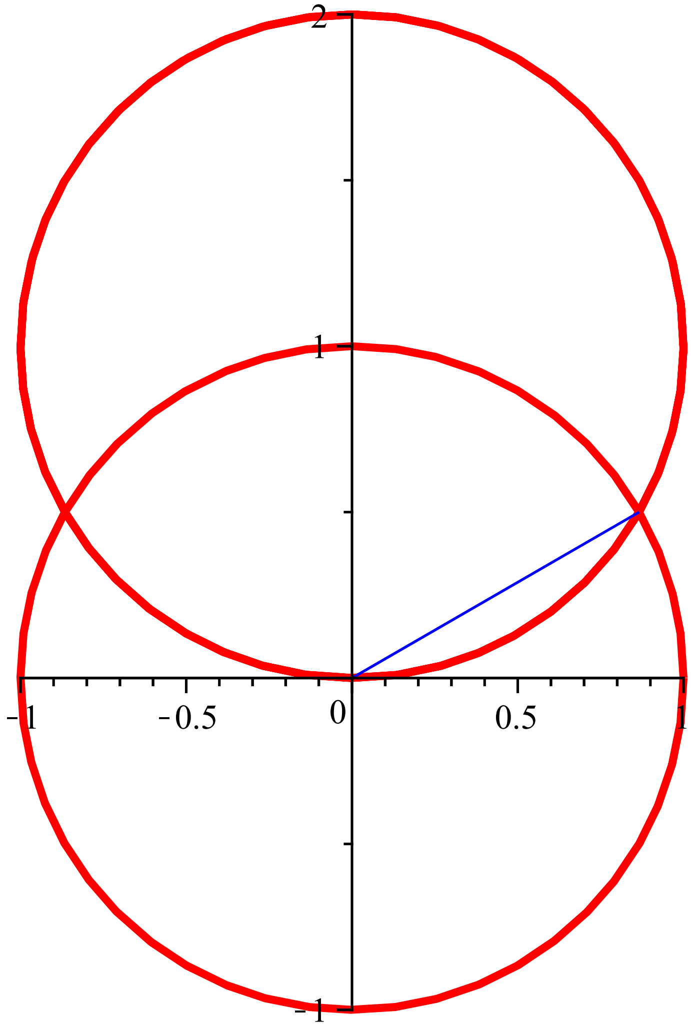
# Solutions to Final 2012 Math 265:

**Problem 1:** Find the area bounded by both circles $x^2+y^2= 1$ and $r=2\sin\theta$.

**Solution:**

```
> with(plots):display(plot([2*sin(t),1],t=0..2*Pi,coords=polar,
  color=red,thickness=3),
          pointplot([[0,0],[cos(Pi/6),sin(Pi/6)]],connect=true,
  color=blue));
```




```
>
```

The intersection points of the circles satisfy $1 = 2\sin t$, or $\sin t = \frac{1}{2}$, which implies $t = \frac{\pi}{6}$ or $t = \pi - \frac{\pi}{6}$. The area of the intersection is

$$2\left(\int_0^{\frac{\pi}{6}}\int_0^{2\sin t} r \,\mathrm{d}r \,\mathrm{d}t + \int_{\frac{\pi}{6}}^{\frac{\pi}{2}}\int_0^{1} r \,\mathrm{d}r \,\mathrm{d}t\right) = 2\left(\int_0^{\frac{\pi}{6}} \frac{1}{2} 4\sin^2 t \, d t + \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{1}{2} dt\right) = 2\left(2\int_0^{\frac{\pi}{6}} \frac{1}{2}(1 - \cos(2t))\, dt\right.$$

$$+\frac{\pi}{6}\Bigg) = 2\left(\frac{\pi}{6} - \left[\frac{1}{2}\sin(2\,t)\right]_0^{\frac{\pi}{6}} + \frac{\pi}{6}\right) =$$

$$= 2\left(\frac{\pi}{3} - \frac{\sqrt{3}}{4}\right)$$

```
> 2*int(int(r,r=0..2*sin(t)),t=0..Pi/6)+2*int(int(r,r=0..1),t=
  Pi/6..Pi/2);
```

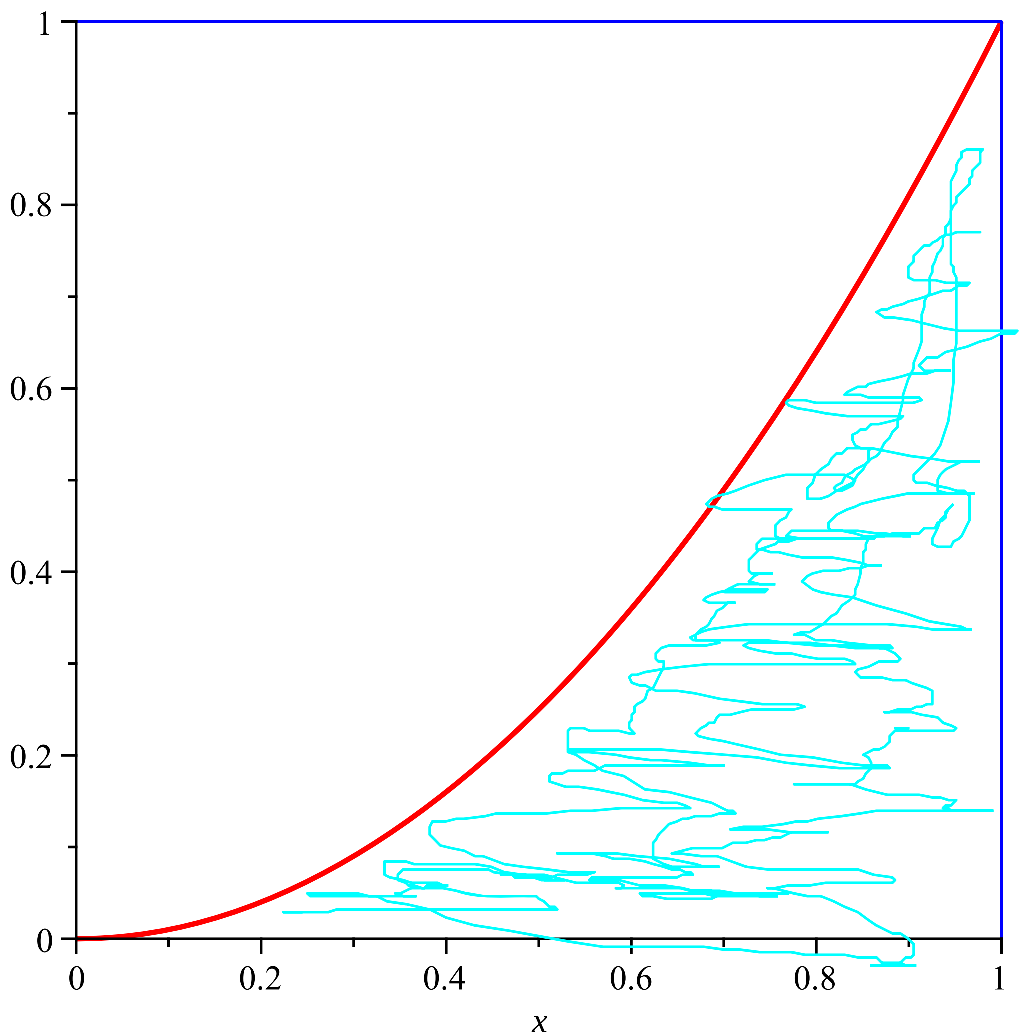
$$-\frac{1}{2}\sqrt{3} + \frac{2}{3}\pi \quad (1)$$

**Problem 2:** Change the order of integration to evaluate the integral

$$\int_0^1 \int_{\sqrt{y}}^1 \exp(3\,x^3 + 3)\ dx\ dy$$

If $x = \sqrt{y}$ then $y = x^2$. The region of integration is

```
> display(plot(x^2,x=0..1,thickness=2),pointplot([[0,1],[1,1],[1,
  0]],connect=true,color=blue));
```



After change of order the integral is

$$\int_0^1\int_0^{x^2} \exp(3\,x^3+3)\;dy\,dx = \int_0^1 x^2\exp(3\,x^3+3)\;dx = [t=3\,x^3+3,\, dt=9\,x^2 dx] = \frac{1}{9}\int_3^6 \exp(t)\;d\,t$$

$$= \frac{1}{9}\left(e^6 - e^3\right)$$

```
> int(int(exp(3*x^3+3),x=sqrt(y)..1),y=0..1);
```

$$\int_0^1\int_{\sqrt{y}}^1 e^{3\,x^3+3}\,dx\,dy \qquad (2)$$

As You can see Maple is not smart enough to change the order of integration and cannot evaluate the integral.

```
> int(int(exp(3*x^3+3),y=0..x^2),x=0..1);
```

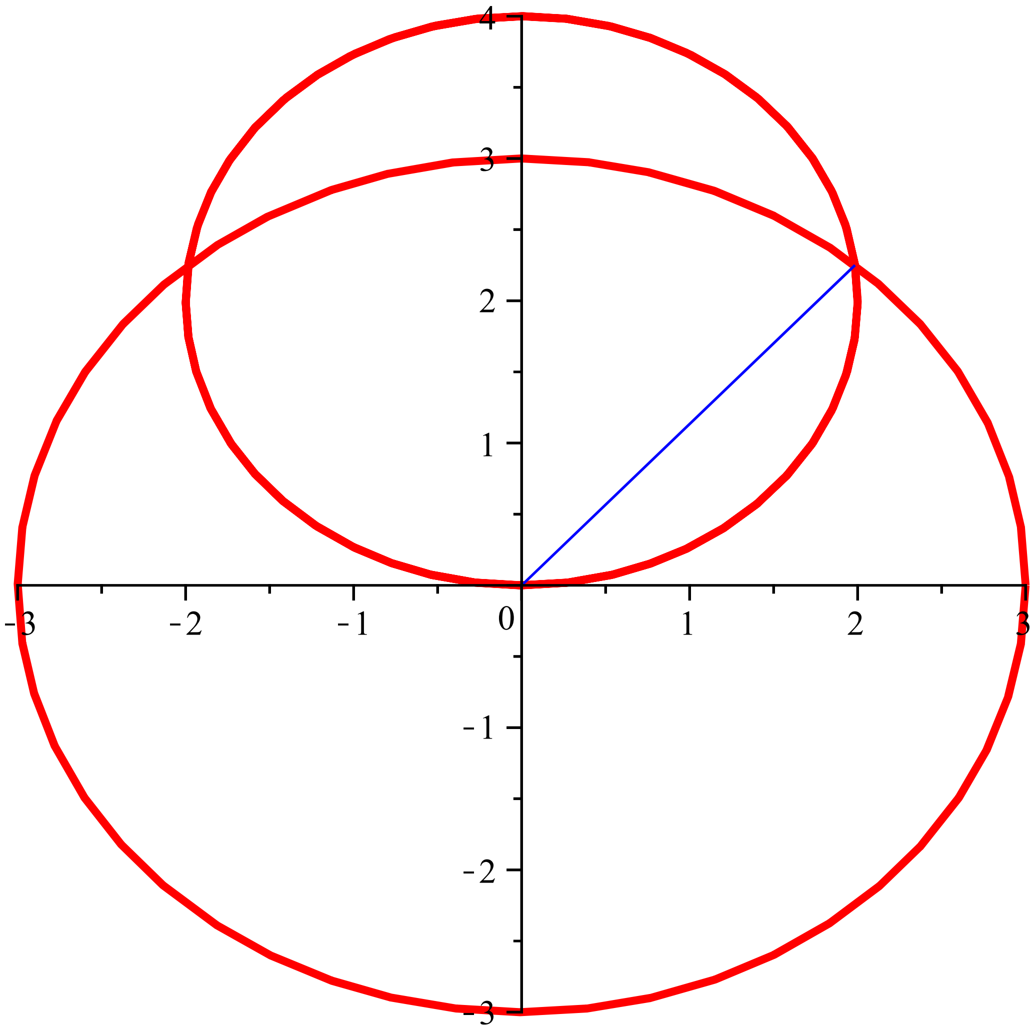
$$-\frac{1}{9}\,e^3 + \frac{1}{9}\,e^6 \qquad (3)$$

**Problem 3:** Find the center of mass of the solid which is the union of the two balls
$$ x^2+y^2+z^2\le 9\ \ ,\ \ z \ge 0\ ,$$
and $$ x^2+(y-2)^2+z^2\le 4\ ,$$
if the density at a point is equal to the distance from the origin. Hint: use the symmetries of the solid;
$\arcsin(3/4)=0.848$.
Solution:
The problem will be much easier if we "rotate" it exchanging the y and z axis:
The first ball becomes $x^2+y^2+z^2 \le 9$ (same equation) , and the second: $x^2+y^2+(z-2)^2 \le 4$
In spherical coordinates the equation of the first ball is $\rho \le 3$ .The second ball:
$x^2+y^2+z^2-4\,z+4 \le 4$ or
$\rho^2 \le 4\,\rho\cos\phi$ or $\rho \le 4\cos\phi$ . The surfaces of the balls intersect when $3 = 4\cos\phi$, or $\cos\phi = \frac{3}{4}$
which corresponds to $\phi = 0.723$,
which we will denote by $\phi_0$. The density is $\rho$ .

```
> evalf(arccos(3/4));
```

$$0.7227342478 \qquad (4)$$

```
> with(plots):display(plot([4*cos(Pi/2-t),3],t=0..2*Pi,coords=
polar,color=red,thickness=3),
      pointplot([[0,0],[3*cos(Pi/2-0.7227342478),3*sin(Pi/2
-0.7227342478)]],connect=true,color=blue));
## we put Pi/2-t since phi starts at the vertical axis
```




$$m=\int_0^{2\pi}\int_0^{\pi}\int_0^{3}\rho\cdot\rho^2\sin\phi\,d\rho\,d\phi\,d\theta+\int_0^{2\pi}\int_0^{\phi_0}\int_3^{4\cos\phi}\rho\cdot\rho^2\sin\phi\,d\rho\,d\phi\,d\theta=$$

$$=2\pi\cdot\left[-\cos\phi\right]_0^{\pi}\left[\frac{1}{4}\rho^4\right]_0^{3}+2\pi\cdot\int_0^{\phi_0}\left[\frac{1}{4}\rho^4\right]_3^{4\cos\phi}\sin\phi\,d\phi=$$

$$=\pi 3^4+\frac{\pi}{2}\cdot\int_0^{\phi_0}\left[4^4\cos^4\phi-3^4\right]\sin\phi\,d\phi=\pi 3^4+\frac{\pi}{2}\cdot 4^4\int_0^{\phi_0}\cos^4\phi\sin\phi\,d\phi-\frac{\pi}{2}\cdot 3^4\int_0^{\phi_0}\sin\phi\,d\phi=$$

$$=\pi 3^4+\frac{\pi}{2}\cdot 4^4\left[-\frac{1}{5}\cos^5\phi\right]_0^{\phi_0}-\frac{\pi}{2}\cdot 3^4\left[-\cos\phi\right]_0^{\phi_0}=\pi 3^4+\frac{\pi}{10}\cdot 4^4\left(1-\cos^5(\phi_0)\right)-\frac{\pi}{2}\cdot 3^4\left(1-\cos(\phi_0)\right)=A$$

```
> int(int(int(r^3*sin(t),r=0..3),t=0..Pi),tt=0..2*Pi)+int(int(int
```

```
(r^3*sin(t),r=3..4*cos(t)),t=0..phi0),tt=0..2*Pi);
```

$$\frac{661}{10}\pi - \frac{128}{5}\cos(\phi 0)^5\pi + \frac{81}{2}\cos(\phi 0)\pi \quad (5)$$

```
> expand(simplify(Pi*3^4+(1/10)*Pi*4^4*(1-cos(phi[0])^5)-(1/2)*Pi*
  3^4*(1-cos(phi[0]))));
```

$$\frac{661}{10}\pi - \frac{128}{5}\pi\cos(\phi_0)^5 + \frac{81}{2}\pi\cos(\phi_0) \quad (6)$$

The moments $M_{xz}$ and $M_{yz}$ are 0 by symmetry. We have to calculate $M_{xy}$

$$M_{xy} = \int_0^{2\pi}\int_0^{\pi}\int_0^{3} \rho\cos\phi\,\rho\cdot\rho^2\sin\phi\,d\rho\,d\phi\,d\theta + \int_0^{2\pi}\int_0^{\phi_0}\int_3^{4\cos\phi} \rho\cos\phi\;\rho\cdot\rho^2\sin\phi\,d\rho\,d\phi\,d\theta =$$

$$= 0\left(\text{since} \int_0^{\pi}\cos\phi\sin\phi\,d\phi = 0\right) + 2\pi\cdot\int_0^{\phi_0}\left[\frac{1}{5}\rho^5\right]_3^{4\cos\phi}\cos\phi\sin\phi\;d\phi =$$

$$= \frac{2\pi}{5}\cdot\int_0^{\phi_0}\left[4^5\cos^5\phi - 3^5\right]\cos\phi\sin\phi\;d\phi = \frac{2\pi}{5}\cdot 4^5\int_0^{\phi_0}\cos^6\phi\sin\phi\;d\phi - \frac{2\pi}{5}\cdot 3^5\int_0^{\phi_0}\cos\phi\sin\phi\;d\phi =$$

$$= \frac{2\pi}{5}\cdot 4^5\left[-\frac{1}{7}\cos^7\phi\right]_0^{\phi_0} - \frac{2\pi}{5}\cdot 3^5\left[-\frac{1}{2}\cos^2\phi\right]_0^{\phi_0} = \frac{2\pi}{35}\cdot 4^5\left[1-\cos^7\phi_0\right] - \frac{\pi}{5}\cdot 3^5\left[1-\cos^2\phi_0\right]$$

$$= B$$

```
> int(int(int(r^4*cos(t)*sin(t),r=0..3),t=0..Pi),tt=0..2*Pi)+int
  (int(int(r^4*cos(t)*sin(t),r=3..4*cos(t)),t=0..phi0),tt=0..2*Pi)
  ;
```

$$\frac{347}{35}\pi - \frac{2048}{35}\cos(\phi 0)^7\pi + \frac{243}{5}\cos(\phi 0)^2\pi \quad (7)$$

```
> expand((2*Pi*(1/35))*4^5*(1-cos^7*phi[0])-(1/5)*Pi*3^5*(1-cos^2*
  phi[0]));
```

$$\frac{347}{35}\pi - \frac{2048}{35}\pi\cos^7\phi_0 + \frac{243}{5}\pi\cos^2\phi_0 \quad (8)$$

The center of mass for "rotated" problem is (0,0,B/A). The center of mass for the original problem is (0,B/A,0).

**Problem 4:** Find the mass of the wire in the shape of the curve:
r(t)= < t, t^2, 2t^2 > ,0 ≤ t ≤ 1
if the density per unit length is the distance from the (y,z) - plane .

Solution: the density at point (x,y,z) is equal to x.

$r(t) = \langle t, t^2, 2t^2\rangle$ , $r'(t) = \langle 1, 2t, 4t\rangle$, $|r'(t)| = \sqrt{1 + 4t^4 + 16t^4} = \sqrt{1 + 20t^2}$

$$m = \int_0^1 t\sqrt{1+20\,t^2}\,\mathrm{d}t = \left[s = 1+20\,t^2,\ ds = 40\,t\,dt\right] = \frac{1}{40}\int_1^{21}\sqrt{s}\,\mathrm{d}\,s = \frac{1}{40}\left[\frac{2}{3}s^{\frac{3}{2}}\right]_1^{21} = \frac{1}{60}\left(21^{\frac{3}{2}}-1\right)$$

```
> simplify(int(t*sqrt(1+20*t^2),t=0..1));
```

$$\frac{7}{20}\sqrt{21}-\frac{1}{60} \qquad \textbf{(9)}$$

**Problem 5:** Evaluate the integral

$\iint_D$ (x^2+y^2) exp(xy) dA ,

where $D$ is a region in the positive quadrant bounded by the curves $xy=1$, $xy=2$, $x^2-y^2=1$, $x^2-y^2=2$.

Solution:

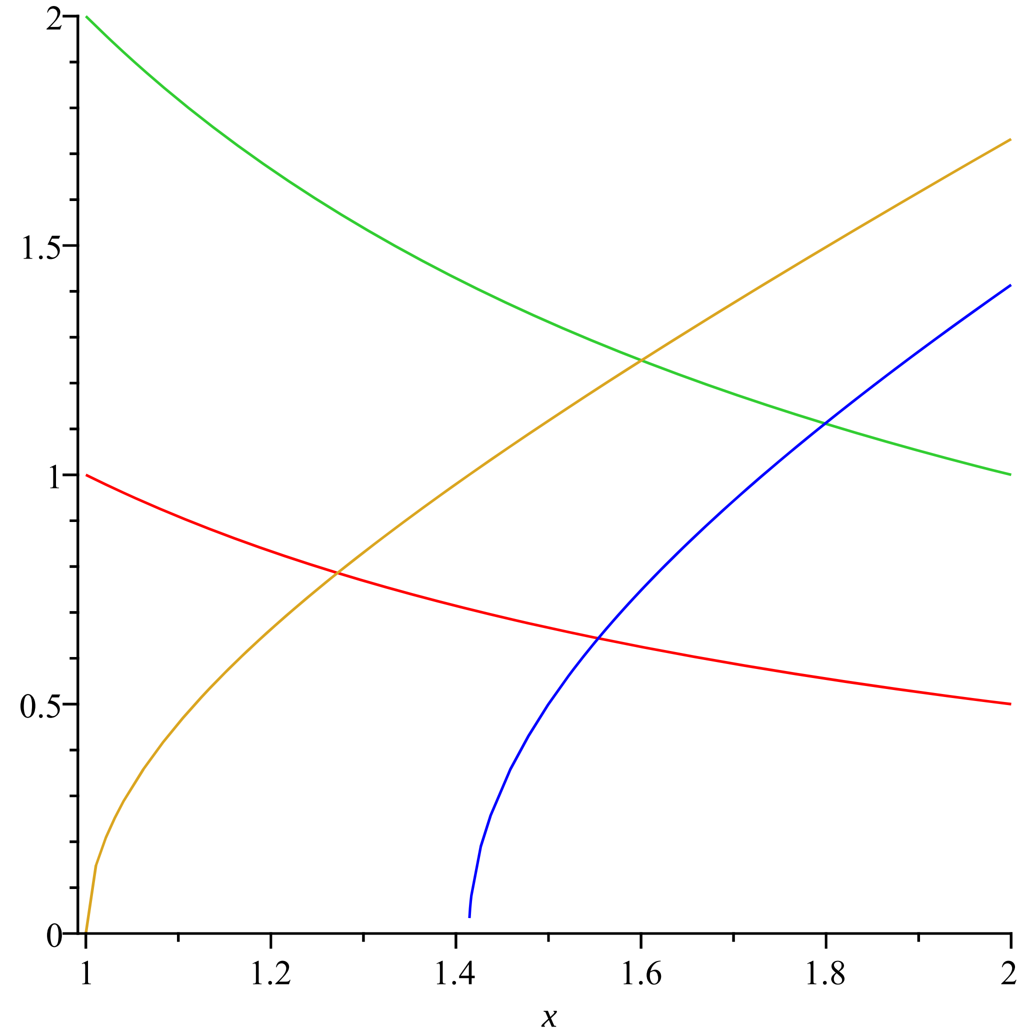
Change of variables: $u = xy,\ v = x^2 - y^2,$

Jacobian $\frac{\partial(u,v)}{\partial\,(x,y)} = \begin{bmatrix} y & 2\,x \\ \mathrm{x} & -2\,y \end{bmatrix} = -2\,y^2 - 2\,x^2 = -2\left(x^2+y^2\right)$

so $\frac{\partial(x,y)}{\partial\,(u,v)} = -\frac{1}{2\left(x^2+y^2\right)}$

Our integral becomes $\int_1^2\int_1^2 \left(x^2+y^2\right)e^u\frac{1}{2\left(x^2+y^2\right)}d\,u\,d\,v = \frac{1}{2}\cdot 1\cdot\left(e^2-e\right)$

```
> plot([1/x,2/x,sqrt(x^2-1),sqrt(x^2-2)],x=1..2);# Original region
  of integration
```




**Problem 6:** Find the area of the surface cut from the paraboloid \$z = 3x^2 + 3y^2\$ by the planes \$z = 48\$ and \$z = 75\$.

Solution: The projection of this area on x,y-plane is region D bounded by circles 48=3x^2 + 3y^2 or $x^2 + y^2 = 16$ and

$75 = 3x^2 + 3y^2$ or

$x^2 + y^2 = 25$

The parametrization : $r(x, y) = < x, y, 3x^2 + 3y^2 >$.

The tangent vectors: $r_x(x, y) = \langle 1, 0, 6x \rangle$

$r_y(x, y) = \langle 0, 1, 6y \rangle$

$r_x(x, y) \times r_y(x, y) = \langle -6x, -6y, 1 \rangle$

$|r_x(x, y) \times r_y(x, y)| = \sqrt{1 + 36x^2 + 36y^2}$

The area is equal to

$$\iint_D \sqrt{1 + 36x^2 + 36y^2}\, dA = \int_0^{2\pi}\int_4^5 \sqrt{1 + 36r^2}\, r\, dr\, d\theta = 2\pi \left[ \frac{2}{3}\, \frac{1}{72} \left(1 + 36r^2\right)^{\frac{3}{2}} \right]_4^5 = \frac{\pi}{3 \cdot 18} \Big( (1$$

$$+36\cdot 25)^{\frac{3}{2}} - (1+36\cdot 16)^{\frac{3}{2}}\Big)$$

```
> int(int(r*sqrt(1+36*r^2),r=4..5),t=0..2*Pi);
```

$$-\frac{577}{54}\sqrt{577}\,\pi + \frac{901}{54}\sqrt{901}\,\pi \quad (10)$$

```
> simplify(Pi*((1+36*25)^(3/2)-(1+36*16)^(3/2))/(3*18));
```

$$-\frac{1}{54}\pi\left(577\sqrt{577} - 901\sqrt{901}\right) \quad (11)$$

**Problem 7:** Evaluate

$$\iint_S < x^2, y^2, z^2 > dS$$

where S is the surface x^2+y^2+ z^2=2az, a>0

Solution:

We will use the divergence theorem: $div\, F = 2x + 2y + 2z$

The equation of the sphere in spherical coordinates is $\rho^2 = 2\,a\rho\cos\phi$ or $\rho = 2\,a\cos\phi$ . *At the same time we can* represent the sphere as

$x^2 + y^2 + z^2 - 2\,az + a^2 = a^2$ or

$x^2 + y^2 + (z - a)^2 = a^2$ *which shows that the sphere is above the x, y plane.*

Because the sphere is symmetrical with respect the planes (x,z) and (y,z) the integrals of 2x and of 2y give 0.

We have to evaluate

$$\int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_0^{2a\cos\phi} 2\rho\cos\phi\,\rho^2\sin\phi\, d\rho\, d\phi\, d\theta = 4\pi\int_0^{\frac{\pi}{2}}\left[\frac{1}{4}\rho^4\right]_0^{2a\cos\phi}\cos\phi\sin\phi\, d\phi =$$

$$16\,a^4\pi\int_0^{\frac{\pi}{2}}\cos^5\phi\sin\phi\, d\phi = 16\,a^4\pi\left[-\frac{1}{6}\cos^6\phi\right]_0^{\frac{\pi}{2}} = \frac{8}{3}a^4\pi$$

```
> int(int(int(2*r^3*cos(t)*sin(t),r=0..2*a*cos(t)),t=0..Pi/2),tt=
0..2*Pi);
```

$$\frac{8}{3}a^4\pi \quad (12)$$

**Problem 8:**

(a)\ Prove that

$$div(F\times G) = G\cdot curl\,F - F\cdot curl\,G$$

for any vector fields $\vec F, \vec G$ on $\mathbb R^3$ with continuous first partial derivatives.

Proof :

Let $F = \langle P, Q, R\rangle$ , $G = \langle S, T, V\rangle$. Then, $F\times G = \begin{bmatrix} i & j & k \\ P & Q & R \\ S & T & V\end{bmatrix} = \langle QV - RT, RS - PV, PT - QS\rangle$

$div(F\times G)=$
$$\frac{\partial Q}{\partial x}V+\frac{\partial V}{\partial x}Q-\frac{\partial R}{\partial x}T-\frac{\partial T}{\partial x}R+\frac{\partial R}{\partial y}S+\frac{\partial S}{\partial y}R-\frac{\partial P}{\partial y}V-\frac{\partial V}{\partial y}P+\frac{\partial P}{\partial z}T+\frac{\partial T}{\partial z}P-\frac{\partial Q}{\partial z}S-\frac{\partial S}{\partial z}Q$$
(*)

On the other hand:

$$curl\,F=\begin{bmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{bmatrix}=\left\langle \frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z},\frac{\partial P}{\partial z}-\frac{\partial R}{\partial x},\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right\rangle$$

$$G\cdot curl\,F=\frac{\partial R}{\partial y}S-\frac{\partial Q}{\partial z}S+\frac{\partial P}{\partial z}T-\frac{\partial R}{\partial x}T+\frac{\partial Q}{\partial x}V-\frac{\partial P}{\partial y}V \qquad (**)$$

$$curl\,G=\begin{bmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ S & T & V \end{bmatrix}=\left\langle \frac{\partial V}{\partial y}-\frac{\partial T}{\partial z},\frac{\partial S}{\partial z}-\frac{\partial V}{\partial x},\frac{\partial T}{\partial x}-\frac{\partial S}{\partial y}\right\rangle$$

$$F\cdot curl\,G=\frac{\partial V}{\partial y}P-\frac{\partial T}{\partial z}P+\frac{\partial S}{\partial z}Q-\frac{\partial V}{\partial x}Q+\frac{\partial T}{\partial x}R-\frac{\partial S}{\partial y}R \qquad (***)$$

It is easy to check that (**)-(***)=(*).

(b) \ Show that for any $f:\mathbb R^3\to \mathbb R$ with continuous second partial derivatives, we have $${\rm curl }\left( \nabla f \right)\equiv 0\ .$$

$$\nabla f=\left\langle \frac{\partial f}{\partial x},\frac{\partial f}{\partial y},\frac{\partial f}{\partial z}\right\rangle$$

$$curl(\nabla f)=\begin{bmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{bmatrix}=\left\langle \frac{\partial^2 f}{\partial y\partial z}-\frac{\partial^2 f}{\partial z\partial y},\frac{\partial^2 f}{\partial z\partial x}-\frac{\partial^2 f}{\partial x\partial z},\frac{\partial^2 f}{\partial x\partial y}-\frac{\partial^2 f}{\partial y\partial x}\right\rangle=\langle 0,0,0\rangle$$

(c) \ Is the vector field
$ \vec F(x,y,z)=\langle xyz, -y^2z, yz^2\rangle \ ,$
a gradient of some function $f:\mathbb R^3\to \mathbb R$ with continuous second partial derivatives?

$$F=\langle xyz,-y^2z,yz^2\rangle\ ,\quad curl\,F=\begin{bmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xyz & -y^2z & yz^2 \end{bmatrix}=\langle z^2+y^2,xy,-xz\rangle\neq\langle 0,0,0\rangle$$

$F$ is not a gradient of any such function (By part (b)).

**Problem 9:** One of the vector fields:

$$ \vec F(x,y,z)=\langle\cos x\cos y\cos z, -\sin x\sin y\cos z, -\sin x \cos y\sin z \rangle\ ,$$

$$ \vec G(x,y,z)=\langle\cos x\cos y\cos z, -\cos x\sin y\cos z, -\sin x \cos y\sin z\rangle\ ,$$

is conservative. Find out which one and then integrate the conservative vector field along the curve $C$ given as $\vec r(u)=\langle u,2u, 3u\rangle$, $u\in[0,2\pi]$.

Solution:

$F = \langle \cos x \cos y \cos z, -\sin x \sin y \cos z, -\sin x \cos y \sin z \rangle$ ,

$$curl\ F = \begin{bmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \cos x \cos y \cos z & -\sin x \sin y \cos z & -\sin x \cos y \sin z \end{bmatrix} =$$

$$\langle \sin x \sin y \sin z - \sin x \sin y \sin z,\ -\cos x \cos y \sin z + \cos x \cos y \sin z,\ -\cos x \sin y \cos z + -\cos x \sin y \cos z \rangle = \langle 0, 0, 0 \rangle$$

Since curl(F) = 0 and F is well defined in the whole space $R^3$ (no holes), F is conservative. We do not need to check G. Now, we look for the potential of F, i.e., a function f such that $F = \nabla f$

$$\frac{\partial f}{\partial x} = \cos x \cos y \cos z \quad , \quad (1)$$

$$\frac{\partial f}{\partial y} = -\sin x \sin y \cos z \quad , \quad (2)$$

$$\frac{\partial f}{\partial z} = -\sin x \cos y \sin z \quad . \quad (3)$$

Integrating (1) in x : $f = \sin x \cos y \cos z + C(y, z)$

$$\frac{\partial f}{\partial y} = -\ \sin x \sin y \cos z + \frac{\partial C}{\partial y}(y, z)$$

Comparing with (2) : $\dfrac{\partial C}{\partial y}(y, z) = 0$ so $C(y, z) = C(z)$

*Now* : $f = \sin x \cos y \cos z + C(z)$

$$\frac{\partial f}{\partial z} = -\ \sin x \cos y \sin z + C'(z)$$

Comparing with (3) : $C'(z) = 0$ so $C(z) = C$

*Now* : $f = \sin x \cos y \cos z + C$ (C can be disregarded for integration)

Now, we will use the FThLI: $\int_C F dr = f(last\ point) - f(first\ point)$

$r(u) = \langle u, 2u, 3u \rangle$, $u \in [0, 2\pi]$

$r(0) = (0, 0, 0)$, $r(2\pi) = (2\pi, 2\pi, 6\pi)$

$f(2\pi, 2\pi, 6\pi) = 0$

$f(0, 0, 0) = 0$

$\int_C F dr = 0$

## Problem 10 :

Evaluate the integral

$$\int_c \left( \frac{\arctan(x^3+3)}{3+x^2} + 2x\cos(x^2+y^2) \right) dx + \left( \frac{\sin(\cos^2(1+y^2))}{y^2+100} + 2y\cos(x^2+y^2) \right) dy$$

where C is the ellipse $(x-4)^2 + \frac{y^2}{4} = 1$ .

Solution:

We use Green's theorem. If the integrand is written as $F = \langle P, Q \rangle$, then

$$\frac{\partial Q}{\partial x} = -2y\sin(x^2+y^2)2x$$

$$\frac{\partial P}{\partial y} = -2x\sin(x^2+y^2)2y$$

so $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0$ and the integral is 0. (The functions under the integral are well defined in the whole plane)

## Problem 11 :

Calculate the flux of $ \vec F(x,y,z)=\langle x^3,y^3,z^3\rangle$ across the surface of the hemisphere $x^2+y^2+z^2=9$, $z\ge 0$, oriented outwards.

Solution: We will use the divergence theorem, although the surface S is not closed. We will close it by adding a flat disk $S_1 : \quad x^2+y^2 \le 9$ in the $z=0$ plane. Then,

$$\iiint_E \text{div F dV} = \iint_S F\, dS + \iint_{S_1} F\, dS$$

$$div\, F = 3x^2 + 3y^2 + 3z^2 = 3\rho^2$$

$$\iiint_E \text{div F dV} = \int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_0^3 3\rho^2\rho^2 \sin\phi\, d\rho\, d\phi\, d\theta = 2\pi\, 3\ \ \frac{1}{5}3^5 = \frac{2\pi}{5}\cdot 3^6$$

$$\iint_{S_1} F\, dS = \iint_{x^2+y^2\le 9} \langle x^3, y^3, 0\rangle < 0, 0, -1 > dA = 0$$

Thus, $\iint_S F\, dS = \frac{2\pi}{5}\cdot 3^6$

## Problem 12 :

Let $S$ be an ellipsoidal disc cut out from the plane $2x+y-2z=1$ by the cylinder $x^2+y^2=4$. Let $C$ be the boundary of $S$.

Find $$\oint_C \langle \cos(1+x^3)+y^3,y^3+y^2\sin(1+ y^3),-2x^3+\cos(1+z^3)\rangle \cdot d\vec r\ .$$

Solution: We will use the Stokes' theorem: $\int_C F\,dr = \iint_S curl(F)\,dS$

$F = \langle \cos(1+x^3) + y^3, y^3 + y^2\sin(1+y^3), -2x^3 + \cos(1+z^3) \rangle$

$$curl\,F = \begin{bmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \cos(1+x^3) + y^3 & y^3 + y^2\sin(1+y^3) & -2x^3 + \cos(1+z^3) \end{bmatrix} = \langle 0, 6x^2, -3y^2 \rangle$$

The normal vector is $r(x,y) = \left\langle x, y, x + \frac{y}{2} - \frac{1}{2} \right\rangle$

$r_x = \langle 1, 0, 1 \rangle$

$r_y = \left\langle 0, 1, \frac{1}{2} \right\rangle$

$r_x \times r_y = \left\langle -1, -\frac{1}{2}, 1 \right\rangle$ It is well oriented (upward)

$$\iint_S curl(F)\,dS = \iint_{x^2+y^2 \le 4} < 0, 6x^2, -3y^2 > < -1, -\frac{1}{2}, 1 > dA =$$

$$= \iint_{x^2+y^2 \le 4} -3(x^2+y^2)\,dA = \int_0^{2\pi}\int_0^2 (-3r^2)\,r\,dr\,d\theta = -6\pi\left[\frac{1}{4}r^4\right]_0^2 = -24\pi$$