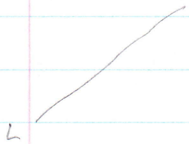
Intro to linear

Chapter 4 Lin Lines & Planes

Sept 12^{th}

4.1 equation of a straight line $y = mx + b$



Ex: The line $y = 2x + 1$ in $\mathbb{R}^2$ can be written as follows: $x = r$ ← parameter
$y = 2r + 1$

then $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} r \\ 2r+1 \end{bmatrix}$

$= \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{\vec{v}_0} + r \underbrace{\begin{bmatrix} 1 \\ 2 \end{bmatrix}}_{\vec{d}}$

vector form or parametric form

Use different letters to represent different parameters

Find the pt of intersections of

$L_1 = \left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix} + r \begin{bmatrix} 1 \\ 2 \end{bmatrix} \middle| r \in \mathbb{R} \right\}$ and

$L_2 = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} + r \begin{bmatrix} 0 \\ 1 \end{bmatrix} \middle| r \in \mathbb{R} \right\}$

use ~~r~~ t for the parameter of L_2 and we solve

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} + r \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + r \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

*If want to find pt of intersection of 2 different use 2 letters

$$\begin{cases} 0 + r = 1 + 0 & \Rightarrow r = 1 \\ 1 + 2r = 1 + r & \Rightarrow t = 2 \end{cases}$$

so the pt of intersection is $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$

4.2

In $\mathbb{R}$: there is only one line

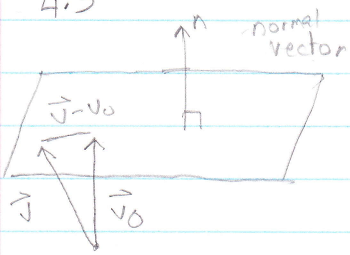
In $\mathbb{R}^2$: 2 distinct line: either (i) lines are parallel
(ii) intersect

in $\mathbb{R}^3$: 2 distinct lines (i) are parallel
(ii) intersect
(iii) skew (they neither are parallel nor distinct)

there is a unique plane containing both lines

no plane containing both, but there are 2 parallel plane, each containing one line

4.3



$$w = \{ \vec{v} \in \mathbb{R}^3 \mid (\vec{v}-\vec{v_0}) \cdot \vec{n} = 0 \}$$

Ex: The plane with $\vec{n} = (1,2,3)$ and containing $(0,1,1)$

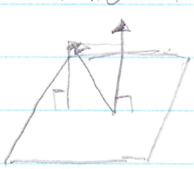
is $\{(x,y,z) \in \mathbb{R}^3 \mid [(x,y,z) - (0,1,1)] \cdot (1,2,3) = 0\}$

$(x, y-1, z-1) \cdot (1,2,3) = 0$

$x + 2y - 2 + 3z - 3 = 0$

$x + 2y + 3z = 5$

Ex: Find the distance from the point $(3,3,3)$ to the plane $x + 2y + 3z = 5$



$$\text{Distance} = \| \text{proj}_{(1,2,3)} ((3,3,3) - (0,1,1)) \|$$

$$= \| \text{proj}_{(1,2,3)} (3,2,2) \|$$

$$= \left\| \frac{3+4+6}{1+4+9} (1,2,3) \right\| = \frac{13}{14}\sqrt{14} = \frac{13}{\sqrt{14}}$$

4.4

Def: The angle between 2 planes in $\mathbb{R}^3$ is the angle between the normal vectors

In $\mathbb{R}^2$: there is only one plane

$\mathbb{R}^3$: 2 distinct planes either (i) are parallel
(ii) intersect

$\mathbb{R}^4$???

n	equations in $\mathbb{R}^n$	geometric object	dimension
1	$ax = b$	point	0
2	$ax + by = c$	line	1
3	$ax + by + cz = d$	plane	2
4	$ax + by + cz + dw = e$	?	3

The resulting dimensional object in $\mathbb{R}^n$ is called a hyperplane

Idea: one equation in $\mathbb{R}^n$ will cut down the dimension by 1

4.5

If $\vec{x} = (x_1, x_2, x_3)$
$\vec{y} = (y_1, y_2, y_3)$

then the cross product of $\vec{x}$ and $\vec{y}$ is

$$\vec{x} \times \vec{y} \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} = (x_2 y_3 - y_2 x_3)\underline{i} - (x_1 y_3 - y_1 x_3) j + (x_1 y_2 + y_1 x_2) k$$

Ex $(0,1,2) \times (-3,4,1) = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 0 & 1 & 2 \\ -3 & 4 & 1 \end{vmatrix} = (-7,-6,3)$

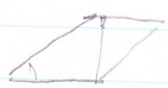
Properties:

* $\vec{u} \times \vec{v} = -\vec{v} \times \vec{u}$
* $(\vec{u} \times \vec{v}) \cdot u = 0$
* $(\vec{u} \times \vec{v}) \cdot \vec{v} = 0$
* $(\vec{u} + \vec{v}) \times \vec{w} = \vec{u} \times \vec{w} + \vec{v} \times \vec{w}$

* $\|\vec{u} \times \vec{v}\| = \|\vec{u}\| \|\vec{v}\| \sin\theta$ where $0 \le \theta \le \pi$ is the angle between $\vec{u}$ and $\vec{v}$

Remark:

* $\|\vec{u} \times \vec{v}\|$ = area of the parallelogram with sides $\vec{u}$ & $\vec{v}$



* Area of triangle with sides $\vec{u}$ and $\vec{v}$ is $\frac{1}{2}\|\vec{u} \times \vec{v}\|$

* In general $\vec{u} \times (\vec{v} \times \vec{w}) \neq (\vec{u} \times \vec{v}) \times \vec{w}$
* Suppose $\vec{u}, \vec{v} \neq \vec{0}$ then $\vec{u}, \vec{v}$ parallel $\Leftrightarrow \vec{u} \times \vec{v} = 0$
* Direction of $\vec{u} \times \vec{v}$ is given by the right hand rule

angle between 2 planes = angle between normal vectors

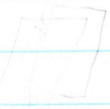
4.6

Ex: Find an equation of the plane (i) containing y-axis
(ii) perpendicular to the plane $4x - y + 3z = 5$

normal vector is perpendicular to $(0,1,0)$ and $(4,-1,3)$

$$(0,1,0) \times (4,-1,3) = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 0 & 1 & 0 \\ 4 & -1 & 3 \end{vmatrix} = (3,0,-4)$$

4.7 Volume of a parallel piped in $\mathbb{R}^3$ with sides $\vec{u}, \vec{v}$, and $\vec{w}$ is $(\vec{u} \times \vec{v}) \cdot \vec{w}$ $(= |(\vec{v} \times \vec{w}) \cdot \vec{u}|$



Area of the base parallelogram $= \|\vec{u} \times \vec{v}\|$

height $= \|\vec{w}\| \cos\theta$

Ex: Find the volume of parallelpiped with sides

$\vec{u} = (2,0,3)$

$\vec{v} = (1,1,-6)$

$\vec{w} = (-1,2,1)$

$$\text{Volume} = \begin{vmatrix} 2 & 0 & 3 \\ 1 & 1 & -6 \\ -1 & 2 & 1 \end{vmatrix} = 2(1+12)+0+3(3) = 35$$