MAT1330 - Calculus for the Life Sciences

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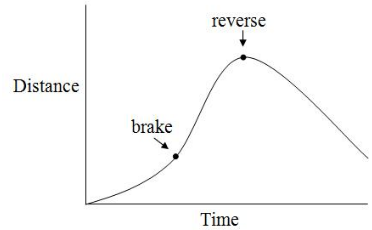
November 25, 2013

7 Stability and the Derivative

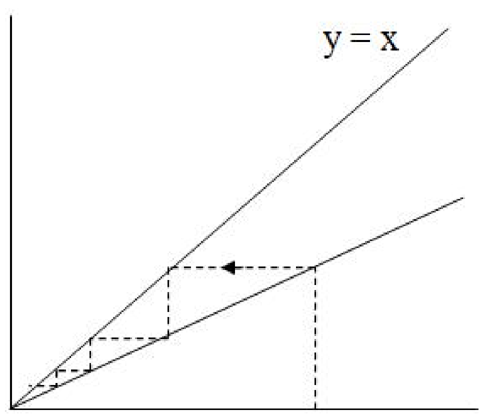
Critical points vs points of inflection
Critical point is like putting in reverse
Points of inflection are like applying the brakes.



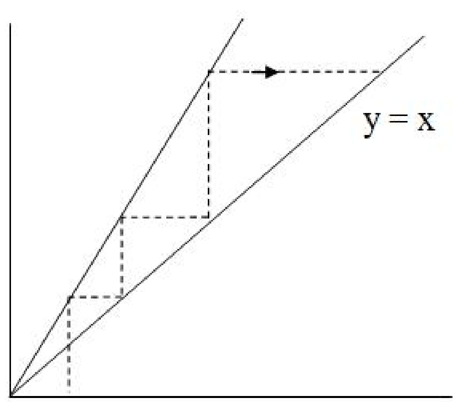
Consider $x_{t+1} = \frac{1}{3}x_t$. Solutions tend toward 0 regardless of where they start.
Consider $x_{t+1} = 3x_t$. Any solution not starting at zero moves away, no matter how close it starts.
Consider $x_{t+1} = -3x_t + 3$ and $x_{t+1} = -\frac{1}{3}x_t + 1$. What happens?

Definition 7.1. *An equilibrium is* <u>*stable*</u> *if solutions near the equilibrium move closer to it and* <u>*unstable*</u> *if solutions move further away.*

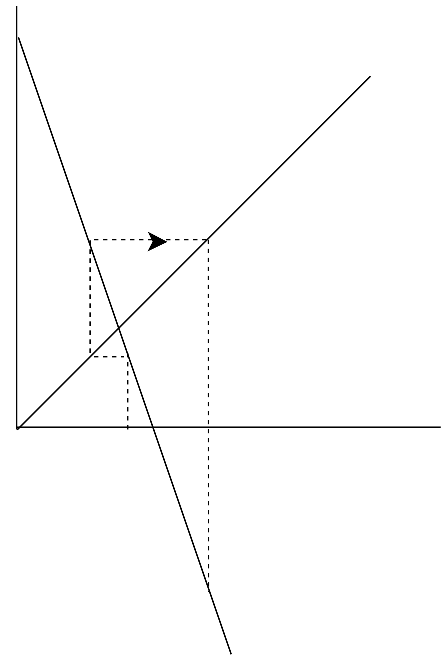
$$f(x) = \frac{1}{3}x$$
$$f'(x) = \frac{1}{3} < 1$$



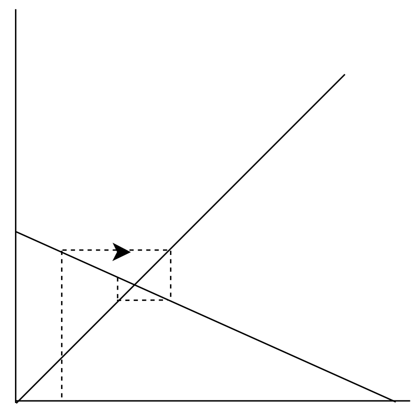
$$g(x) = 3x$$
$$g'(x) = 3 > 1$$



$h(x) = -3x + 3$
$h'(x) = -3 < -1$



$j(x) = -\frac{1}{3}x + 1$
$j'(x) = -\frac{1}{3}$
$-1 < -\frac{1}{3} < 1$



<u>Slope Criterion</u> - An equilibrium is x^* is stable if $|f'(x^*)| < 1$ and unstable if $|f'(x^*)| > 1$

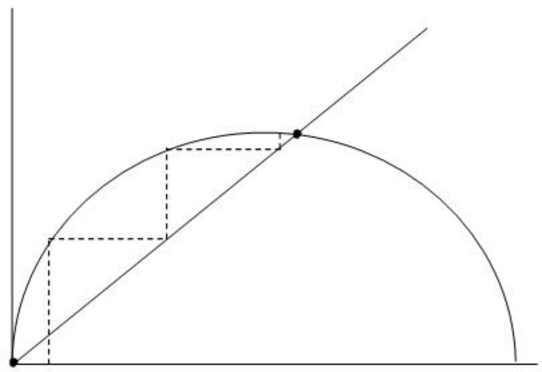
Eg) $x_{t+1} = 1 - (x_t - 1)^2$. Find the equilibria and determine their stability.

$$\begin{aligned}
\textit{Equilibria} : x &= 1 - (x-1)^2 \\
x &= 1 - (x^2 - 2x + 1) \\
x &= 1 - x^2 + 2x - 1 \\
x^2 - x &= 0 \\
x(x-1) &= 0 \\
x &= 0, 1 \\
y' &= -2(x-1) \\
y' &= 0 \to x = 1 \qquad y(1) = 1 \\
y'' &= -2 < 0 \text{ concave down}
\end{aligned}$$

Stability?

$y'(0) = -2(0-1) = 2 > 1 \therefore$ unstable

$y'(1) = 0 < 1 \therefore$ stable



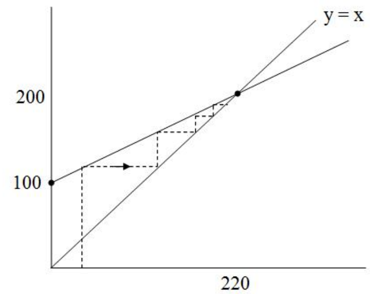
Eg) Influenza: $x_{t+1} = \frac{1}{2}x_t + 100$

$$y' = \frac{1}{2} < 1 \therefore \text{ equlibrium is stable}$$

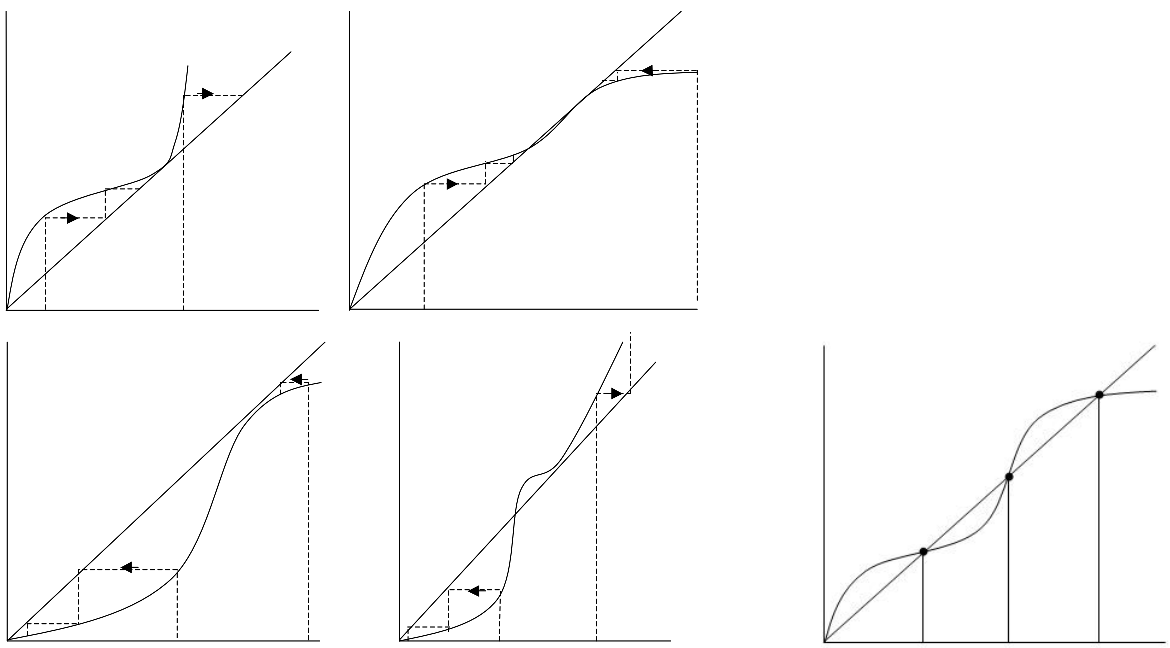
$$\text{Equilibria: } x = \frac{1}{2}x + 100$$

$$\frac{1}{2}x = 100$$

$$x = 200$$



Question: What can we say about stability if the derivative is ± 1?
Answer: Nothing. Any behaviour is possible.



Depending on how the graph touches the line, the equilibrium may be stable or unstable or semistable (on one side) or neither.

7.1 The Logistic Equation

Eg) Consider a population of rabbits, x, on an island. Rabbits breed and grow so $x_{t+1} = rx_t$ where r is the growth rate. However, if the rabbits keep breeding forever, they'll fill the island, so we need some limit on this.

$$\text{Try } x_{t+1} = rx_t - rx_t^2 \qquad y = rx - rx^2 \qquad y(0) = 0$$

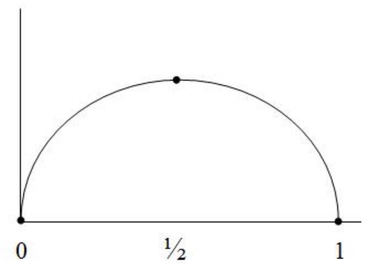
$$x_{t+1} = rx_t(1 - x_t) \qquad y' = r - 2rx = 0 \qquad x = \frac{1}{2}$$

$$y'' = -2r \qquad \to \text{concave down}$$

Question: What happens?

Answer: It depends on r.

$$
\begin{aligned}
\text{Equilibria: } x &= rx - rx^2 \\
rx^2 + (1-r)x &= 0 \\
x(rx + (1-r)) &= 0 \\
x &= 0, rx = r - 1 \\
x &= 1 - \frac{1}{r}
\end{aligned}
$$



But the second only exists if $1 - \frac{1}{r} > 0 \rightarrow 1 > \frac{1}{r} \rightarrow r > 1$ (or else we have negative rabbits).
What happens if $r < 1$?
The growth rate isn't sufficient for reproduction and the species dies out.
Stability?

$$
\begin{aligned}
y' &= r - 2rx \\
y'(0) &= r \\
\therefore 0 \text{ is stable if } r < 1 &\text{ and unstable if } r > 1 \\
y'\left(1 - \frac{1}{r}\right) &= r - 2r\left(1 - \frac{1}{r}\right) \\
&= r - 2\cancel{r}\frac{(r-1)}{\cancel{r}} \\
&= r - 2r + 2 \\
&= 2 - r \\
\text{Stable if } -1 &< 2 - r < 1 \\
-3 &< -r < -1 \\
1 &< r < 3
\end{aligned}
$$

$$\therefore \text{ for } 1 < r < 3, \ 0 \text{ is unstable and } 1 - \frac{1}{r} \text{ is stable.}$$

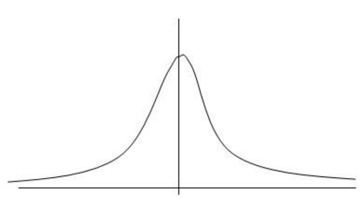
What happens if $r > 3$?
Question: Both equilibria are unstable, so where do solutions go?
Answer: It depends on r. They might go to a periodic orbit, or they might produce chaos and never settle down.

7.2 Maxima and Minima

Eg) Consider $y = e^{-x^2}$ (the bell curve)

$$
\begin{aligned}
y &= e^{-x^2} \\
y' &= -2xe^{-x^2} \\
y' &= 0 \rightarrow x = 0 \\
y(0) &= 1 \\
y'' &= -2e^{-x^2} + 4x^2e^{-x^2} \\
y''(0) &= -2 < 0 \qquad \rightarrow \text{ concave down} \\
y''(x) &= e^{-x^2}(-2 + 4x^2) = 0 \rightarrow x^2 = \frac{1}{2} \rightarrow x = \pm\frac{1}{\sqrt{2}}
\end{aligned}
$$

$\therefore x = 0$ is a maximum. The function has no minimum.



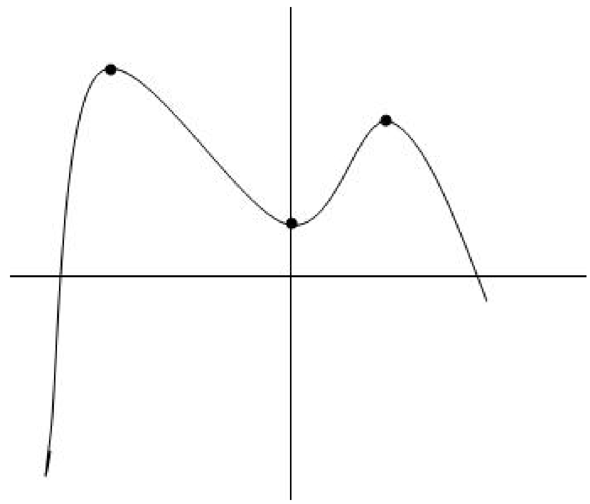
Eg) Sketch $y = -\frac{x^4}{4} - \frac{x^3}{3} + x^2 + 1$.

$$y(0) = 1$$
$$y' = -x^3 - x^2 + 2x = 0$$
$$-x(x^2 + x - 2) = 0$$
$$-x(x-1)(x+2) = 0$$
$$x = 0, 1, -2$$
$$y'' = -3x^2 - 2x + 2$$
$y''(0) = 2$ concave up

$y''(1) = -3$ concave down

$y''(-2) = -12 + 4 + 2 < 0$ concave down

$$y(1) = 1.4167 = \frac{17}{12}$$
$$y(-2) = \frac{11}{3} = 3.67$$



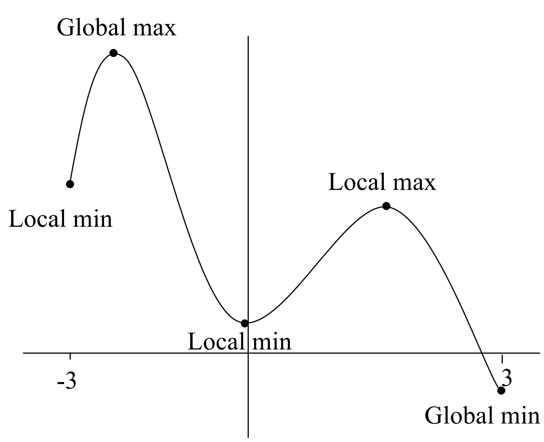
Question: What is the maximum and minimum of the function?

Answer: Depends on the scale. Globally, the max is at $x = -2$ and the min is $-\infty$. But locally, $x = 0$ is a minimum and $x = 1$ is a maximum. We call these <u>local</u> maxima and minima.

Note: if the domain is restricted, the boundary points may also be local maxima or minima.

Eg) Find the local and global maxima and minimum of

$$y = -\frac{x^4}{4} - \frac{x^3}{3} + x^2 + 1$$
$$-3 \leq x \leq 3$$



Eg) Sketch $y = |x| \quad -2 \leq x \leq 1$ and determine the global maximum and minimum.

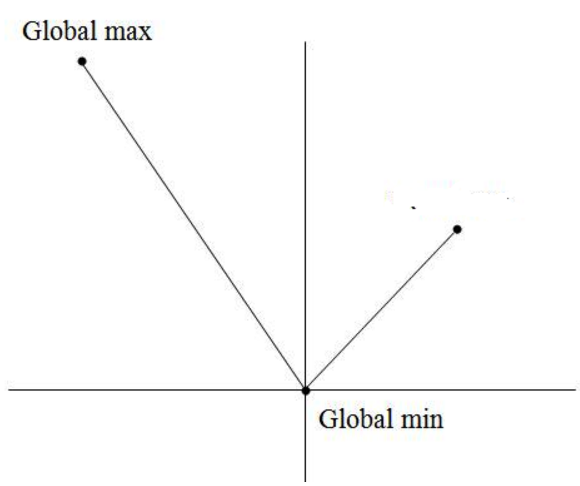
$$y = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$
$$y' = \begin{cases} 1 & x > 0 \\ -1 & x < 0 \end{cases}$$

$y' \neq 0$ ever.

But y' DNE at $x = 0$.

$\therefore$ points to check are $x = -2, 0, 1$

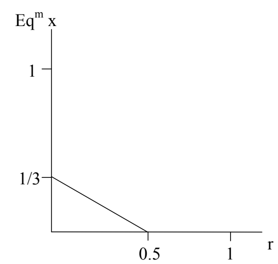
$$y(-2) = 2$$
$$y(0) = 0$$
$$y(1) = 1$$



Eg) The number of trees in a forest is $x_{t+1} = 1.5x_t - 1.5x_t^2$ per year. If we can chop down rx_t trees per year, how can we maximize the number of trees without destroying the forest?

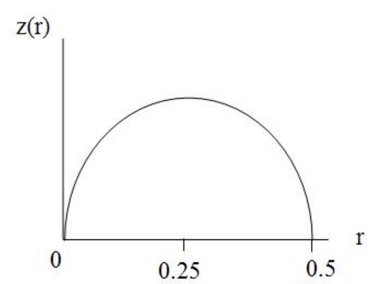
If $r = 0$ then we have no wood.
If r is large then we have lots of wood this year but destroy future trees.

$$\begin{aligned} x_{t+1} &= 1.5x_t - 1.5x_t^2 - rx_t \\ \text{Equilibria:}\quad x &= 1.5x - 1.5x^2 - rx \\ 0 &= 1.5x^2 - 0.5x + rx \\ 0 &= x(1.5x + (r - 0.5)) \\ x = 0 \qquad x &= \frac{r-0.5}{-1.5} = \frac{0.5-r}{1.5} \end{aligned}$$



We need $r < 0.5$. Therefore, if we chop down more than half the forest, we will end up with no wood.

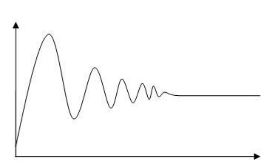
$$\begin{aligned} \text{The amount we harvest is } x(r) &= \frac{r(0.5-r)}{1.5} \\ \text{To maximize, take the derivative: } z'(r) &= \frac{0.5-2r}{1.5} \\ z'(r) &= 0 \rightarrow r = 0.25 \\ z(0.25) &= 0.0416666... \\ z(0.26) &= 0.0416 \\ z(0.24) &= 0.0416 \end{aligned}$$



A larger harvest depletes the forest and a smaller harvest gives us less wood. This is called a sustainable yield.

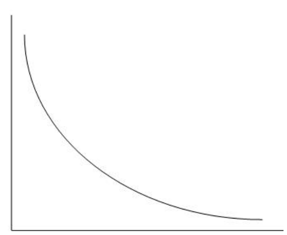
7.3 Limits at Infinity

Recall our disease $y = \frac{1}{3} - \frac{1}{3}e^{-0.05x}\cos x$ with oscillating behaviour.



What happens to the disease eventually?

That is, what is $\lim_{x\to\infty} y$?

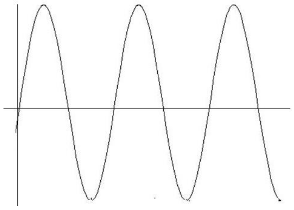


First, let's look at $\lim_{x\to\infty} e^{-0.05x}$

As x gets bigger, $e^{-0.05x}$ gets smaller.

Then $\lim_{x\to\infty} e^{0.05x} = \infty \qquad \therefore \lim_{x\to\infty} \frac{1}{e^{0.05x}} = \frac{1}{\infty} = 0.$

Question: What about $\lim_{x\to\infty} \cos x$?



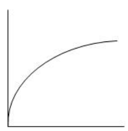
Answer: It has no such limit.

But that turns out not to be a problem because

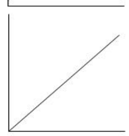
$$\begin{aligned}\lim_{x\to\infty} y &= \lim_{x\to\infty}\frac{1}{3} - \lim_{x\to\infty}\frac{1}{3}\lim_{x\to\infty} e^{-0.05x}\lim_{x\to\infty}\cos x\\ &= \frac{1}{3} - \frac{1}{3}\lim_{x\to\infty} e^{-0.05x}\lim_{x\to\infty}\cos x\\ &= \frac{1}{3} - \frac{1}{3}\cdot 0\cdot \lim_{x\to\infty}\cos x\\ \text{Since} -1 \le \lim_{x\to\infty}\cos x \le 1, &\qquad 0\cdot\lim_{x\to\infty}\cos x = 0\\ \therefore \lim_{x\to\infty} y = \frac{1}{3}\end{aligned}$$

Eg)

$$\lim_{x\to\infty}\sqrt{x} = \lim_{x\to\infty} x^{\frac{1}{2}} = \infty$$

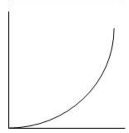


$$\lim_{x\to\infty} x = \infty$$



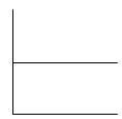
$$\lim_{x\to\infty} x^2 = \infty$$

$$\lim_{x\to\infty} x^3 = \infty$$

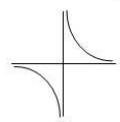


$$\vdots$$

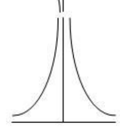
$$\lim_{x\to\infty} x^0 = \lim_{x\to\infty} 1 = 1$$



$$\lim_{x\to\infty}\frac{1}{x} = 0$$



$$\lim_{x\to\infty}\frac{1}{x^2} = 0$$

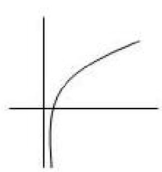


$$\lim_{x\to\infty}\frac{1}{x^3} = 0$$

$$\vdots$$

$$\therefore \lim_{x\to\infty} x^n = \begin{cases}\infty & n>0\\ 1 & n=0\\ 0 & n<0\end{cases}$$

Eg) $\lim_{x\to\infty}\ln x = \infty$



Eg) Find $\lim_{x\to\infty}\dfrac{\sqrt{x}+e^{-x}}{x^{-1}}$.

$$\lim_{x\to\infty}\frac{\sqrt{x}+e^{-x}}{x^{-1}} = \frac{\lim_{x\to\infty}\sqrt{x}+\lim_{x\to\infty}e^{-x}}{\lim_{x\to\infty}x^{-1}}$$
$$= \frac{\infty+0}{0}$$
$$= (\infty+0)\cdot\frac{1}{0}$$
$$= (\infty+0)\cdot\infty$$
$$= \infty$$

<u>Limits of Sequences</u> $a_{t+1} = f(a_t)$ differs from an ordinary function in that populations are only defined at integer values of t. The solution is a list of numbers $a_0, a_1, a_2, ...$ known as a <u>sequence</u>.

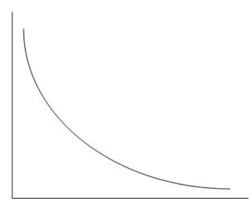
Eg) Our influenza example: $a_{n+1} = \frac{1}{2}a_n + 200$.

The solution is $a_n = \frac{1}{2^n}a_0 - \frac{1}{2^{n-1}}100 + 100$.

Define the <u>associated function</u> $a(x) = \frac{1}{2^x}a_0 - \frac{1}{2^{x-1}}100 + 200$. This function is defined for all x (like how the updating function is defined for all x, not just a select few values).

If the associated function has a limit, then the sequence has the same limit (note: the reverse isn't true, as we will see).

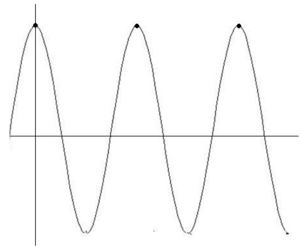
$$\lim_{x\to\infty} a(x) = \lim_{x\to\infty}\left(\frac{1}{2^x}a_0 - \frac{1}{2^{x-1}}100 + 200\right)$$
$$\frac{1}{2^x} = 2^{-x} = e^{\ln 2^{-x}} = e^{-x\ln 2}$$
$$\text{and remember that } \lim_{x\to\infty} e^{-bx} = 0$$
$$\therefore \lim_{x\to\infty}\frac{1}{2^x} = 0$$
$$\frac{1}{2^{x-1}} = 2^{1-x} = e^{\ln 2^{1-x}} = e^{(1-x)\ln 2} = e^{\ln 2 - x\ln 2}$$
$$= e^{\ln 2}e^{-x\ln 2}$$
$$\lim_{x\to\infty}\frac{1}{2^{x-1}} = \lim_{x\to\infty} e^{\ln 2}\lim_{x\to\infty} e^{-x\ln 2}$$
$$= e^{\ln 2}\cdot 0 = 0$$
$$\therefore \lim_{x\to\infty} a(x) = 0 - 0 + 200 = 200$$



Eg) $a_n = \cos(2\pi n)$

The associated function $a(t) = \cos(2\pi t)$ has no limit. However,

$$a_0 = \cos 0 = 1$$
$$a_1 = \cos 2\pi = 1$$
$$a_2 = \cos 4\pi = 1$$
$$a_3 = \cos 6\pi = 1$$
$$\therefore \lim_{n\to\infty} a_n = 1$$



Therefore, the sequence may have a limit even when the associated function doesn't.

7.4 L'Hopital's Rule

Suppose we want to find $\lim_{x\to\infty}\frac{x}{e^{2x}}$. If we proceed as before, then $\lim_{x\to\infty}\frac{x}{e^{2x}} = \frac{\infty}{\infty} = ?$

<u>L'Hopital's Rule</u>: If $\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{\infty}{\infty}$ or $\frac{0}{0}$ then $\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$.

Eg) Find $\lim_{x\to\infty} \frac{x}{e^{2x}}$.

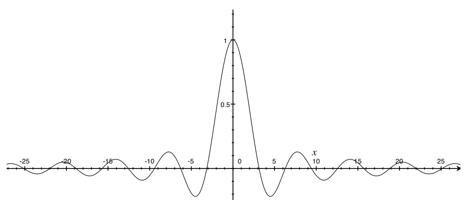
$$\lim_{x\to\infty} \frac{x}{e^{2x}} = \frac{\infty}{\infty}$$
$$\therefore \lim_{x\to\infty} \frac{x}{e^{2x}} \overset{L'H}{=} \lim_{x\to\infty} \frac{1}{2e^{2x}} = \frac{1}{\infty} = 0$$

Eg) Find $\lim_{x\to 0} \frac{x}{e^{2x}-1}$.

$$\lim_{x\to 0} \frac{x}{e^{2x}-1} = \frac{0}{0}$$
$$\lim_{x\to 0} \frac{x}{e^{2x}-1} \overset{L'H}{=} \lim_{x\to 0} \frac{1}{2e^{2x}} = \frac{1}{2\cdot 1} = \frac{1}{2}$$

Eg) Find $\lim_{x\to 0} \frac{\sin x}{x}$.

$$\lim_{x\to 0} \frac{\sin x}{x} = \frac{0}{0}$$
$$\lim_{x\to 0} \frac{\sin x}{x} \overset{L'H}{=} \lim_{x\to 0} \frac{\cos x}{1} = 1$$



Eg) Find $\lim_{x\to\infty} \frac{e^{3x}+2x}{4x+5e^{3x}}$.

$$\lim_{x\to\infty} \frac{e^{3x}+2x}{4x+5e^{3x}} = \frac{\infty}{\infty}$$
$$\lim_{x\to\infty} \frac{e^{3x}+2x}{4x+5e^{3x}} \overset{L'H}{=} \lim_{x\to\infty} \frac{3e^{3x}+2}{4+15e^{3x}} = \frac{\infty}{\infty}$$
$$\overset{L'H}{=} \lim_{x\to\infty} \frac{9e^{3x}}{45e^{3x}} = \frac{9}{45} = \frac{1}{5}$$

Note: L'Hopital's rule doesn't guarantee an answer, it just allows us to rewrite the expression, therefore, we can use it as often as needed.

Eg) Find $\lim_{x\to 3} \frac{e^x - e^3\cos(3-x)}{x-3}$.

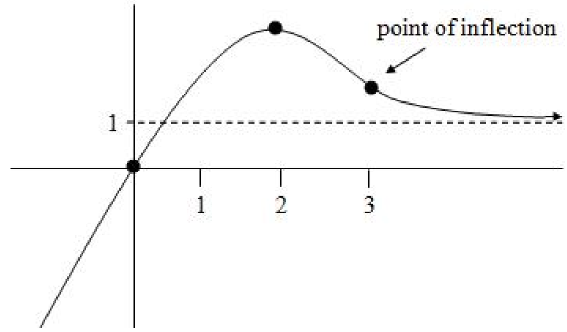
$$\lim_{x\to 3} \frac{e^x - e^3\cos(3-x)}{x-3} = \frac{e^3-e^3}{3-3} = \frac{0}{0}$$
$$\lim_{x\to 3} \frac{e^x - e^3\cos(3-x)}{x-3} \overset{L'H}{=} \lim_{x\to 3} \frac{e^x - e^3\sin(3-x)}{1} = e^3 - e^3\cdot\sin 0 = e^3$$

Eg) Our disease pandemic $y(t) = t\cdot 2^{-t}$. What happens eventually?

$$y(t) = t\cdot 2^{-t} = \frac{t}{2^t} = \frac{t}{e^{t\ln 2}} \qquad \text{since } 2^t = e^{\ln 2^t} = e^{t\ln 2}$$
$$\lim_{t\to\infty} y(t) = \frac{\infty}{\infty} \overset{L'H}{=} \lim_{t\to\infty} \frac{1}{\ln 2 e^{t\ln 2}} = \frac{1}{\infty} = 0$$

Eg) Graph $y = \frac{x+e^x-1}{e^x}$.

$$
\begin{aligned}
y(0) &= \frac{0+1-1}{1} = 0 \\
y' &= \frac{e^x(1+e^x) - (x+e^x-1)e^x}{e^{2x}} \\
&= \frac{e^x(2-x)}{e^{2x}} = \frac{2-x}{e^x} \\
y' &= 0 \rightarrow x = 2 \\
y(2) &= \frac{1+e^2}{e^2} = 1.135 \\
y'' &= \frac{-e^x - (2-x)e^x}{e^{2x}} = \frac{x-3}{e^x} \\
y''(2) &= \frac{-1}{e^2} < 0 \therefore \text{concave down} \\
y'' &= 0 \rightarrow x = 3 \text{ is a point of inflection} \\
\lim_{x\to\infty} y &= \frac{\infty}{\infty} \\
\lim_{x\to\infty} \frac{x+e^x-1}{e^x} &\overset{L'H}{=} \lim_{x\to\infty} \frac{1+e^x}{e^x} = \frac{\infty}{\infty} \\
\lim_{x\to\infty} \frac{1+e^x}{e^x} &\overset{L'H}{=} \lim_{x\to\infty} \frac{e^x}{e^x} = 1 \\
\lim_{x\to-\infty} \frac{x+e^x-1}{e^x} &= \frac{-\infty+0-1}{0} = -\infty \cdot \frac{1}{0} = -\infty \cdot \infty = -\infty
\end{aligned}
$$



Note that we could not use L'Hopital's rule here.

Important: We can only use L'Hopital's rule when the limit is $\frac{\infty}{\infty}$ or $\frac{0}{0}$. If the limit is $\frac{0}{\infty}$ or $\frac{\infty}{0}$, it cannot be used.

What we have is an extra technique for graphing: limits at ∞ and any points where the domain doesn't exist.

Summary of graphing techniques

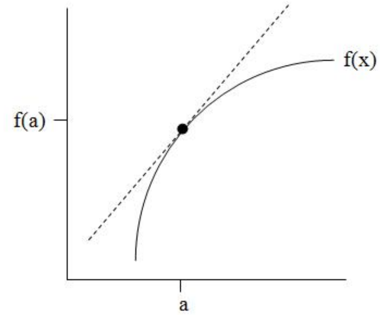
- Check Domain (any points where the function doesn't exist)
- y-intercept ($x = 0$)
- derivatives
- $y' = 0 \rightarrow$ critical points
  - $y' > 0 \rightarrow$ increasing
  - $y' < 0 \rightarrow$ decreasing
- second derivative
  - $y'' > 0 \rightarrow$ concave up
  - $y'' < 0 \rightarrow$ concave down
  - $y'' = 0 \rightarrow$ point of inflection
- $\lim_{x\to\infty} y$
- $\lim_{x\to-\infty} y$
- $\lim_{x\to a} y$ if a is a point where the function doesn't exist.

  If the above limits require it, use L'Hopital's rule.

7.5 Approximating Functions

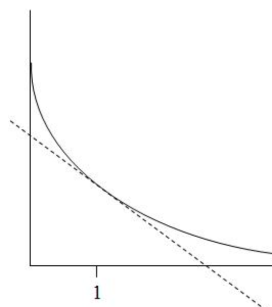
A function can be approximated by its tangent line, close to the point of interest.
How can we find the tangent line? It has slope $f'(a)$, therefore

$$\begin{aligned} y - y_0 &= f'(a)(x - x_0) \\ y - f(a) &= f'(a)(x - a) \\ y &= f(a) + f'(a)(x - a) \end{aligned}$$



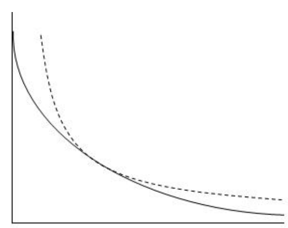
Eg) Find the tangent line to e^{-x} at $x = 1$. How good an approximation is this to the function at 1.1? At 0?

$$\begin{aligned} y_1 &= f(1) + f'(1)(x - 1) \\ &= e^{-1} + (-e^{-1})(x - 1) \\ &= e^{-1} - e^{-1}(x - 1) \\ y &= 0.368 - 0.368(x - 1) \\ y_1(1.1) &= 0.368 - 0.268(0.1) \\ &= 0.3312 \\ f(1.1) &= e^{-1.1} = 0.33287 \end{aligned}$$



This is a good approximation if we're close to $a = 1$.

$$\begin{aligned} y(0) &= 0.368 + 0.368 = 0.736 \\ f(0) &= e^0 = 1 \therefore \text{ not so good further away} \end{aligned}$$



The linear approximation is $y_1 = f(a) + f'(a)(x - a)$.
The quadratic approximation is $y_2 = f(a) + f'(a)(x - a) + \frac{f''(a)}{2}(x - a)^2$
(The $\frac{1}{2}$ occurs because $\frac{d}{dx}(x^2) = 2x$)
Eg) For e^{-x}, the quadratic approximation is

$$\begin{aligned} y_2 &= f(1) + f'(1)(x - 1) + \frac{f''(1)}{2}(x - 1)^2 \\ &= e^{-1} + (-e^{-1})(x - 1) + \frac{e^{-1}}{2}(x - 1)^2 \\ &= 0.368 - 0.368(x - 1) + \frac{0.368}{2}(x - 1)^2 \\ y_2(1.1) &= 0.368 - 0.368(0.1) + \frac{0.368}{2}(0.1)^2 \\ &= 0.33304 \end{aligned}$$

This is a better approximation than the linear approximation.

Eg) Find the linear and quadratic approximations to $f(x) = \sin x$ at $x = \frac{\pi}{4}$ and compare to $x = 0.7$.

$$\begin{aligned} f(x) &= \sin x & f\left(\frac{\pi}{4}\right) &= \frac{1}{\sqrt{2}} \\ f'(x) &= \cos x & f'\left(\frac{\pi}{4}\right) &= \frac{1}{\sqrt{2}} \\ f''(x) &= -\sin x & f''\left(\frac{\pi}{4}\right) &= -\frac{1}{\sqrt{2}} \end{aligned}$$

Linear approximation:

$$\begin{aligned} y_1 &= f\left(\frac{\pi}{4}\right) + f'\left(\frac{\pi}{4}\right)(x - \frac{\pi}{4}) \\ &= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}(x - \frac{\pi}{4}) \\ y_1(0.7) &= 0.6467 \\ f(0.7) &= .6442 \end{aligned}$$

Quadratic approximation:

$$\begin{aligned} y_2 &= f\left(\frac{\pi}{4}\right) + f'\left(\frac{\pi}{4}\right)\left(x - \frac{\pi}{4}\right) + \frac{f''\left(\frac{\pi}{4}\right)}{2}\left(x - \frac{\pi}{4}\right)^2 \\ &= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\left(x - \frac{\pi}{4}\right) - \frac{1}{2\sqrt{2}}\left(x - \frac{\pi}{4}\right)^2 \\ y_2(0.7) &= 0.6441 \end{aligned}$$

Taylor Polynomials: The quadratic approximation is better than the linear. A cubic would be even better:

$$y = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3$$

where $3! = 3 \cdot 2 \cdot 1$.

In general, $n! = n(n-1)(n-2)(n-3)...(3)(2)(1)$ with the understanding that $0! = 1$.

Eg)

$$\begin{aligned} 5! &= 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120 \\ 3! &= 2 \cdot 1 = 2 \\ 1! &= 1 \end{aligned}$$

The n^{th} approximation of all would thus be

$$y_n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + ... + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

This is called the Taylor polynomial of degree n.

Eg) Find the Taylor approximation of e^x at $a = 0$.

$$\begin{aligned} f(x) &= e^x & f(0) &= 1 \\ f'(x) &= e^x & f'(0) &= 1 \\ f''(x) &= e^x & f''(0) &= 1 \\ &\vdots & &\vdots \end{aligned}$$

$$\therefore e^x \approx 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + ... + \frac{1}{n!}x^n$$

Incidentally, this is why $\lim_{x\to\infty} e^x = \infty$

$e^1 \approx 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!} = 2.718$

Eg) Find the Taylor approximation of $-\ln(1-x)$ at $a = 0$.

$$
\begin{aligned}
f(x) &= -\ln(1-x) & f(0) &= 0\\
f'(x) &= -\frac{1}{1-x}(-1) = \frac{1}{1-x} = (1-x)^{-1} \qquad \text{(chain rule)} & f'(0) &= 1\\
f''(x) &= -(1-x)^{-2}(-1) = (1-x)^{-2} & f''(0) &= 1\\
f'''(x) &= -2(1-x)^{-3}(-1) = 2(1-x)^{-3} & f'''(0) &= 2\\
f^{iv}(x) &= -3\cdot 2\cdot(1-x)^{-4}(-1) = 3\cdot 2(1-x)^{-4} & f^{iv}(0) &= 3!\\
f^{v}(x) &= 4\cdot 3\cdot 2(1-x)^{-5} & f^{v}(0) &= 4!
\end{aligned}
$$

$$
\begin{aligned}
y_n &= 0 + x + \frac{x^2}{2!} + 2\frac{x^3}{3!} + 3!\frac{x^4}{4!} + 4!\frac{x^5}{5!} + ... + (n-1)!\frac{x^n}{n!}\\
&= x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5} + ... + \frac{x^n}{n}
\end{aligned}
$$

Eg) Show that $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + ... = 0.693$

$$
\begin{aligned}
x = -1 : -1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + ... \pm \frac{1}{n} = -\ln(1-(-1)) = -\ln 2\\
\therefore \quad 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + ... \approx -(-\ln 2) = \ln 2 = 0.693
\end{aligned}
$$

Eg) Find the second degree Taylor approximation of $\sin x$ at base point 3. Use this to find the approximated value of $\sin 3.3$. Compare this with the Taylor approximation at base 0.

$$
\begin{aligned}
f(x) &= \sin x & f(3) &= \sin 3\\
f'(x) &= \cos x & f'(3) &= \cos 3\\
f''(x) &= -\sin x & f''(3) &= -\sin 3
\end{aligned}
$$

$$
\begin{aligned}
\sin x \approx & f(a) + f'(a)(x-a) + \frac{f''(a)(x-a)^2}{2!}\\
y_2(x) &= \sin 3 + \cos 3(x-3) - \frac{\sin 3(x-3)^2}{2!}\\
y_2(3.3) &= 0.14112 - 0.98999(0.3) - 0.14112\frac{(0.3)^2}{2}\\
&= -0.16222\\
\sin(3.3) &= -0.1577
\end{aligned}
$$

Therefore, this is fairly close.

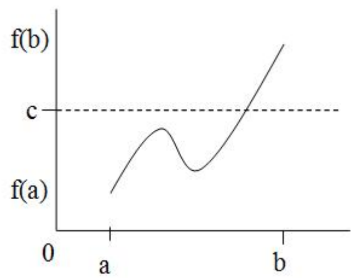
$$
\begin{aligned}
f(x) &= \sin x & f(0) &= 0\\
f'(x) &= \cos x & f'(0) &= 1\\
f''(x) &= -\sin x & f''(0) &= 0\\
f'''(x) &= -\cos x & f'''(0) &= -1
\end{aligned}
$$

$$
\begin{aligned}
y_2 &= 0 + 1(x) + 0 = x \\
y_2(3.3) &= 3.3 \gg -0.1577 \\
y_3 &= x - \frac{x^3}{3!} \\
y_3(3.3) &= -2.6875 \ll -0.1577 \\
y_7 &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \\
y_7(3.3) &= -0.2738 \qquad \rightarrow \text{closer}
\end{aligned}
$$

Therefore, approximations are good when close to the base point. The further away you move, the more terms you need in the approximation.

7.6 The Intermediate Value Theorem

Theorem 7.2. *If $f(x)$ is continuous for $a \leq x \leq b$ and c is between $f(a)$ and $f(b)$, then there is some x_0 between a and b such that $f(x_0) = c$.*

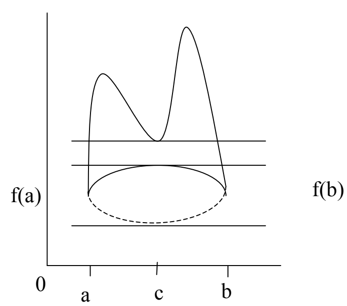


Eg) If you walk down a mountain from top to bottom, no matter what path you take, you must be halfway at some point.

Eg) Show that the equation $\ln x = 3 - 2x$ has a solution.

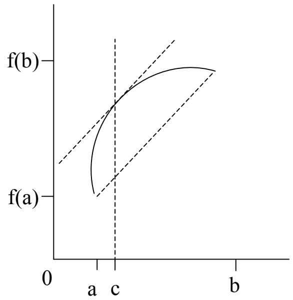
$$
\begin{aligned}
\text{Define } f(x) &= \ln x - 3 + 2x \\
f(1) &= 0 - 3 + 2 = -1 < 0 \\
f(2) &= \ln 2 - 3 + 4 = 1 + \ln 2 > 0 \\
\therefore f(x_0) &= 0 \text{ for some } x_0 \text{ satisfying } 1 < x_0 < 2
\end{aligned}
$$

Theorem 7.3. *Rolle's Theorem: If $f(x)$ is differentiable for all x satisfying $a \leq x \leq b$ and $f(a) = f(b)$, then there exists some c such that $a < c < b$ and $f'(c) = 0$.*



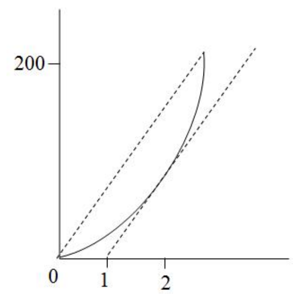
But this idea holds for any derivative, not just zero.

Theorem 7.4. *Mean Value Theorem: If $f(x)$ is differentiable for $a \leq x \leq b$ then there exists some c with $a < c < b$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$*



Eg) A freeway is 220 km long, with a speed limit of 100 km/h. If you make it to the end of the freeway in 2 hours and ten minutes, did you break the speed limit?

$$\begin{aligned} 2 \text{ hr } 10 \text{ min} &= 2\frac{1}{6} \text{ hrs} = 2.1667 \text{ hrs} \\ \text{velocity} &= \frac{\text{distance}}{\text{time}} \\ MVT \rightarrow & \frac{220-0}{2.1667-0} = 101.5 \end{aligned}$$



Therefore yes, you must have. (This is how some real speed cameras work)

7.7 Newton's Method

Newton's method is a procedure for finding approximate solutions to equations that can't be solved. We want to solve $f(x) = 0$. Let x^* be the exact solution, satisfying $f(x^*) = 0$. Let x_0 be some initial approximation of x^*. We want to find x_1, another approximation, which is closer to x^*.

The linear approximation $\hat{f}(x)$ of $f(x)$ near x_0 is

$$\hat{f}(x) = f(x_0) + f'(x_0)(x - x_0)$$

$\hat{f}(x)$ is a simple linear function, so we can solve $\hat{f}(x) = 0$. Then

$$f(x_0) + f'(x_0)(x - x_0) = 0$$

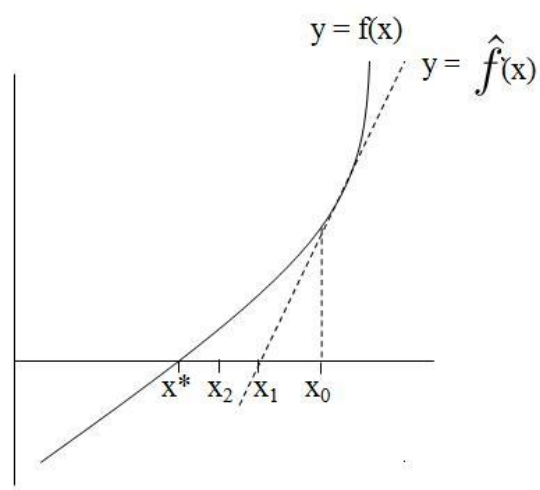
$$x = x_0 - \frac{f(x_0)}{f'(x_0)}$$

The solution of this equation is the new approximation x_1:

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

We can repeat this process to find $x_2, x_3, x_4, \ldots$ successive approximations of the exact solution, each of which is closer to x^* than the one before.

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}, \qquad x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}, \qquad x_4 = \ldots$$



Eg) Find $\sqrt{5}$ accurate to 5 decimal places.
We need to solve $x^2 - 5 = 0$.

$$\text{Set } f(x) = x^2 = 5$$
$$\text{Then } f'(x) = 2x$$

What to choose for x_0? We know that $\sqrt{4} = 2$ and $\sqrt{5}$ is close to $\sqrt{4}$ so let's try $x_0 - 2$.

$$x_0 = 2 \qquad\qquad 1 \text{ d.p}$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = x_0 - \frac{x_0^2 - 5}{2x_0} = \frac{x_0}{2} + \frac{2.5}{x_0}$$

$$x_1 = \frac{2}{2} + \frac{2.5}{2} = 2.25 \qquad\qquad 2 \text{ d.p}$$

$$x_2 = \frac{x_1}{2} + \frac{2.5}{x_1} = \frac{2.25}{2} + \frac{2.5}{2.25} = 2.23611$$

Question: Is this enough?

$$x_3 = \frac{x_2}{2} + \frac{2.5}{x_2} = 2.236067977915$$

Therefore, 3 decimal places must be correct.

$$x_4 = \frac{x_3}{2} + \frac{2.5}{x_3} = 2.236067977$$

Therefore, we have 9 decimal places of $\sqrt{5}$ after 3 steps.
When Newton's method works, it converges to the exact solution very fast.

Newton's method is a DTDS $x_{t+1} = x_t - \dfrac{f(x_t)}{f'(x_t)}$

The solution is $x^*$is an equation: $x^* = x^* - \dfrac{f(x^*)}{f'(x^*)}$

$0 = \dfrac{f(x^*)}{f'(x^*)}$ But $f(x^*) = 0$ so we have $0 = 0$... so long as $f'(x^*) \neq 0$

Stability of x^*? Updating function is $h(x) = x - \dfrac{f(x)}{f'(x)}$

$h'(x) = 1 - \dfrac{f'(x)f'(x) - f(x) \cdot f''(x)}{[f'(x)]^2} = \dfrac{[f'(x)]^2 - [f'(x)]^2 + f(x)f''(x)}{[f'(x)]^2} = \dfrac{f(x)f''(x)}{[f'(x)]^2}$

At equilibrium, we have $h'(x^*) = \dfrac{f(x^*)f''(x^*)}{[f'(x^*)]^2} = 0$ since $f(x^*) = 0$ and $f'(x^*) \neq 0$.

Since $|h'(x^*)| < 1$, this equilibrium is stable.

In fact, the smaller $|h'(x^*)|$, the faster the convergence. Since $|h'(x^*)| = 0$ for Newton's method, convergence is as fast as possible and we say that the equilibrium is superstable. But, it is only locally stable. So the initial point needs to be close to x^* or else it may fail.

Eg) Use Newton's method to find roots of $f(x) = (x-2)^3 - (x-1)^2 + 2$ starting at $x_0 = 2$ and $x_0 = 1.57$.

$$f(x) = (x-2)^3 - (x-1)^2 + 2$$
$$f'(x) = 3(x-2)^2 - 2(x-1)$$

$x_0 = 2 \quad f(2) = 1 \quad f'(2) = -2$

$x_1 = 2 - \dfrac{1}{-2} = 2.5$

$x_2 = 2.444$

$x_3 = 2.445$

$x_0 = 1.57$

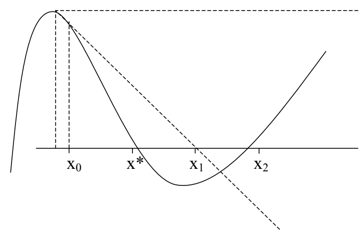
$x_1 = 4.2961$

$x_2 = 3.9447$

$x_3 = 3.8195$

$x_4 = 3.8023$

If $f(x)$ has several roots close to each other, then Newton's method may converge to a different root, not the desired one. If $f(x)$ has a local min or max near the solution x^*, then we need to choose x_0 close to x^*. Otherwise, the next approximation may shoot off to infinity since at points of min/max $f' = 0$ and we are dividing by f'.



8 Differential equations

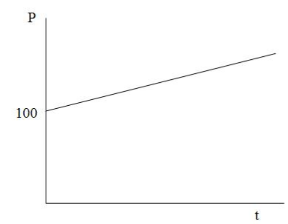
In nature, it's usually easier to measure change than it is to measure quantities directly.

Eg) The speedometer in your car measures the speed (distance per time) rather than your position.

Eg) A population grows at the rate of 50 new individuals per year. The population P is thus increasing and satisfies $\frac{dP}{dt} = 50$. This is called a differential equation.
We'd like to find the solution, but first we need more information. Suppose additionally that there are 1000 individuals in the population originally. That is, $P(0) = 1000$. This is called an initial condition, just as for discrete-time dynamical systems.
Every differential equation must have an initial condition.

Question: How can we solve this?
Answer: Let's guess, since this one is easy.
The slope is 50, so the solution is a line with slope 50. That is, $P(t) = 50t + c$.
Now we apply our initial condition:

$$\begin{aligned} P(0) = 50(0) + c &= 1000 \\ c &= 1000 \\ \therefore P(t) = 50t + 1000 & \end{aligned}$$

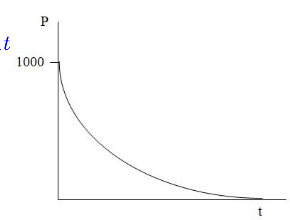


Note that the population is increasing. This is because the derivative is positive.

Eg) Suppose now that our population dies at a rate r, proportional to the number of individuals in the population at any given time. Then we have $\frac{dP}{dt} = -rP$. If $r = 0.01$, this is $\frac{dP}{dt} = -0.01P$ with $P(0) = 1000$.

Check that $P(t) = 1000e^{-0.01t}$ is a solution (we'll discover why shortly).

$$\begin{aligned} P'(t) &= -0.01 \cdot 1000e^{-0.01t} \\ &= -0.01P \\ P(0) &= 1000e^{0} \\ &= 1000 \end{aligned}$$



Notice that the population is decreasing. This is because the derivative is negative.

Eg) Suppose we combine both examples. Our population grows by 50 new individuals, but dies at rate

$0.01P$.

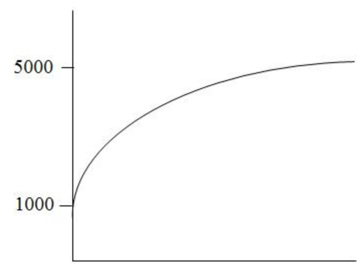
$$\frac{dP}{dt} = 50 - 0.01P \qquad\qquad P(0) = 1000$$

Where

$$\begin{aligned} \frac{dP}{dt} &= \text{rate of change} \\ 50 &= \text{ increase in population} \\ -0.01P &= \text{ decrease in population} \\ P(0) &= \text{ initial condition} \end{aligned}$$

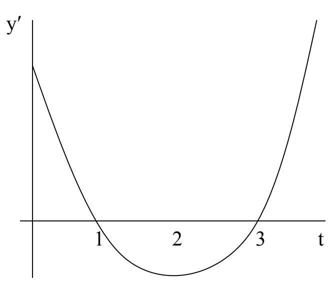
Check that the solution is $P(t) = 5000 - 4000e^{-0.01t}$.

$$\begin{aligned} P'(t) &= -4000(-0.01)e^{-0.01t} \\ &= 40e^{-0.01t} \\ 50 - 0.01P &= 50 - 0.01(5000 - 4000e^{-0.01t}) \\ &= 50 - 50 + 40e^{-0.01t} \\ &= 40e^{-0.01t} \\ P(0) &= 5000 - 4000 \\ &= 1000 \end{aligned}$$



Graphical solutions of differential equations

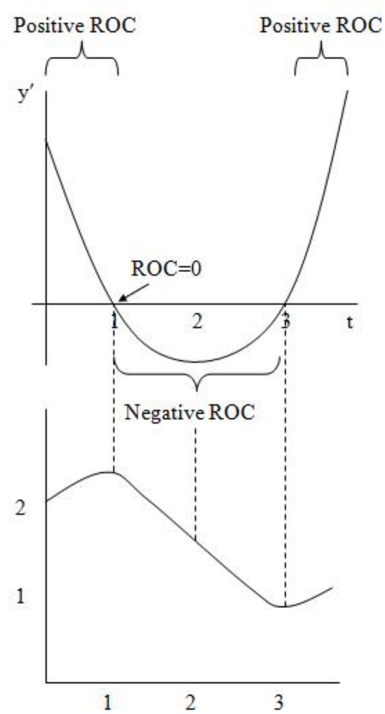
Eg) Suppose the derivative looks like this graph. Draw the original function satisfying $y(0) = 2$.



Remember

- $y' > 0 \quad \rightarrow y$ is increasing
- $y' = 0 \quad \rightarrow y$ has a critical point
- $y' < 0 \quad \rightarrow y$ is decreasing

ROC=rate of change



8.1 Solving Differential Equations

A differential equation is of the form $\frac{dF}{dt} = f(t)$ where we know $f(t)$ but not $F(t)$. To solve for $F(t)$, we integrate $f(t)$ with respect to t. Thus, the solution is

$$F(t) = \int f(t)dt + c \qquad \text{where } c \text{ is an arbitrary constant}$$

Definition 8.1. *The function F is called the* <u>*antiderivative*</u> *or* <u>*integral*</u> *of f. The integral function is the opposite of the derivative, except for the arbitrary constant. So*

$$\int \frac{dg}{dt}dt = g(t) + c \qquad \textit{and} \qquad \frac{d}{dt}\int g(t)dt = g(t)$$

The arbitrary constant occurs because the derivative of any constant is zero, so we can't rule out the existence of one when integrating. It turns out to be quite important, so don't forget it.

Eg) Find $\int 2xdx$.

$$\begin{aligned} y &= x^2 \\ y' &= 2x \\ f(x) &= 2x \\ f(x) &= \int 2xdx \\ &= x^2 + c \\ \text{since } \frac{d}{dx}(x^2+1) &= 2x \end{aligned}$$

Eg)

$$\begin{aligned} \int x^3 dx &= \frac{x^4}{4} + c && \text{since } \frac{d}{dx}(x^4) = 4x^3 \\ \int x dx &= \frac{x^2}{2} + c && \text{since } \frac{d}{dx}(x^2) = 2x \\ \int x^{37} dx &= \frac{x^{38}}{38} + c && \text{since } \frac{d}{dx}(x^{38}) = 38x^{37} \\ \text{Power rule: } \int x^n dx &= \frac{x^{n+1}}{n+1} + c \text{ if } n \neq -1 \end{aligned}$$

Eg) Find $\int \frac{1}{x^3}dx$.

$$\begin{aligned} \int \frac{1}{x^3}dx &= \int x^{-3}dx \\ &= \frac{x^{-3+1}}{-3+1} + c \\ &= \frac{x^{-2}}{-2} + c \\ &= -\frac{1}{2x^2} + c \end{aligned}$$

Eg) Find $\int \frac{1}{\sqrt[3]{x}}dx$.

$$\begin{aligned} \int \frac{1}{\sqrt[3]{x}}dx &= \int x^{-\frac{1}{3}}dx \\ &= \frac{x^{-\frac{1}{3}+1}}{-\frac{1}{3}+1} + c \\ &= \frac{x^{\frac{2}{3}}}{\frac{2}{3}} + c \\ &= \frac{3x^{\frac{2}{3}}}{2} + c \end{aligned}$$

Integral rules: Suppose that $\int f(x)dx = F(x) + c$ and $\int g(x)dx = G(x) + d$. Then,

$$\begin{aligned}
\int af(x)dx &= a\int f(x)dx && \text{constant products are preserved} \\
&= aF(x) + ac \\
&= aF(x) + c' \\
\int [f(x) + g(x)]dx &= \int f(x)dx + \int g(x)dx && \text{integral of a sum is the sum of integrals} \\
&= F(x) + c + G(x) + d \\
&= F(x) + G(x) + e
\end{aligned}$$

Eg)

$$\int (3x^7 - 2x^{-5} + 9)dx = \frac{3x^8}{8} - \frac{2x^{-4}}{-4} + 9x + c = \frac{3x^8}{8} + \frac{x^{-4}}{2} + 9x + c$$

Eg) $\frac{dP}{dt} = 50$ from before. Find $P(t)$.

$$\begin{aligned}
\therefore P(t) &= \int 50dt \\
&= 50t + c \\
P(0) &= 50(0) + c = 1000 \\
c &= 1000 \\
\therefore P(t) &= 50t + 1000
\end{aligned}$$

Remember that velocity is the derivative of position: $v = \frac{dP}{dt}$ and acceleration is the derivative of velocity: $a = \frac{dv}{dt}$

Eg) A rock is thrown down from a building with initial speed 4 m/s. Find its velocity as a function of time.

$$\begin{aligned}
a &= -9.8 && \text{due to gravity} \\
\text{initial condition } v(0) &= -4 && \text{since direction is down} \\
\frac{dV}{dt} &= -9.8 \\
v &= \int -9.8dt \\
&= -9.8t + c \\
v(0) &= c = -4 \\
\therefore v(t) &= -9.8t - 4
\end{aligned}$$

Eg) Now suppose the building is 150 m. tall. When will the rock hit the ground and how fast will it be travelling when it does?

$$
\begin{aligned}
P(0) &= 150 \\
\frac{dP}{dt} &= -9.8t - 4 \\
P(t) &= \int (-9.8t - 4)dt \\
&= -9.8\frac{t^2}{2} - 4t + c \\
&= -4.9t^2 - 4t + c \\
P(0) &= c = 150 \\
P(t) &= -4.9t^2 - 4t + 150
\end{aligned}
$$

The rock hits the ground when $P(t) = 0$

$$
\begin{aligned}
-4.9t^2 - 4t + 150 &= 0 \\
4.9t^2 + 4t - 150 &= 0 \\
t &= \frac{-4 \pm \sqrt{4^2 - 4(4.9)(-150)}}{9.8} \\
&= \frac{-4 \pm \sqrt{2956}}{9.8} \\
&= 5.1397 \quad \text{or} \quad -5.956
\end{aligned}
$$

Therefore, the rock hits the ground after 5.1 seconds.
At this time, the velocity is $v(5.1397) = -9.8(5.1397) - 4 = -54.369$ m/s

Eg) During the earlier years of the AIDS epidemic, the US CDC reported that the number of new cases per year in the US satisfied

$$\text{rate of new AIDS cases } \approx 523.8t^2$$

where t is the time in years since the beginning of 1981. There were 340 people infected at the end of 1980 and there were 433,760 at the end of 2005. Therefore, is this model accurate?

$$
\begin{aligned}
\frac{dA}{dt} &= 523.8t^2 \\
A(t) &= \int 523.8t^2 dt \\
&= 523.8\frac{t^3}{3} + c \\
&= 174.6t^3 + c \\
A(0) &= c = 340 \\
\therefore A(t) &= 174.6t^3 + 340
\end{aligned}
$$

In the 26 years between 1980 and 2006, we expect

$$
\begin{aligned}
A(25) &= 174.6(25)^3 + 340 \\
&= 2,728,465 \text{ people living with AIDS}
\end{aligned}
$$

However, there were only 433,760 people. What happened to the rest?
Answer: Death: 550,394 people have died of HIV/AIDS in the US.
Therefore, the actual total is 984,154.
Therefore, education, prevention and treatment have likely saved 2 million lives.

8.2 Integration Techniques

Remember that integration is essentially an educated guess. If we want to integrate an arbitrary function, we need to find another function whose derivative is the function we want. So let's look at some derivatives that we know.

$$\begin{aligned}
\frac{d}{dx}e^x &= e^x \\
\frac{d}{dx}\sin x &= \cos x \\
\frac{d}{dx}\cos x &= -\sin x \\
\frac{d}{dx}\ln x &= \frac{1}{x}
\end{aligned}$$

Therefore, we have some integrals:

$$\begin{aligned}
\int e^x dx &= e^x + c \\
\int \cos x dx &= \sin x + c \\
\int \sin x dx &= -\cos x + c \\
\text{But } \int \frac{1}{x} dx &\neq \ln x + c \qquad \text{Why not?}
\end{aligned}$$

Answer: The domain of $\frac{1}{x}$ is $x < 0$ and $x > 0$ whereas the domain of $\ln x$ is $x > 0$.
We fix this by using absolute values:

$$\int \frac{1}{x} dx = \ln|x| + c$$

Let's check:

$$\begin{aligned}
\ln|x| &= \begin{cases} \ln x & x > 0 \\ \ln(-x) & x < 0 \end{cases} \\
\frac{d}{dx}\ln|x| &= \begin{cases} \frac{1}{x} & x > 0 \\ \frac{1}{-x}(-1) & x < 0 \end{cases} \\
&= \begin{cases} \frac{1}{x} & x > 0 \\ \frac{1}{x} & x < 0 \end{cases} \\
&= \frac{1}{x}
\end{aligned}$$

Eg)

$$\int \left(3e^x - 5\cos x + \frac{7}{x}\right) dx = 3\int e^x dx - 5\int \cos x dx + 7\int \frac{1}{x} dx = 3e^x - 5\sin x + 7\ln|x| + c$$

Substitution: One trick for dealing with more complicated integrals is to make a substitution for part of the expression.

Eg) $\int 2x(x^2+3)^8 dx$

Instead of multiplying out, let $u = x^2 + 3$. Our goal is to get rid of all the x terms (including the dx) and have an expression using only u's.

$$\begin{aligned}
u &= x^2+3 \\
\frac{du}{dx} &= 2x \\
dx &= \frac{du}{2x} \\
\int 2x(x^2+3)^8 dx &= \int 2xu^8 \frac{du}{2x} = \int \cancel{2x} \frac{u^8 du}{\cancel{2x}} \\
&= \int u^8 du \\
&= \frac{u^9}{9} + c \\
&= \frac{(x^2+3)^9}{9} + c
\end{aligned}$$

Once we decide on a substitution, we need to take the derivative, because we need to isolate dx. Don't forget to put the answer in terms of the original variable.

Eg) Find $\int x^2 e^{x^3} dx$.

Possible Substitutions

- $u = x^2 e^{x^3}$ - Can't eliminate
- $u = x^3$ - Derivative is present
- $u = x^2$ - Derivative isn't present, can't integrate when we substitute
- $u = e^{x^3}$ - Derivative is present, but not obvious
- $u = x$ - Only results in a change of variable

$$\begin{aligned}
\int x^2 e^{x^3} dx &= \int x^2 e^u \frac{du}{3x^2} \\
&= \int \frac{e^u}{3} du \\
&= \frac{1}{3} \int e^u du \\
&= \frac{1}{3} e^u + c \\
&= \frac{1}{3} e^{x^3} + c
\end{aligned}$$

$$\begin{aligned}
u &= x^3 \\
\frac{du}{dx} &= 3x^2 \\
dx &= \frac{du}{3x^2}
\end{aligned}$$

Eg) Find $\int \tan x dx$.

$$\begin{aligned}
\int \tan x dx &= \int \frac{\sin x}{\cos x} dx \\
&= \int \frac{\sin x}{u} \frac{du}{-\sin x} \\
&= -\int \frac{1}{u} du \\
&= -\ln|u| + c \\
&= -\ln|\cos x| + c
\end{aligned}$$

$$\begin{aligned}
u &= \cos x \\
\frac{du}{dx} &= -\sin x \\
dx &= -\frac{du}{\sin x}
\end{aligned}$$

Eg) Find $\int \frac{e^{3x}}{e^{3x}-1} dx$.

$$\int \frac{e^{3x}}{e^{3x}-1}dx = \int \frac{e^{3x}}{u}\frac{du}{3e^{3x}}$$
$$= \frac{1}{3}\int \frac{1}{u}du$$
$$= \frac{1}{3}\ln|u| + c$$
$$= \frac{1}{3}\ln|e^{3x}-1| + c$$

$$u = e^{3x} - 1$$
$$\frac{du}{dx} = 3e^{3x}$$
$$dx = \frac{du}{3e^{3x}}$$

Eg) Find $\int e^{-2x}dx$.

$$\int e^{-2x}dx = \int e^{u}\frac{du}{2}$$
$$= -\frac{1}{2}\int e^{u}du$$
$$= -\frac{1}{2}e^{u} + c$$
$$= -\frac{1}{2}e^{-2x} + c$$

$$u = -2x$$
$$\frac{du}{dx} = -2$$
$$dx = \frac{du}{-2}$$

Eg) Find $\int \sin(4t-3)dt$.

$$\int \sin(4t-3)dt = \int \sin u\frac{du}{4}$$
$$= \frac{1}{4}\int \sin u du$$
$$= -\frac{1}{4}\cos u + c$$
$$= -\frac{1}{4}\cos(4t-3) + c$$

$$u = 4t - 3$$
$$\frac{du}{dt} = 4$$
$$dt = \frac{du}{4}$$

Substitution depends on (a) guessing the right thing to substitute and (b) hoping that all the original variables cancel. However, substitutions can always be used to eliminate constants.

Eg) The number of people in a village not infected by ebola decreases at rate $\frac{dy}{dt} = -4^t$ where t is the time in days. If there are 100,000 people in the village initially, how long before everyone is infected?

$$\frac{dy}{dt} = -4^t = -e^{\ln 4^t} = -e^{t\ln 4}$$
$$y = -\int e^{t\ln 4}dt$$
$$= -\int e^{u}\frac{du}{\ln 4}$$
$$= -\frac{1}{\ln 4}\int e^{u}du$$
$$= -\frac{1}{\ln 4}e^{u} + c$$
$$= -\frac{1}{\ln 4}e^{t\ln 4} + c$$

$$u = t\ln 4$$
$$\frac{du}{dt} = \ln 4$$
$$dt = \frac{du}{\ln 4}$$

$$
\begin{aligned}
y(0) = -\frac{1}{\ln 4} + c &= 100,000 \\
c &= 100,000 + \frac{1}{\ln 4} \\
&= 100,000.7213 \\
y &= -\frac{1}{\ln 4}e^{t\ln 4} + 100,000 + \frac{1}{\ln 4} \\
y = 0 \to \frac{1}{\ln 4}e^{t\ln 4} &= 100,000.7213 \\
e^{t\ln 4} &= 100,000.7213 \cdot \ln 4 \\
t\ln 4 &= \ln(100,000.7213\ln 4) \\
t &= \frac{\ln(100,000.7213\ln 4)}{\ln 4} \\
&= 8.54 \text{ days}
\end{aligned}
$$

Question: When does substitution not work?

Answer:

1) When we can't cancel all of the original variables.

2) When we can't integrate the new function.

Eg) $\int x^2 e^x dx$

Possible substitutions: $u = x^2$, $u = e^x$, $u = x$.

- $u = x^2$

$$
\begin{aligned}
\int x^2 e^x dx &= \int u e^x \frac{du}{2x} \\
&= \int u e^x \frac{du}{2x} \\
\text{But } u = x^2 &\text{ so } x = \sqrt{u} \\
&= \int u e^{\sqrt{u}} \frac{du}{2\sqrt{u}} \\
&= \frac{1}{2}\int u^{-\frac{1}{2}} e^{\sqrt{u}} du \\
&\text{We can't integrate this}
\end{aligned}
$$

$$
\frac{du}{dx} = 2x \qquad dx = \frac{du}{2x}
$$

- $u = e^x$

$$
\begin{aligned}
\int x^2 e^x dx &= \int x^2 u \frac{du}{e^x} \\
&= \int x^2 u \frac{du}{u} \\
&= \int x^2 du \\
&\text{Illegal - we need everything in terms of } u \\
x &= \ln u \\
&= \int (\ln u)^2 du \\
&\text{We can't integrate this}
\end{aligned}
$$

$$
\text{Since } u = e^x \qquad \frac{du}{dx} = e^x \qquad dx = \frac{du}{e^x}
$$

- $u = x$

$$\frac{du}{dx} = 1$$
$$dx = du$$
$$\int x^2 e^u du = \int u^2 e^u du$$

Same problem as original (just a change of variable)

So substitution just doesn't work in this case.
Integration by parts: Remember the product rule for derivatives:

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$
$$\therefore u\frac{dv}{dx} = \frac{d}{dx}(uv) - v\frac{du}{dx}$$
$$\int u\frac{dv}{dx}dx = \int \frac{d}{dx}(uv)dx - \int v\frac{du}{dx}dx$$
$$\int uv'dx = uv - \int vu'dx$$

since this is the direct integral of a derivative

Eg) $\int x \cos x dx$ Note that we can't do this with substitution.

$$u = x \qquad v' = \cos x$$
$$u' = 1 \qquad v = \sin x$$
$$\int x \cos x dx = x \sin x - \int 1 \sin x dx$$
$$= x \sin x - \cos x + c$$
$$\text{Check: } \frac{d}{dx}(x \sin x - \cos x + c) = \sin x + x \cos x - \sin x$$
$$= x \cos x$$

Eg) $\int x^4 \ln x dx$

$$u = x^4 \qquad v' = \ln x$$
$$u' = 4x^3 \qquad v = ???$$

This doesn't work.

$$u = \ln x \qquad v' = x^4$$
$$u' = \frac{1}{x} \qquad v = \frac{x^5}{5}$$
$$\int x^4 \ln x dx = \frac{x^5}{5} \ln x - \int \frac{1}{x}\frac{x^5}{5}dx$$
$$= \frac{x^5}{5} \ln x - \frac{1}{5}\int x^4 dx$$
$$= \frac{x^5}{5} \ln x - \frac{1}{5}\frac{x^5}{5} + c$$
$$= \frac{x^5}{5} \ln x - \frac{1}{25}x^5 + c$$

Eg) $\int x^2 e^x dx$

$$\begin{aligned} u &= e^x \qquad v' = x^2 \\ u' &= e^x \qquad v = \frac{x^3}{3} \\ \int x^2 e^x dx &= \frac{x^3}{3}e^x - \int \frac{x^3}{3} e^x dx \end{aligned}$$

This is a worse integral.

$$\begin{aligned} u &= x^2 \qquad v' = e^x \\ u' &= 2x \qquad v = e^x \\ \int x^2 e^x dx &= x^2 e^x - \int 2x e^x dx \end{aligned}$$

This is a better integral.

Do Integration by parts again:

$$\begin{aligned} u &= 2x \qquad v' = e^x \\ u' &= 2 \qquad v = e^x \\ \int x^2 e^x dx &= x^2 e^x - \left[2xe^x - \int 2e^x dx\right] \\ &= x^2 e^x - 2xe^x + 2e^x + c \end{aligned}$$

Eg) $\int \ln x dx$

$$\begin{aligned} u &= \ln x \qquad v' = 1 \\ u' &= \frac{1}{x} \qquad v = x \\ \int \ln x dx &= x \ln x - \int \frac{1}{x} x dx \\ &= x \ln x - \int 1 dx \\ &= x \ln x - x + c \end{aligned}$$

Or we can use substitution:

$$\begin{aligned} \text{Let } u &= \ln x \qquad x = e^u \\ \frac{du}{dx} &= \frac{1}{x} \\ dx &= x du \\ \int \ln x dx &= \int u x du = \int u e^u du \\ w &= u \qquad v' = e^u \\ w' &= 1 \qquad v = e^u \\ \int \ln x dx &= u e^u - \int e^u du \\ &= u e^u - e^u + c \\ &= \ln x e^{\ln x} - e^{\ln x} + c \\ &= x \ln x - x + c \end{aligned}$$

Eg) $I = \int e^x \sin x dx$

$$
\begin{aligned}
u &= e^x \qquad v' = \sin x \\
u' &= e^x \qquad v = -\cos x \\
I &= -e^x \cos x + \int e^x \cos x dx \\
u &= e^x \qquad v' = \cos x \\
u' &= e^x \qquad v = \sin x \\
I &= -e^x \cos x + \left[e^x \sin x - \int e^x \sin x dx \right] \\
I &= -e^x \cos x + e^x \sin x - \int e^x \sin x dx
\end{aligned}
$$

$\int e^x \sin x dx$ is the original problem

$$
\begin{aligned}
I &= -e^x \cos + e^x \sin x - I \\
2I &= -e^x \cos x + e^x \sin x \\
I &= \frac{1}{2}[-e^x \cos x + e^x \sin x] + c
\end{aligned}
$$

Eg) $\int t^{-2}(4t - 3t^3)dt$

$$
\begin{aligned}
u &= t^{-2} \qquad v' = 4t - 3t^3 \\
u' &= -2t^{-3} \qquad v = 2t^2 - \frac{3t^4}{4} \\
I &= t^{-2}\left(2t^2 - \frac{3t^2}{4}\right) + \int 2t^{-3}\left(2t^2 - \frac{3t^4}{4}\right) dt
\end{aligned}
$$

worse integral

$$
\begin{aligned}
u &= 4t - 3t^3 \qquad v' = t^{-2} \\
u' &= 4 - 9t^2 \qquad v = -t^{-1} \\
I &= t^{-1}(4t - 3t^3) - \int t^{-1}(4 - 9t^2)dt \\
u &= 4 - 9t^2 \qquad v' = t^{-1} \\
u' &= -18t \qquad v = \ln |t| \\
I &= t^{-1}(4t - 3t^2) - \left[(4 - 9t^2) \ln t + 18 \int t \ln |t| dt \right]
\end{aligned}
$$

complicated

What about substitution?

$$
\begin{aligned}
u &= 4t - 3t^3 \\
\frac{du}{dt} &= 4 - 9t^2 \\
\int t^{-2}(4t - 3t^3)dt &= \int t^{-2} u \frac{du}{4 - 9t^2}
\end{aligned}
$$

doesn't simplify

$$
\begin{aligned}
u &= 4 - 3t^2 \\
\frac{du}{dt} &= -6t \\
dt &= \frac{du}{-6t} \\
\int t^{-2}(4t - 3t^3)dt &= \int t^{-1} u \frac{du}{-6t} \\
&= -\frac{1}{6} \int t^{-2} u du \\
&= -\frac{1}{6} \int \frac{3u}{4 - u} \qquad \text{because } t^2 = \frac{4 - u}{3}
\end{aligned}
$$

Can't solve this

Easy answer: expand the brackets

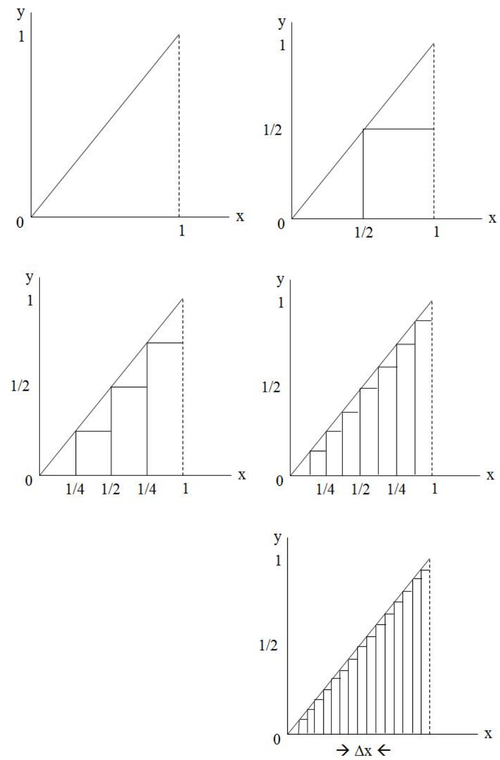
$$
\begin{aligned}
\int t^{-2}(4t - 3t^3)dt &= \int \frac{4t - 3t^3}{t^2} dt \\
&= \int \left(\frac{4}{t} - 3t\right) dt \\
&= 4 \ln |t| - \frac{3t^2}{2} + c
\end{aligned}
$$

Be aware: sometimes the simple answer is staring you in the face and you can miss it.

8.3 Riemann Sums

A derivative is a rate of change. What's an integral?
Answer: An integral is an area.
How to find the area under a triangle?
Secret knowledge: $A = \frac{bh}{2} = \frac{1 \cdot 1}{2} = \frac{1}{2}$
Approximately rectangles:

$$
\begin{aligned}
A_1 &\approx 0 + \frac{1}{2} \cdot f\left(\frac{1}{2}\right) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4} = 0.25 \\
A_2 &\approx 0 + \frac{1}{4} \cdot f\left(\frac{1}{4}\right) + \frac{1}{4} \cdot f\left(\frac{1}{2}\right) + \frac{1}{4} \cdot f\left(\frac{3}{4}\right) \\
&= \frac{1}{4} \cdot \frac{1}{4} + \frac{1}{4} \cdot \frac{1}{2} + \frac{1}{4} \cdot \frac{3}{4} \\
&= \frac{3}{8} = 0.375 \\
A_3 &\approx \frac{1}{8}\left[f\left(\frac{2}{8}\right) + f\left(\frac{3}{8}\right) + \cdots + f\left(\frac{7}{8}\right)\right] \\
&= \frac{7}{16} = 0.4375 \\
A_n &\approx \Delta x \sum_{i=0}^{n-1} f(x_i) \\
x_0 &= 0,\ x_1 = 0 + \Delta x,\ x_2 = 0 + 2\Delta x, \ldots
\end{aligned}
$$



Definition 8.2. *The Riemann integral of a function f on the interval from a to b is*

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=0}^{n} f(x_i)\Delta x$$

where the values $x_0, x_1, ..., x_n$ break the interval from a to b into n pieces, each of length $\Delta x = \frac{b-a}{n}$

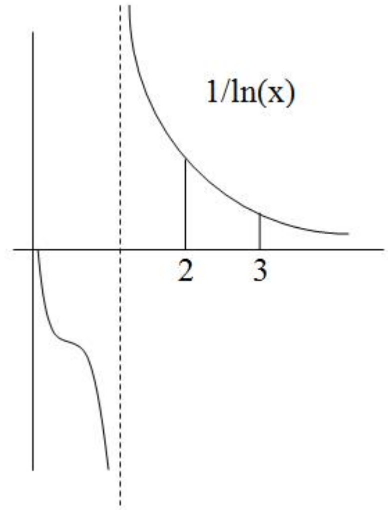
Eg)

$$
\begin{aligned}
\int_0^1 x dx &= \left[\frac{x^2}{2} + c\right]_0^1 \rightarrow \text{This means evaluate the answer at 1 minus evaluating the answer at 0} \\
&= \left(\frac{1^2}{c} + c\right) - \left(\frac{0^2}{2} + c\right) \\
&= \frac{1}{2} \text{ precisely}
\end{aligned}
$$

Eg) $\int_2^3 \frac{1}{\ln x} dx$ We can't integrate this function.

Break into five pieces: $\Delta x = \dfrac{3-2}{5} = \dfrac{1}{5}$

$$x_0 = f(2) = \frac{1}{\ln 2}$$
$$x_1 = f\left(2+\frac{1}{5}\right) = \frac{1}{\ln\left(\frac{11}{5}\right)}$$
$$x_2 = f\left(2+\frac{2}{5}\right) = \frac{1}{\ln\left(\frac{12}{5}\right)}$$
$$x_3 = f\left(2+\frac{3}{5}\right) = \frac{1}{\ln\left(\frac{13}{5}\right)}$$
$$x_4 = f\left(2+\frac{4}{5}\right) = \frac{1}{\ln\left(\frac{14}{5}\right)}$$
$$\int_2^3 \frac{1}{\ln x}dx \approx \frac{1}{5}\left[\frac{1}{\ln 2} + \frac{1}{\ln\left(\frac{11}{5}\right)} + \frac{1}{\ln\left(\frac{12}{5}\right)} + \frac{1}{\ln\left(\frac{13}{5}\right)} + \frac{1}{\ln\left(\frac{14}{5}\right)}\right]$$
$$= 1.1742$$



Theorem 8.3. *(The Fundamental Theorem of Calculus) Suppose* $\frac{dF}{dt} = f(t)$*. The indefinite integral is* $\int f(t)dt = F(t) + c$*.*

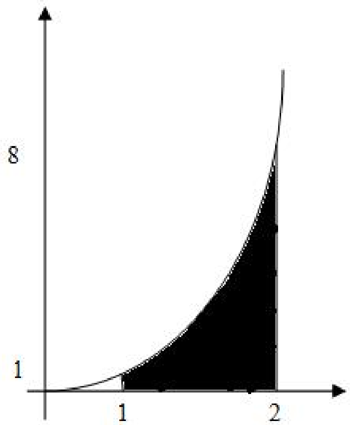
The definite integral is

$$\int_a^b f(t)dt = [F(t)+c]_{evaluated\ at\ b} - [F(t)+c]_{evaluated\ at\ a}$$
$$= (F(b)+c) - (F(a)+c) = F(b) - F(a).$$

Eg)

$$\int_1^2 x^3 dx$$

$$\int_1^2 x^3 dx = \left.\frac{x^4}{4}\right|_1^2$$
$$= \frac{2^4}{4} - \frac{1^4}{4}$$
$$= \frac{16}{4} - \frac{1}{4}$$
$$= 4 - \frac{1}{4}$$
$$= 3\frac{3}{4}$$
$$= \frac{15}{4}$$
$$= 3.753$$



Eg)

$$\int_0^\pi \sin x dx$$

$$\begin{aligned}\int_0^\pi \sin x dx &= [-\cos x]_0^\pi \\ &= (-\cos\pi) - (-\cos 0) \\ &= 1 + 1 \\ &= 2\end{aligned}$$

9 Review

Fundamentals

- Absolute value
- Exponential/trig functions/logarithms

Discrete-time dynamical systems

- Cobwebbing

Derivatives

- Limits
- Continuity
- Differentiability
- Product Rule
- Second Derivative
- Chain Rule

Graphing Techniques - Stability

- Logistic equation
- Maximum and minimum
- Intermediate value theorem
- Limits at infinity
- L'Hopital's rule

Approximations

- Tangent line approximation
- Taylor polynomials
- Newton's method

Differential Equations

- Integration
- Substitution
- Integration by parts
- Riemann sums
- Fundamental theorem of calculus

Example Sketch $y = x^2 \ln x$.
Example Find $\int x^2 \ln x dx$.