University of Ottawa		Université d'Ottawa
Faculty of Engineering		Faculté de genie
Department of	uOttawa	Départment de
Civil Engineering	L'Université canadienne Canada's university	Génie Civil

CVG 2181

Numerical Methods in Civil Engineering

Final Exam **April 27th, 2010**

Time: **3 Hours**

OPEN BOOK. Non-programmable calculators are permitted.

If you do not understand a question, clearly state an assumption and proceed.

In all problems, use **<u>four digits</u>** after the decimal point.

Answer all questions. Mark distribution is provided on the exam.

At the end of the exam, when time is up:

- Stop working and turn your exam upside down.
- Remain silent.
- **<u>Do not move or speak until all exams have been picked up</u>**.

Questions (20 points) Briefly answer the following questions:

Q1(3) What is the difference between implicit and explicit methods? Which one is more stable and which one is faster? How can you prevent instability of explicit schemes?

Answer: In implicit methods, the unknowns are on both sides of the equation and usually a system of equations should be solved. In explicit methods, the unknowns can be found directly. Implicit methods are more stable and explicit methods are faster. One should use a small step size in explicit methods for stability.

Q2(2) Use the first order forward difference method for the time derivative and an implicit second order centered scheme for the space derivative to discretize this equation:

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}$$

Answer:

$$\frac{T_i^{t+\Delta t} - T_i^t}{\Delta t} = \alpha \left(\frac{T_{i+1}^t - 2T_i^t + T_{i-1}^t}{\Delta x^2} \right)$$

Q3(2) What is the main difference between one-step and multi-step methods? Is the fifth order Runge-Kutta method a one-step method or a multi-step method?

Answer: The multistep methods use the results of previous steps to find the new value while in single step methods, initial values only at one time are used. RK5 is a single step method.

Q4(2) What is the purpose of using modifiers in solving ODEs?

Answer: They improve accuracy and stability.

Q5(2) What is the slope used in the Heun method? How can you improve it?

Answer: An average of the slope at points i and i+1. It can be improved by multiple iterations on the corrector step.

Q6(2) What are the two approaches for estimation of error in adaptive methods?

Answer: Reducing the step size and increasing the order of accuracy.

Q7(2) You are going to use the forth order Adams Molton method. (a) How many steps do you need to calculate with a one-step method before you can start Adams-Molton method? (b) What numerical method do you recommend for those initial steps? Why?

Answer: You must generate three points. RK4 is recommended to keep the order of accuracy equal to four.

Q8(2) Which of the following statements is wrong?

(a) Iterative methods lead to less round-off errors and need less storage.
(b) The result of the Newton method for non-linear systems may depend on initial conditions.
(c) A singular matrix can be consistent.
(d) It is not necessary to sort the data in ascending order for the Newton's polynomial.
(e) All statements are true.

Answer: (e)

Q9(2) For what polynomials the 1/3 Simpson method leads to an exact solution? How much will the error decrease if you divide the grid size by 2?

Answer: It is exact for polynomials of degree up to 3. The global error is of order 4, so the error is approximately reduced by 2^4=16.

Q10(1) How can you solve the following non-linear problem using Excel?

f1(x,y)=0
f2(x,y)=0

Answer: Using solver, you can minimize $f1^2+f2^2$

P1 (20 points)

The following data is available

x	f(x)
0	1
1	3
3	5
4	4

You are going to estimate the value of the function at x=2 using **cubic splines.** The boundary conditions are as following

- The second derivative at the beginning of the first segment is equal to zero.
- The second derivative at the end of the last segment is equal to the second derivative at the beginning of that segment.

(1) Find the cubic polynomial needed to calculate the value of the function at x=2.

(2) Evaluate the value of the function at x=2.

$$h_0 = 1$$
$$h_1 = 2$$
$$h_2 = 1$$
$$2(h_0 + h_1) = 6$$
$$2(h_1 + h_2) = 6$$
$$S = \begin{bmatrix} h_0 & 2(h_0+h_1) & h_1 & 0 \\ 0 & h_1 & 2(h_1+h_2) & h_2 \end{bmatrix} = \begin{bmatrix} 1 & 6 & 2 & 0 \\ 0 & 2 & 6 & 1 \end{bmatrix}$$

$$f[x_0, x_1] = (3-1)/(1-0) = 2$$
$$f[x_1, x_2] = (5-3)/(3-1) = 1$$
$$f[x_2, x_3] = (4-5)/(4-3) = -1$$
$$RHS = 6\begin{bmatrix} f[x_1, x_2] - f[x_0, x_1] \\ f[x_2, x_3] - f[x_1, x_2] \end{bmatrix} = \begin{bmatrix} -6 \\ -12 \end{bmatrix}$$
$$\Rightarrow \begin{bmatrix} 1 & 6 & 2 & 0 \\ 0 & 2 & 6 & 1 \end{bmatrix} \begin{bmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{bmatrix} = \begin{bmatrix} -6 \\ -12 \end{bmatrix}$$
$$\Rightarrow \begin{cases} S_0 + 6S_1 + 2S_2 = -6 \\ 2S_1 + 6S_2 + S_3 = -12 \end{cases}$$
and
$$S_0 = 0$$
$$S_3 = S_2$$
Thus
$$\begin{cases} 6S_1 + 2S_2 = -6 \\ 2S_1 + 7S_2 = -12 \end{cases}$$
$$\Rightarrow S_1 = -0.4737 \quad and \quad S_2 = -1.5790$$

Coefficients :
$$a_1 = \frac{S_2 - S_1}{6h_1} = \frac{-1.5789 - (-0.4737)}{6 \times 2} = -0.0921$$
$$b_1 = \frac{S_1}{2} = -0.2369$$
$$c_1 = \frac{y_2 - y_1}{h_1} - \frac{2h_1S_1 + h_1S_2}{6} = 1.8421$$
$$d_1 = y_1 = 3$$
Thus
$$g_1(x) = -0.0921(x-1)^3 - 0.2369(x-1)^2 + 1.8421(x-1) + 3$$
$$\Rightarrow g_1(2) = 4.5131$$

P2 (15 points)

Consider the following ODE

$$\frac{dy}{dx} = 3y - 15x + 5 \quad , \quad y(0) = 1$$

(a) Use the First order Euler method to calculate the value of y at x=0.3 using a step size Δx=0.1

(b) Then use the Fourth order Adams method with two iterations on the corrector step to calculate the value of y at x=0.4.

$h = 0.1$

$y_0 = 1$

$x_0 = 0$

$f_0 = 3y_0 - 15x_0 + 5 = 3\times 1 - 15\times 0 + 5 = 8$

Step 1

$y_1 = y_0 + h\times f_0 = 1 + 0.1\times 8 = 1.8$

$x_1 = 0.1$

Step 2:

$f_1 = 3y_1 - 15x_1 + 5 = 8.9$

$y_2 = y_1 + h\times f_1 = 2.69$

$x_2 = 0.2$

Step 3:

$f_2 = 3y_2 - 15x_2 + 5 = 10.07$

$y_3 = y_2 + h\times f_2 = 3.697$

$x_3 = 0.3$

Step 4:

Predictor:

$$y_{i+1}^0 = y_i^m + h\left(\frac{55}{24} f_i^m - \frac{59}{24} f_{i-1}^m + \frac{37}{24} f_{i-2}^m - \frac{9}{24} f_{i-3}^m\right)$$

$$y_4^0 = y_3 + h(\frac{55}{24} f_3 - \frac{59}{24} f_2 + \frac{37}{24} f_1 - \frac{9}{24} f_0)$$

$$\Rightarrow y_4^0 = 3.697 + \frac{0.1}{24}(55 \times 11.591 - 59 \times 10.07 + 37 \times 8.9 - 9 \times 8) = 4.9498$$

$$x_4 = 0.4$$

$$f_4^0 = 3y_4^0 - 15x_4 + 5 = 13.8494$$

Corrector, first iteration:

$$y_4^1 = y_3 + h(\frac{9}{24} f_4^0 + \frac{19}{24} f_3 - \frac{5}{24} f_2 + \frac{1}{24} f_1)$$

$$\Rightarrow y_4^1 = 3.697 + \frac{0.1}{24}(9 \times 13.8494 + 19 \times 11.591 - 5 \times 10.07 + 8.9) = 4.9613$$

$$f_4^1 = 3y_4^1 - 15x_4 + 5 = 13.8838$$

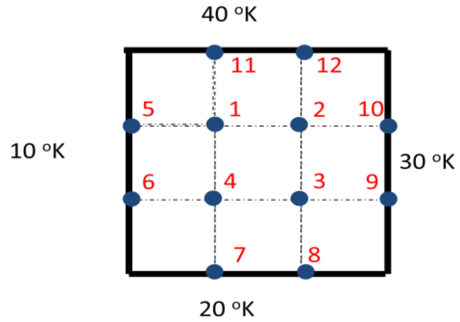
Corrector, second iteration:

$$y_4^2 = y_3 + h(\frac{9}{24} f_4^1 + \frac{19}{24} f_3 - \frac{5}{24} f_2 + \frac{1}{24} f_1)$$

$$\Rightarrow y_4^1 = 3.697 + \frac{0.1}{24}(9 \times 13.8839 + 19 \times 11.591 - 5 \times 10.07 + 8.9) = 4.9626$$

P3 (15 points)

Consider a 3m x 3m concrete slab, with an initial temperature of 5 °C. The temperature at the



Using the unsteady heat conduction equation

$$\frac{\partial T}{\partial t} = \alpha\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right)$$

with Δt=0.1 and α=0.5,

(a) Find the temperature at points 1, 2, 3 and 4 at time t=0.1

(b) Then find the temperature at point 1 at time t=0.2.

Use the first order forward difference method for the time derivative and an explicit second order centered scheme for the space derivative.

$$T_1^0 = T_2^0 = T_3^0 = T_4^0 = 5$$

$$\frac{T_1^{t+\Delta t} - T_1^t}{\Delta t} = \alpha\left(\frac{T_2^t - 2T_1^t + T_5^t}{\Delta x^2} + \frac{T_{11}^t - 2T_1^t + T_4^t}{\Delta y^2}\right)$$

Therefore

$$T_1^{t+\Delta t} = T_1^t + \alpha\Delta t\left(\frac{T_2^t - 2T_1^t + T_5^t}{\Delta x^2} + \frac{T_{11}^t - 2T_1^t + T_4^t}{\Delta y^2}\right)$$

$$\Rightarrow T_1^{0.1} = T_1^0 + \alpha\Delta t\left(\frac{T_2^0 - 2T_1^0 + T_5^0}{\Delta x^2} + \frac{T_{11}^0 - 2T_1^0 + T_4^0}{\Delta y^2}\right)$$

$$\Rightarrow T_1^{0.1} = 5 + 0.5\times 0.1\left(\frac{5 - 2\times 5 + 10}{1^2} + \frac{40 - 2\times 5 + 5}{1^2}\right) = 7$$

Similarly

$$T_2^{t+\Delta t} = T_2^t + \alpha\Delta t\left(\frac{T_{10}^t - 2T_2^t + T_1^t}{\Delta x^2} + \frac{T_{12}^t - 2T_2^t + T_3^t}{\Delta y^2}\right)$$

$$T_3^{t+\Delta t} = T_3^t + \alpha\Delta t\left(\frac{T_9^t - 2T_3^t + T_4^t}{\Delta x^2} + \frac{T_2^t - 2T_3^t + T_8^t}{\Delta y^2}\right)$$

$$T_4^{t+\Delta t} = T_4^t + \alpha\Delta t\left(\frac{T_3^t - 2T_4^t + T_6^t}{\Delta x^2} + \frac{T_1^t - 2T_4^t + T_7^t}{\Delta y^2}\right)$$

Which lead to

$$T_2^{0.1} = 5 + 0.5\times 0.1(30 - 2\times 5 + 5 + 40 - 2\times 5 + 5) = 8$$
$$T_3^{0.1} = 5 + 0.5\times 0.1(30 - 2\times 5 + 5 + 5 - 2\times 5 + 20) = 7$$
$$T_4^{0.1} = 5 + 0.5\times 0.1(5 - 2\times 5 + 10 + 5 - 2\times 5 + 20) = 6$$

and

$$T_1^{0.2} = T_1^{0.1} + \alpha\Delta t\left(\frac{T_2^{0.1} - 2T_1^{0.1} + T_5^{0.1}}{\Delta x^2} + \frac{T_{11}^{0.1} - 2T_1^{0.1} + T_4^{0.1}}{\Delta y^2}\right)$$
$$\Rightarrow T_1^{0.2} = 7 + 0.5\times 0.1(8 - 2\times 7 + 10 + 40 - 2\times 7 + 6) = 8.8$$

P4 (15 points)

We want to calculate

$$I = \int_{-1}^{1}\int_{-1}^{1} \frac{1}{2+xy}dydx$$

by the two-point Gauss quadrature, using the fact that $I = \int_{-1}^{1} f(x)dx$ where $f(x) = \int_{-1}^{1} \frac{1}{2+xy}dy$.

a) Calculate $f\left(-\frac{1}{\sqrt{3}}\right)$ and $f\left(\frac{1}{\sqrt{3}}\right)$ (9 points)

b) Calculate I (6 points)

Note: $\frac{1}{\sqrt{3}} \approx 0.5774$

Solution:

$c_1 = c_2 = 1$

$t_1 = -\frac{1}{\sqrt{3}}; t_2 = \frac{1}{\sqrt{3}}$

a)

$$f\left(-\frac{1}{\sqrt{3}}\right) = \int_{y=-1}^{1} \frac{1}{2+\left(-\frac{1}{\sqrt{3}}\right)y} = \frac{1}{2+\left(-\frac{1}{\sqrt{3}}\right)\left(-\frac{1}{\sqrt{3}}\right)} + \frac{1}{2+\left(-\frac{1}{\sqrt{3}}\right)\left(\frac{1}{\sqrt{3}}\right)} = 1.0285$$

$$f\left(\frac{1}{\sqrt{3}}\right) = \int_{y=-1}^{1} \frac{1}{2+\left(\frac{1}{\sqrt{3}}\right)y} = \frac{1}{2+\left(\frac{1}{\sqrt{3}}\right)\left(-\frac{1}{\sqrt{3}}\right)} + \frac{1}{2+\left(\frac{1}{\sqrt{3}}\right)\left(\frac{1}{\sqrt{3}}\right)} = 1.0285$$

b)

$$I = \int_{-1}^{1} f(x)dx = f\left(-\frac{1}{\sqrt{3}}\right) + f\left(+\frac{1}{\sqrt{3}}\right) = 2.0571$$

P5 (15 points)

Calculate two $O(h^6)$ estimates of the derivative of $f(x) = \frac{x}{1+x^2}$ at x=0 using a) $O(h^2)$ central difference and Richardson extrapolation and b) $O(h^4)$ central difference and Richardson extrapolation. Begin with h=0.4.

Solution:

a) **Second Order Central difference :**

$$f'(x) = \frac{f(x+h) - f(x-h)}{2h}$$

1- $O(h^2)$

$$h = 0.4$$

$$f'(0) = \frac{f(0.4) - f(-0.4)}{2(0.4)} = 0.8620$$

$$h = 0.2$$

$$f'(0) = \frac{f(0.2) - f(-0.2)}{2(0.2)} = 0.9615$$

$$h = 0.1$$

$$f'(0) = \frac{f(0.1) - f(-0.1)}{2(0.1)} = 0.9900$$

2- O(h^4)

$$h = 0.4 / h = 0.2$$

$$f'(0) = \frac{4}{3}(0.9615) - \frac{1}{3}(0.8620) = 0.9946$$

$$h = 0.2 / h = 0.2$$

$$f'(0) = \frac{4}{3}(0.9900) - \frac{1}{3}(0.9615) = 0.9996$$

3- O(h^6)

$$f'(0) = \frac{16}{15}(0.9996) - \frac{1}{15}(0.9946) = 0.9999$$

J	1	2	3		
H	0.4	0.2	0.1		
	0.86206897	0.96153846	0.99009901	k=1	O(h2)
		0.99469496	0.99961919	k=2	O(h4)
			0.99994747	k=3	O(h6)

b) <u>Fourth order central difference</u>

$$f'(x) = \frac{-f(x+2h) + 8f(x+h) - 8f(x-h) + f(x-2h)}{12h}$$

1- O(h^4)

$h = 0.4$

$$f'(0) = \frac{-f(0.8) + 8f(0.4) - 8f(-0.4) + f(-0.8)}{12(0.4)} = 0.9461$$

$h = 0.2$

$$f'(0) = \frac{-f(0.4) + 8f(0.2) - 8f(-0.2) + f(-0.4)}{12(0.2)} = 0.9946$$

O(h^6)

$$f'(0) = \frac{16}{15}(0.9946) - \frac{1}{15}(0.9461) = 0.9979$$

J	1	2		
H	0.4	0.2		
	0.94617325	0.99469496	k=2	O(h4)
		0.99792974	k=3	O(h6)

Bonus (10 points)

Solve problem 4 using composite trapezoidal rule (h=0.5) instead of the three-points Gaussian quadrature.

$$h = 0.5; n = 4$$

$$I = \int_{-1}^{1} f(x)dx = \frac{2}{8}\left(f(-1) + 2f(-0.5) + 2f(0) + 2f(0.5) + f(1) \right)$$

$$f(-1) = \int_{-1}^{1} \frac{1}{2+(-1)y} = \frac{1}{4}\left(\frac{1}{2+(-1)(-1)} + 2\frac{1}{2+(-1)(-0.5)} + 2\frac{1}{2+(-1)(0)} + 2\frac{1}{2+(-1)(0.5)} + \frac{1}{2+(-1)(1)} \right) = 1.1666$$

$$f(-0.5) = \int_{-1}^{1} \frac{1}{2+(-0.5)y} = \frac{1}{4}\left(\frac{1}{2+(-0.5)(-1)} + 2\frac{1}{2+(-0.5)(-0.5)} + 2\frac{1}{2+(-0.5)(0)} + 2\frac{1}{2+(-0.5)(0.5)} + \frac{1}{2+(-0.5)(1)} \right) = 1.1665$$

$$f(0) = \int_{-1}^{1} \frac{1}{2+(0)y} = \frac{1}{4}\left(\frac{1}{2+(0)(-1)} + 2\frac{1}{2+(0)(-0.5)} + 2\frac{1}{2+(0)(0)} + 2\frac{1}{2+(0)(0.5)} + \frac{1}{2+(0)(1)} \right) = 1$$

$$f(0.5) = \int_{-1}^{1} \frac{1}{2+(0.5)y} = \frac{1}{4}\left(\frac{1}{2+(0.5)(-1)} + 2\frac{1}{2+(0.5)(-0.5)} + 2\frac{1}{2+(0.5)(0)} + 2\frac{1}{2+(0.5)(0.5)} + \frac{1}{2+(0.5)(1)} \right) = 0.8888$$

$$f(1) = \int_{-1}^{1} \frac{1}{2+(1)y} = \frac{1}{4}\left(\frac{1}{2+(1)(-1)} + 2\frac{1}{2+(1)(-0.5)} + 2\frac{1}{2+(1)(0)} + 2\frac{1}{2+(1)(0.5)} + \frac{1}{2+(1)(1)} \right) = 0.8181$$

finally:

$$I = \frac{1}{4}\left(1.1666 + 2(1.1655) + 2(1) + 2(0.8888) + 0.8181\right) = 2.0114$$