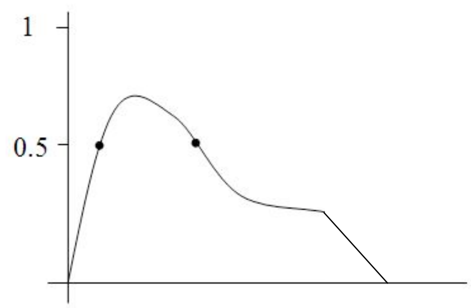
Eg) Remember our pandemic example where $y(t) = t \cdot 2^{-t}$. Suppose we find a cure that satisfies $y(t) = \frac{5-t}{4}$ and we initiate this cure on day 4 of the outbreak. This creates a new function consisting of the disease for $0 \leq t < 4$ and the cure for $t \geq 4$.
1) Is the new function continuous?
2) What happens to the disease now?
3) What would happen if we waited until day 5?
4) What would happen if we appplied the cure on day 3?

$$y(t) = \begin{cases} t \cdot 2^{-t} & 0 \leq t < 4 \\ \frac{5-t}{4} & t \geq 4 \end{cases}$$

$$\lim_{t \to 4^-} y(t) = 4 \cdot 2^{-4} = 0.25$$

$$\lim_{t \to 4^+} y(t) = \frac{5-4}{4} = 0.25$$

$\therefore$ the function is continuous at $t = 4$.

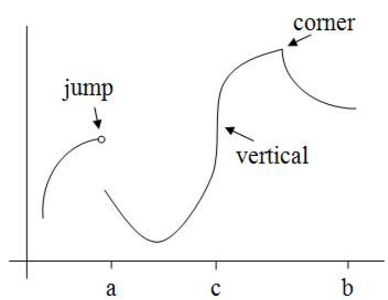


Homework: Answer questions 3 and 4.

6 Computing Derivatives

A function is not differentiable if it

- Has a jump
- Has a corner
- Is vertical



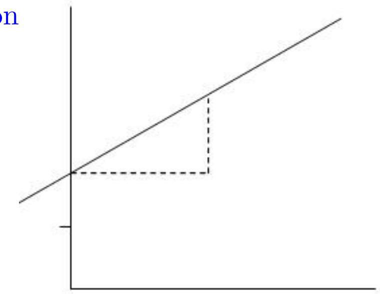
Otherwise, most functions are differentiable.

6.1 Linear functions

Eg) $y = 3x + 2$. Find the derivative using the definition.

$$\begin{aligned}\frac{dy}{dx} &= \lim_{\Delta x\to 0}\frac{y(x+\Delta x)-y(x)}{\Delta x} \\ &= \lim_{\Delta x\to 0}\frac{[3(x+\Delta x)+2]-[3x+2]}{\Delta x} \\ &= \lim_{\Delta x\to 0}\frac{3x+3\Delta x+2-3x-2}{\Delta x} \\ &= \lim_{\Delta x\to 0}\frac{3\Delta x}{\Delta x} \\ &= \lim_{\Delta x\to 0} 3 = 3\end{aligned}$$

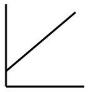
note: we won't need this form soon



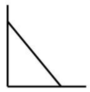
But we could have guessed this already, since $y = 3x + 2$ has slope 3. Therefore, the derivative of a linear function $f(x) = mx + b$ is $f'(x) = m$.

Eg) Find the derivative of

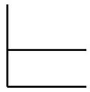
$y = 2 + 5x$ $\qquad f'(x) = 5$



$y = -3 - 17x$ $\qquad f'(x) = -17$



$y = 4$ $\qquad f'(x) = 0$

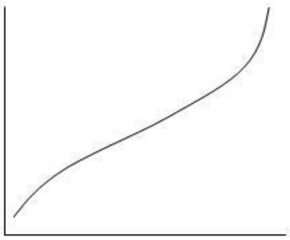


This is true in general: If the derivative is

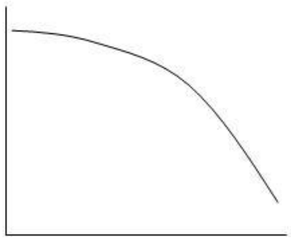
- positive, then the function is increasing.
- negative, then the function is decreasing.
- zero, then the function is neither increasing nor decreasing.

These last points are very important.

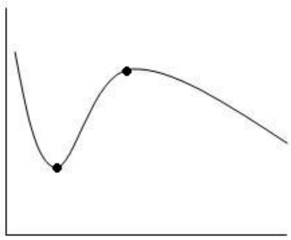
Definition 6.1. *If $f'(x) = 0$ or if f' is not defined at x (in the domain), then we say that f has a critical point at x.*



Positive derivative



Negative Derivative



Two Critical Points

6.2 Derivatives of sums, powers and polynomials

The derivative of a sum is the sum of the derivatives. Thus

$$\frac{d}{dx}[f(x)+g(x)] = \frac{df}{dx} + \frac{dg}{dx}$$

Eg)

$$\begin{aligned} f(x) &= 3x - 4 \qquad & g(x) &= 5 - 7x \\ f'(x) &= 3 & g'(x) &= -7 \end{aligned}$$

$$\frac{d}{dx}[f+g] = 3 - 7 = -4$$

Eg)

$$\begin{aligned} f(x) &= 37x + 200 \qquad\qquad g(x) = -10{,}000 \\ f'(x) &= 37 \\ g'(x) &= 0 \\ \frac{d}{dx}[f+g] &= 37 + 0 = 37 \end{aligned}$$

6.3 Derivatives of power functions

We already know the derivative of x is 1. What about x^2, x^3, x^4?

$$\begin{aligned} \frac{d}{dx}(x^2) &= \lim_{\Delta x \to 0} \frac{(x+\Delta x)^2 - x^2}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{x^2 + 2x\Delta x + (\Delta x)^2 - x^2}{\Delta x} \\ &= \lim_{\Delta x \to 0} 2x + \Delta x \\ &= 2x \\ \frac{d}{dx}(x^3) &= \lim_{\Delta x \to 0} \frac{(x+\Delta x)^3 - x^3}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{(x+\Delta x)(x^2 + 2x\Delta x + \Delta x^2) - x^3}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{x^3 + 2x^2\Delta x + x\Delta x^2 + x^2\Delta x + 2x\Delta x^2 + \Delta x^3 - x^3}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{3x^2\Delta x + 3x\Delta x^2 + \Delta x^3}{\Delta x} \\ &= 3x^2 \end{aligned}$$

Homework: Show that $\frac{d}{dx}(x^4) = 4x^3$.

Therefore, we have

$f(x)$	x^1	x^2	x^3	x^4
$f'(x)$	1	$2x$	$3x^2$	$4x^3$

The general formula is $\frac{d}{dx}x^n = nx^{n-1}$.

Eg)

$$\begin{aligned} \frac{d}{dx}(x^{57}) &= 57x^{56} \\ \frac{d}{dx}(x^{-3}) &= -3x^{-4} \\ \frac{d}{dx}\left(\frac{1}{\sqrt{x}}\right) &= \frac{d}{dx}(x^{-\frac{1}{2}}) = -\frac{1}{2}x^{-\frac{3}{2}} \\ \frac{d}{dx}(1) &= \frac{d}{dx}(x^0) = 0x^{-1} = 0 \end{aligned}$$

The derivative of a constant product is the constant times the derivative.

$$\frac{d}{dx}[cf(x)] = c\frac{df}{dx}$$

That is, constants come "outside" of derivatives.

Eg)

$$\frac{d}{dx}\left(5 \cdot x^7\right) = 5(7)x^6 = 35x^6$$

6.4 Derivatives of polynomials

A polynomial is a function with sums and constant products of power functions.
For example, $p(x) = 3x^4 - 2x^2 + 1.2x - 7$.
The degree of a polynomial is the highest power.
For example, degree of $p(x) = 4$.
Therefore, we can write the sum and constant product rules for derivatives to differentiate polynomials.

Eg) $p(x) = 3x^4 - 2x^2 + 1.2x - 7$. Find $p'(x)$.

$$\begin{aligned}
p'(x) &= \frac{d}{dx}[3x^4 - 2x^2 + 1.2x - 7] \\
&= \frac{d}{dx}(3x^4) - \frac{d}{dx}(2x^2) + \frac{d}{dx}(1.2x) - \frac{d}{dx}(7) \\
&= 3\frac{d}{dx}x^4 - 2\frac{d}{dx}x^2 + 1.2\frac{d}{dx}x - 7\frac{d}{dx}(1) \\
&= 3(4x^3) - 2(2x) + 1.2(1) - 0 \\
&= 12x^3 - 4x + 1.2
\end{aligned}$$

Eg)

$$\begin{aligned}
q(x) &= x^5 + 2x^3 - \frac{1}{12}x^2 - 394 \\
q'(x) &= 5x^4 + 6x^2 - \frac{1}{6}x
\end{aligned}$$

General rule: The derivative of any constant is zero.

6.5 The product rule

The derivative of a product is not the product of the derivatives.

Eg)

$$\begin{aligned}
h(x) &= x^2 = x \cdot x \\
h'(x) &= 2x \neq 1 \cdot 1
\end{aligned}$$

Suppose $p(x) = f(x)g(x)$. Then

$$p'(x) = f(x)g'(x) + f'(x)g(x)$$

OR

$$\frac{dp}{dx} = f(x)\frac{dg}{dx} + \frac{df}{dx}g(x)$$

Eg)

$$\begin{aligned}
h(x) &= x^2 = x \cdot x \\
h'(x) &= x(1) + (1)x = 2x
\end{aligned}$$

Eg)

$$\begin{aligned}
p(x) &= (2x^2 + 3x + 1)(5x^3 - x^2 + 4x - 8) \\
p'(x) &= (2x^2 + 3x + 1)(15x^2 - 2x + 4) + (4x + 3)(5x^3 - x^2 + 4x - 8)
\end{aligned}$$

Eg)

$$
\begin{aligned}
w(x) &= (x^2+13x+11)\cdot x^{-1} = \frac{u}{v} \\
w'(x) &= (x^2+13x+11)(-x^{-2}) + (2x+13)x^{-1} \\
&= \frac{-(x^2+13x+11)}{x^2} + \frac{x(2x+13)}{x^2} \\
&= \frac{-(x^2+13x+11)+x(2x+13)}{x^2} \\
&= \frac{-uv'+vu'}{v^2} = \frac{vu'-uv'}{v^2}
\end{aligned}
$$

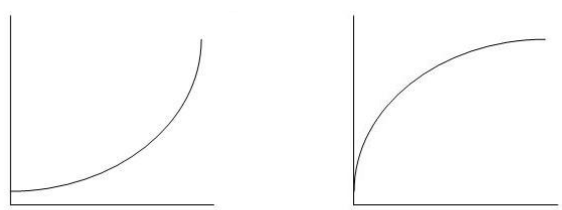
This is the quotient rule.

Eg) $w = \dfrac{x^3-2x^2}{1+x}$. Find $w'(x)$.

$$
\begin{aligned}
w' &= \frac{(1+x)(3x^2-4x)-(x^3-2x^2)(1)}{(1+x)^2} \\
&= \frac{3x^2-4x+3x^3-4x^2-x^3+2x^2}{(1+x)^2} \\
&= \frac{2x^3+x^2-4x}{(1+x)^2}
\end{aligned}
$$

6.6 The Second Derivative

Consider these two graphs:



Both functions increase, but the rate of increase (aka the slope, aka the derivative) in the first gets steeper and steeper, whereas in the second it gets smaller and smaller.
Remember that if a function increases then its derivative is positive and if a function is decreasing then its derivative if negative.
We can apply this idea to the derivative itself. In the first function, the derivative is increasing, so the derivative of the derivative (called the second derivative) is positive.
In the other function, the second derivative is negative.

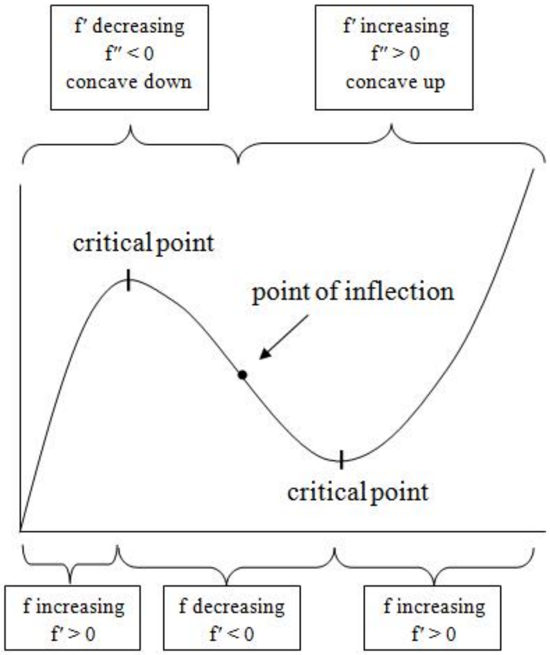
- Second derivative: $f''(x)$ or $\frac{d^2f}{dx^2}$
- Third derivative: $f'''(x)$ or $\frac{d^3f}{dx^3}$
- nth derivative: $f^{(n)}(x)$ or $\frac{d^nf}{dx^n}$

If $f'' < 0$ a graph is said to be concave down.

If $f'' > 0$ a graph is said to be concave up.

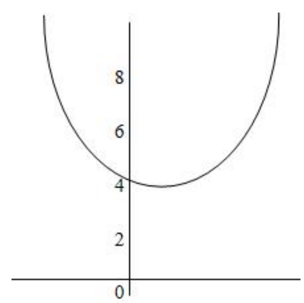
If $f''(x_0) = 0$ then x_0 is a point of inflection.

Thus, with critical points ($f' = 0$), limits, and concavity, we can graph most functions.



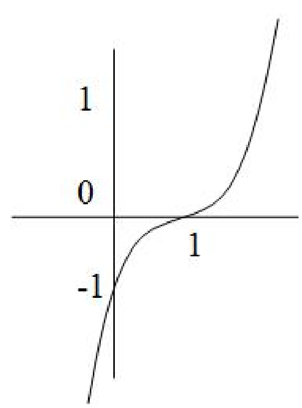
Eg) Graph the function $y = 2x^2 - 3x + 4$.

$$\begin{aligned} y' &= 4x - 3 \\ y' &= 0 \rightarrow x = \frac{3}{4} \text{ is a critical point} \\ y'' &= 4 > 0 \quad \therefore \text{ concave up} \\ y(0) &= 4 \\ y\left(\frac{3}{4}\right) &= 2.875 \end{aligned}$$



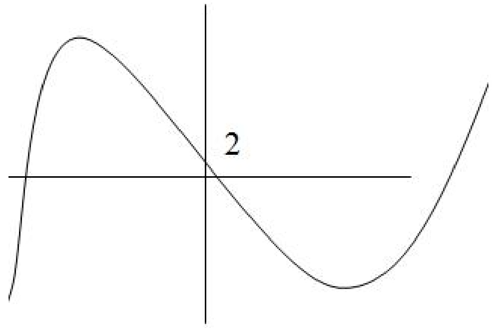
Eg) Graph the function $y = x^3 - 3x^2 + 3x - 1$.

$$\begin{aligned} y(0) &= 1 \\ y' &= 3x^2 - 6x + 3 \\ &= 3(x^2 - 2x + 1) \\ &= 3(x-1)^2 \\ y' &= 0 \rightarrow x = 1 \\ y(1) &= 0 \\ y'' &= 6x - 6 = 0 \rightarrow x = 1 \\ y''(0) &= -6 \quad \rightarrow \text{ concave down} \\ y''(2) &= 6 \quad \rightarrow \text{ concave up} \\ y''(1) &= 0 \quad \rightarrow \text{ point of inflection} \end{aligned}$$



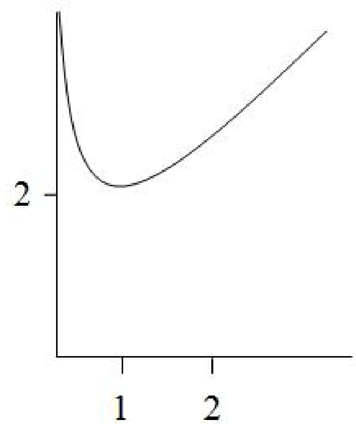
Eg) Graph the function $y = 2x^3 - 3x^2 - 36x + 2$.

$$\begin{aligned}
y' &= 6x^2 - 6x - 36 = 0 \\
&= 6(x^2 - x - 6) = 0 \\
&= 6(x+2)(x-3) = 0 \\
&\rightarrow x = -2, 3 \\
y(-2) &= 46 \\
y(3) &= -79 \\
y(0) &= 2 \\
y'' &= 12x - 6 = 0 \\
x &= \frac{1}{2} \text{ is a point of inflection} \\
y''(-2) &= -30 \qquad \rightarrow \text{ concave down} \\
y''(3) &= 30 \qquad \rightarrow \text{ concave up}
\end{aligned}$$



Eg) Graph the function $y = \sqrt{x} + \frac{1}{\sqrt{x}}$.

$$\begin{aligned}
&\text{Domain } x \geq 0, \qquad x \neq 0 \rightarrow x > 0 \\
y &= x^{\frac{1}{2}} + x^{-\frac{1}{2}} \\
y' &= \frac{1}{2}x^{-\frac{1}{2}} - \frac{1}{2}x^{-\frac{3}{2}} \\
&= \frac{x-1}{2x^{\frac{3}{2}}} \qquad x = 1 \text{ is a critical point} \\
y'' &= -\frac{1}{4}x^{-\frac{3}{2}} + \frac{3}{4}x^{-\frac{5}{2}} \\
&= \frac{-x+3}{4x^{5/2}} \\
y''(1) &= \frac{2}{4} \qquad \rightarrow \text{ concave up} \\
y(1) &= 1 + \frac{1}{1} = 2 \\
\lim_{x \to 0^+} y &= 0 + \frac{1}{0^+} = \infty
\end{aligned}$$



6.7 Derivatives of exponential and logarithmic functions

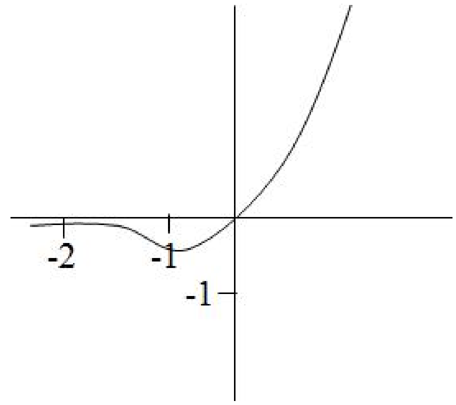
Question: Why is e so special?
Answer: e^x is the only function (other than zero) which is its own derivative.

$$\frac{d}{dx}(e^x) = e^x$$

What about $\ln x$? $\frac{d}{dx}(\ln x) = \frac{1}{x}$

Eg) Graph the function $y = xe^x$

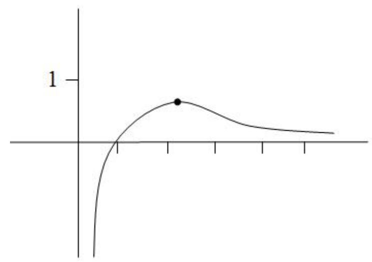
$$
\begin{aligned}
y(0) &= 0 \\
y' &= e^x + xe^x \text{ (product rule)} \\
y' &= 0 \to e^x(1+x) = 0 \to x = -1 \\
y(-1) &= -e^{-1} = -0.367 \\
y'' &= e^x + e^x + xe^x \\
&= 2e^x + xe^x \\
y'' &= 0 \to e^x(2+x) = 0 \to x = -2 \\
y(-2) &= -2e^{-2} = -0.27 \\
y''(-1) &= e^{-1}(1) > 0 \text{ concave up}
\end{aligned}
$$



Eg) Graph the function $y = \frac{\ln x}{x}$

Domain: $x > 0$

$$
\begin{aligned}
y' &= \frac{x\left(\frac{1}{x}\right) - \ln x}{x^2} \text{ (quotient rule)} \\
&= \frac{1 - \ln x}{x^2} \\
y' &= 0 \to \ln x = 1 \to x = e \\
y(e) &= \frac{1}{e} = 0.367 \\
y'' &= \frac{x^2\left(-\frac{1}{x}\right) - (1 - \ln x)2x}{x^4} \text{ (quotient rule)} \\
&= \frac{-x - 2x + 2x\ln x}{x^4} \\
&= \frac{-3 + 2\ln x}{x^3} \\
y'' &= 0 \to \ln x = \frac{3}{2} \to x = 4.48 \\
y''(e) &= \frac{-3+2}{e^3} < 0 \\
y(1) &= 0
\end{aligned}
$$



6.8 The Chain Rule

Suppose $f(x) = f(g(x))$ is a composition of two functions, then

$$
\frac{df}{dx} = \frac{df}{dg}\frac{dg}{dx}
$$

$$
f'(x) = f'(g(x))g'(x)
$$

outer derivative derivative from within

Eg) $y = e^{3x}$. Find the derivative.

$$
\begin{aligned}
f(g(x)) = y = e^{3x} &\rightarrow g = 3x \text{ and } f = e^{g} \\
\frac{dy}{dx} &= \frac{df}{dg}\frac{dg}{dx} \\
&= e^{g} \cdot 3 \\
&= e^{3x} \cdot 3 \\
&= 3e^{3x}
\end{aligned}
$$

Eg) $y = e^{x^2}$. Find the derivative.

$$y' = e^{x^2} \cdot 2x$$

outer derivative　　　　derivative from within

Eg) $y = (4x^2 - 3x - 2)^{13}$. Find the derivative.

$$y' = 13(4x^2 - 3x - 2)^{12}(8x - 3)$$

outer derivative　　　　derivative from within

Eg)

$$
\begin{aligned}
y &= \frac{1}{1 + \ln x} \qquad f = \frac{1}{g} \text{ and } g = 1 + \ln x \\
y' &= \frac{-1}{(1 + \ln x)^2} \cdot \frac{1}{x} \text{ (or we could use the quotient rule)}
\end{aligned}
$$

We can apply the chain rule over and over if we want since

$$
\begin{aligned}
y(x) &= f(g(h(x))) \\
\frac{dy}{dx} &= \frac{df}{dg}\frac{dg}{dh}\frac{dh}{dx}
\end{aligned}
$$

Eg) $y = \ln[(x^2 + 2x)^5 + 3]$. Find the derivative.

$$
\begin{aligned}
f &= \ln g \qquad g = h^5 + 3 \qquad h = x^2 + 2x \\
y' &= \frac{1}{g} \cdot 5h^4 \cdot (2x + 2) \\
&= \frac{1}{(x^2 + 2x)^5 + 3} \cdot 5(x^2 + 2x)^4 \cdot (2x + 2)
\end{aligned}
$$

Eg) $y = 3^t$. Find the derivative.

$$
\begin{aligned}
y &= e^{\ln 3^t} \\
&= e^{t \ln 3} \\
y' &= \ln 3 \cdot e^{t \ln 3} \\
&= \ln 3 \cdot 3^t
\end{aligned}
$$

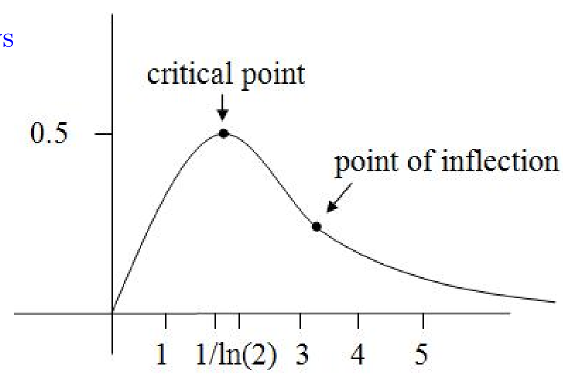
In general, if $y = a^t$, then $y' = \ln a \cdot a^t$.
Specifically, $y = e^t \to y' = \ln e \cdot e^t = e^t$ since $\ln e = 1$.

Eg) Our disease pandemic is $y = t \cdot 2^{-t}$.

Graph the function and find the time when the disease is at its worst and what proportion of the population is infected.

$$
\begin{aligned}
y(0) &= 0 \\
y &= \frac{t}{2^t} \\
y' &= \frac{2^t \cdot 1 - t \cdot \ln 2 \cdot 2^t}{(2^t)^2} \\
&= \frac{2^t(1 - t \cdot \ln 2)}{2^{2t}} \\
&= \frac{1 - t \ln 2}{2^t} \\
y' &= 0 \to 1 - t\ln 2 = 0 \to t = \frac{1}{\ln 2} \text{ (is the exact answer)} = 1.44 \text{ days approximately} \\
y'' &= \frac{2^t(-\ln 2) - (1 - t\ln 2) \cdot \ln 2 \cdot 2^t}{(2^t)^2} \\
&= \frac{2^t(-\ln 2 - \ln 2 + t(\ln 2)^2)}{(2^t)^2} \\
&= \frac{-2\ln 2 + t(\ln 2)^2}{2^t}
\end{aligned}
$$

$$
\begin{aligned}
y'' &= 0 \to -2 + t\ln 2 = 0 \to t = \frac{2}{\ln 2} = 2.885 \text{ days} \\
y\left(\frac{1}{\ln 2}\right) &= \frac{1}{\ln 2} \cdot 2^{-\frac{1}{\ln 2}} \text{ exact} \\
&= 0.530737845 \text{ percent} \\
y''\left(\frac{1}{\ln 2}\right) &= \frac{-2\ln 2 + \frac{1}{\ln 2}(\ln 2)^2}{2^{1/\ln 2}} \\
&= \frac{-2\ln 2 + \ln 2}{2^{1/\ln 2}} \\
&= \frac{-\ln 2}{2^{1/\ln 2}} < 0 \qquad \to \text{ concave down}
\end{aligned}
$$



6.9 Derivatives of trigonometric functions

$$
\frac{d}{dx}\sin x = \cos x \qquad\qquad \frac{d}{dx}\cos x = -\sin x
$$

Eg)

$$
\begin{aligned}
y &= x\sin x \\
y' &= \sin x + x\cos x \\
y'' &= \cos x + \cos x - x\sin x \\
&= 2\cos x - x\sin x
\end{aligned}
$$

Eg) Graph the function $y = e^{-x}\sin x$

$y(0) = 0$

$$y' = e^{-x}\cos x - e^{-x}\sin x$$

$$y' = 0 \to e^{-x}(\cos x - \sin x) = 0 \to 1 - \tan x = 0 \to \tan x = 1 \to x = \frac{\pi}{4}, \frac{5\pi}{4}, \frac{9\pi}{4}, \ldots, -\frac{3\pi}{4}, -\frac{7\pi}{4}, \ldots$$

$$y'' = e^{-x}(-\sin x) - e^{-x}\cos x - [e^{-x}\cos x - e^{-x}\sin x] = \cancel{-e^{-x}\sin x} - e^{-x}\cos x - e^{-x}\cos x + \cancel{e^{-x}\sin x} = -2e^{-x}\cos x$$

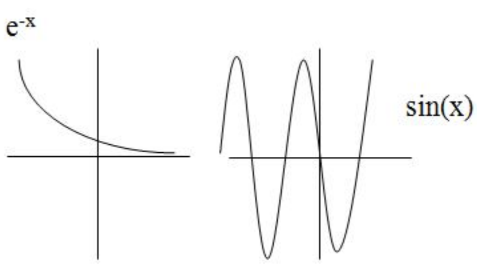
$$y''\left(\frac{\pi}{4}\right) = -2e^{-\frac{\pi}{4}} \cdot \frac{1}{\sqrt{2}} < 0 \qquad y\left(\frac{\pi}{4}\right) = e^{-\frac{\pi}{4}}\frac{1}{\sqrt{2}} = 0.322$$

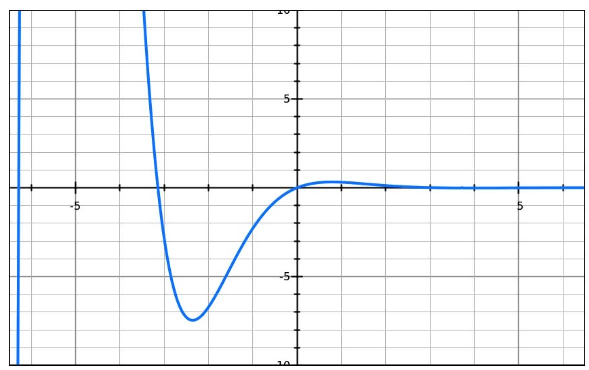
$$y''\left(\frac{5\pi}{4}\right) = -2e^{-\frac{5\pi}{4}}\left(-\frac{1}{\sqrt{2}}\right) > 0 \qquad y\left(\frac{5\pi}{4}\right) = e^{-\frac{5\pi}{4}}\left(-\frac{1}{\sqrt{2}}\right) = -0.0139$$

$$y''\left(\frac{9\pi}{4}\right) = -2e^{-\frac{9\pi}{4}}\left(\frac{1}{\sqrt{2}}\right) < 0 \qquad y\left(\frac{9\pi}{4}\right) = e^{-\frac{9\pi}{4}}\left(\frac{1}{\sqrt{2}}\right) = 0.000602$$

$$y''\left(-\frac{3\pi}{4}\right) = -2e^{\frac{3\pi}{4}}\left(-\frac{1}{\sqrt{2}}\right) > 0 \qquad y\left(-\frac{3\pi}{4}\right) = e^{\frac{3\pi}{4}}\left(-\frac{1}{\sqrt{2}}\right) = -7.46$$

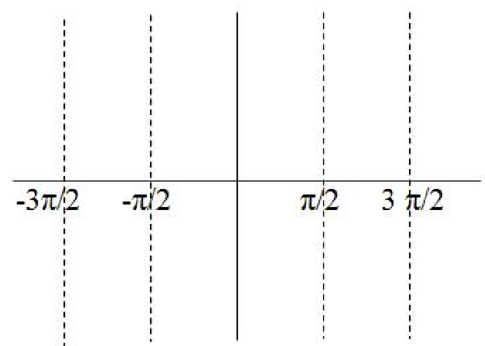
$$y''\left(-\frac{7\pi}{4}\right) = -2e^{\frac{7\pi}{4}}\left(\frac{1}{\sqrt{2}}\right) < 0 \qquad y\left(-\frac{7\pi}{4}\right) = e^{\frac{7\pi}{4}}\left(\frac{1}{\sqrt{2}}\right) = 172.64$$





$y = e^{-x}\sin x$

Eg) Plot $y = \tan x$



$$y = \tan x$$

$$y = \frac{\sin x}{\cos x}$$

$$x \neq \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \ldots$$

$y(0) = 0$

$$y' = \frac{\cos x(\cos x) - \sin x(-\sin x)}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

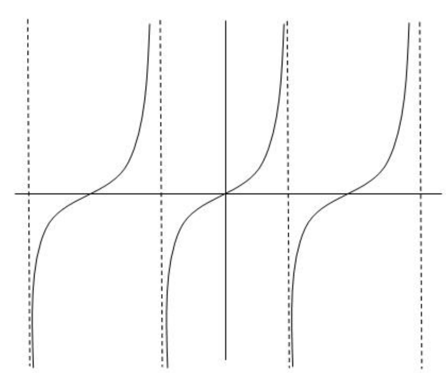
$$= \frac{1}{\cos^2 x} > 0 \text{ always (therefore no critical points)}$$

$$y'' = \frac{d}{dx}(\cos x)^{-2}$$

$$= -2(\cos x)^{-3} \cdot (-\sin x) \qquad \text{chain rule}$$

$$y'' = \frac{2\sin x}{(\cos x)^3}$$

$y'' = 0 \to \sin x = 0 \to x = 0, \pm\pi, \pm 2\pi, \pm 3\pi, \ldots$ are points of inflection.



Eg) A disease follows the time-course $y = \frac{1}{3} - \frac{1}{3}e^{-0.05x}\cos x$. Describe the nature of the disease. What happens eventually?

$$
\begin{aligned}
y(0) &= 0 \\
y' &= \frac{0.05}{3}e^{-0.05x}\cos x - \frac{1}{3}e^{-0.05x}(-\sin x) \\
y' &= 0 \to 0.05 + \tan x = 0 \\
\to x &= \pi + \tan^{-1}(-0.05), 2\pi + \tan^{-1}(-0.05), 3\pi + \tan^{-1}(-0.05) \\
&= 3.0916, 6.2332, 9.3748, \ldots \\
y' &= \frac{1}{3}e^{-0.05x}(0.05\cos x + \sin x) \\
y'' &= \frac{-0.05}{3}e^{-0.05x}(0.05\cos x + \sin x) + \frac{1}{3}e^{-0.05x}(-0.05\sin x + \cos x) \\
&= \frac{1}{3}e^{-0.05x}(-(0.05)^2\cos x - 0.05\sin x - 0.05\sin x + \cos x) \\
&= \frac{1}{3}e^{-0.05x}(0.9975\cos x - 0.1\sin x)
\end{aligned}
$$

$$
\begin{aligned}
y''(3.0916) &= -0.2859 & y(3.0916) &= 0.6185 \\
y''(6.2332) &= 0.2444 & y(6.2332) &= 0.0895 \\
y''(9.3748) &= -0.2089 & y(9.3748) &= 0.5416
\end{aligned}
$$