



Université d'Ottawa • University of Ottawa

Faculté des sciences
Mathématiques et de statistique

Faculty of Science
Mathematics and Statistics

Solution to Final Exam for MAT2377, Winter 2013 Probability and Statistics for Engineers.

Time : 3 hours

Professor : M. Zarepour

Name : _____

Student Number : _____

Calculators are permitted. It is an open book exam.
There are 4 short answer questions and 12 multiple choice questions.

Submit your answers for the multiple choice questions in the following table.

Question	Answer	Question	Answer
1		7	
2		8	
3		9	
4		10	
5		11	
6		12	

- [12] 1. Let X be a random variable with the probability density function

$$f(x) = c|x|, \text{ for } -1 < x < 1$$

and 0 otherwise.

- (a) Find the value for c .

Solution.

Since $\int_{-1}^1 |x| dx = 2 \int_0^1 x dx = 1$, we have $c = 1$.

- (b) Find $P(X \geq 0.5 | X \geq 0)$.

Solution.

$$\begin{aligned} P(X \geq 0.5 | X \geq 0) &= \frac{P(X > 0.5)}{P(X > 0)} \\ &= \frac{\int_{0.5}^1 x dx}{\int_0^1 x dx} = \frac{3/8}{1/2} = 0.75 \end{aligned}$$

- (c) Compute $P(X \geq \mu)$, where $\mu = E[X]$.

Solution. Since

$$E(X) = \int_{-1}^1 x|x| dx = \int_{-1}^0 (-x^2) dx + \int_0^1 x^2 dx = \mu = 0.$$

You can also see this easily without any calculations. Just notice that f is a symmetric function on $(-1, 1)$ and this shows $E(X) = 0$.

$$P(X > 0) = \int_0^1 |x| dx = \int_0^1 x dx = 0.5.$$

- [10] 2. Crystalline forms of certain chemical compounds are used in various electronic devices. It is often more desirable to have large crystals rather than small ones. In a laboratory study, 14 crystals of the same initial size were allowed to grow for certain periods of time. The following data gives the weight y of the crystal (in grams) and the period x of time (in hours) which was used for each crystal.

Time	Weight	Time	Weight
2	0.08	16	8.4
4	1.12	18	8.81
6	4.43	20	10.81
8	4.98	22	11.16
10	4.92	24	10.12
12	7.18	26	13.12
14	5.57	28	15.04

For this data, we have :

$$\bar{x} = 15, \quad \bar{y} = 7.55, \quad \sum_{i=1}^{14} (x_i - \bar{x})^2 = 910, \quad \sum_{i=1}^{14} (x_i - \bar{x})(y_i - \bar{y}) = 458.12$$

$$\sum_{i=1}^{14} (y_i - \bar{y})^2 = 244.16$$

The time and weight are stored in columns C1, respectively C2. Below is the result of the linear regression analysis produced by Minitab :

Regression Analysis: C2 versus C1

The regression equation is C2 = 0.001 + 0.503 C1

Predictor	Coef	SE Coef	T	P
Constant	0.0014	0.5994	0.00	0.998
C1	0.50343	0.03520	14.30	0.000

S = 1.06177 R-Sq = 94.5% R-Sq(adj) = 94.0%

Analysis of Variance

Source	DF	SS	MS	F	P
Regression	1	230.63	230.63	204.58	0.000
Residual Error	12	13.53	1.13		
Total	13	244.16			

Assume the linear regression model $y = \beta_0 + \beta_1 x + \varepsilon$.

(a) Find a 95% confidence interval for β_1 . Conclude why the linear regression model fits to this data set.

Solution.

We have $t_{0.025,12} = 2.179$ we have

$$0.50343 \pm (2.179)(0.0352) = (0.4267292, 0.5801308)$$

is the 95% confidence interval for β_1 . Since the interval does not include 0, we can not accept that $\beta_1 = 0$. From the graphs, normality of residuals are acceptable (see the linear trend in quantile-quantile plot).

(b) Write down the estimated regression line and use it to find a 95% prediction interval for the weight in grams for a period of $x = 7$

Solution.

$$\hat{Y} = \hat{\beta}_0 + \hat{\beta}_1 x = 0.0014 + (0.50343)(7) = 3.52541.$$

The 95% prediction interval is

$$3.52541 \pm (2.179)(1.06177) \sqrt{1 + \frac{1}{14} + \frac{(7-15)^2}{910}} = 3.52541 \pm 2.47215$$

$$(1.05195, 5.99625) \approx (1.052, 6).$$

- [12] 3. A manufacturer of sprinkler systems for fire protection in office buildings claims that the true average system-activation temperature is 130. Assume the distribution of activation temperatures is normal with standard deviation $\sigma = 1.5$. A government regulator is interested in testing the manufacturers claim using the hypothesis $H_0 : \mu = 130$ versus $H_1 : \mu \neq 130$. A random sample of $n = 25$ sprinklers is selected and the activation temperature is recorded.

(a) The random sample of $n = 25$ specimens yielded a sample mean of $\bar{x} = 131.08$. Compute the p -value of the hypothesis test and provide the conclusion with $\alpha = 0.05$.

Solution.

We have

$$Z = \frac{\bar{x} - 130}{1.5/\sqrt{25}} = 3.6.$$

$$p - value = 2P(Z > 3.6) = 0.00032.$$

Therefore we need to reject H_0 .

(b) With a significance level of $\alpha = 0.05$., compute the probability of committing a type II error if the true mean is $\mu = 132$.

Solution.

We have

$$\bar{X} \sim N(132, 1.5^2/25 = 0.09)$$

and

$$\begin{aligned}\beta &= P(130 - (1.96)(1.5)/5 \leq \bar{X} \leq 130 + (1.96)(1.5)/5) \\ &= P(129.412 \leq \bar{X} \leq 130.588) = P\left(\frac{129.412 - 132}{1.5/5} \leq Z \leq \frac{130.588 - 132}{1.5/5}\right) \\ &= P(-8.626667 \leq Z \leq -4.706667) \approx 0.\end{aligned}$$

(c) Suppose an auditor questions the validity of the study design and wishes to conduct another analysis. How many measurements should be taken to estimate the mean to within 0.5 with 95% confidence?

$$n = \frac{z_{\alpha/2}\sigma^2}{E^2} = \frac{(1.96)^2(1.5^2)}{0.5^2} = 34.57 \approx 35.$$

- [12] 4. For each of 18 preserved cores from oil-well carbonate reservoirs, the amount of residual gas saturation after a solvent injection was measured at water flood-out. Amount of saturations are recorded as follows

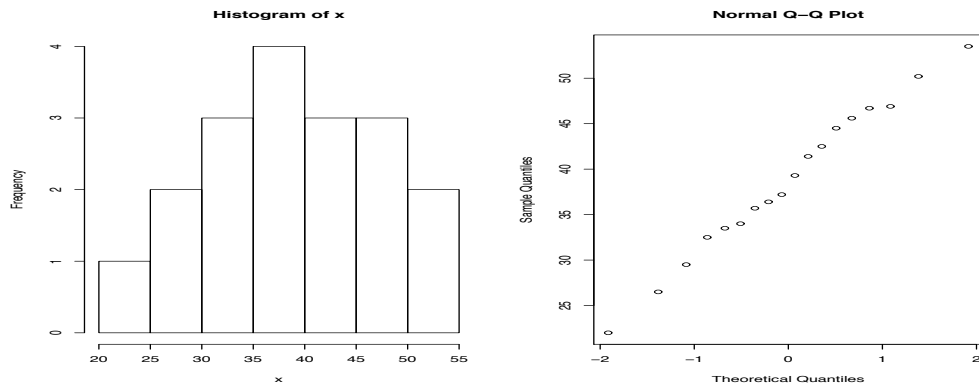
26.5, 41.4, 44.5, 29.5, 37.2, 35.7, 34.0, 42.5, 33.5,

46.7, 46.9, 39.3, 45.6, 53.5, 22.0, 32.5, 36.4, 50.2.

Summary statistics on the amount of saturation (measured as pore volume) were computed from minitab as follows :

Variable	N	Mean	SE Mean	StDev	Minimum	Q1	Median
gass saturation	18	38.77	1.9822	8.41	22	33.62	38.25

The normal probability plot and histogram for the saturation data are presented below :



- (a) Based on the previous histogram and normal probability plot, would it appear reasonable to assume the saturation amount is normally distributed? Discuss.

Solution. Since the qq-plot seems to be straight and the histogram shows roughly a symmetric shape we may believe this quantity follows a normal distribution.

(b) Is there sufficient evidence at $\alpha = 0.05$ to conclude the solvent injection results in a mean saturation amount of less than 40?

Solution.

$$H_0 : \mu = 40 \quad vs \quad H_1 : \mu < 40.$$

$$T = \frac{\bar{x} - 40}{s/\sqrt{n}} = \frac{38.77 - 40}{1.9822} = -0.6205.$$

$$P(t(17) < -0.6205) \in (0.25, 0.4).$$

So we accept H_0 .

(c) Construct a 90% confidence interval for the mean saturation amount.

Solution. The confidence interval is

$$\bar{x} \pm t_{0.05,17}s/\sqrt{n} = 38.77 \pm 1.74(1.9822) = 38.77 \pm 3.45.$$

Multiple Choice Questions

Submit your answers for the multiple choice questions in the table found on the front page. Correct answers to each question worth 4.5 marks.

1. A manufacturer of calculators buys integrated circuits from supplies A,B and C. Fifty percents of the circuits come from A, 30% from B and 20% from C. One percent of the circuits supplied by A have been defective in the past, 3% of B's have been defective and 4% of C have. A circuit is selected at random and found to be defective. What is the probability it was manufactured by B ?

(a) 0.409 (b) 0.591 (c) 0.519 (d) 0.333 (e) 0.67.

Solution.

$$P(A) = 0.5, P(B) = 0.3, P(C) = 0.3.$$

Let D =Defective. We have

$$P(D|A) = 0.01, P(D|B) = 0.03, P(D|C) = 0.04.$$

We need to find

$$P(B|D) = \frac{P(B|D)P(D)}{P(D|A)P(A) + P(D|B)P(B) + P(D|C)P(C)} = \frac{9}{22} = 0.409.$$

Answer is (a).

2. In August, the probability that a thunderstorm will occur on any particular day is 0.1. What is the probability that the first thunderstorm in August will occur on August 12 ?

(a) 0.3138 (b) 0.03138 (c) 0.6962 (d) 0.43047 (e) none of the preceeding.

Solution.

$$(0.9)^{11}(0.1) = 0.03138106.$$

Answer is (b).

3. In a communication system there is one error every 10 seconds, in average. If the number of errors have a Poisson distribution calculate the probability that in 30 seconds we have at least one error.
- (a) $1 - 4e^{-3}$ (b) $1 - 2e^{-1}$ (c) $1 - e^{-1}$ (d) $1 - 3e^{-3}$
 (e) $1 - e^{-3}$.

Solution. X has a Poisson distribution with $\mu = E(X) = 3$ errors/(30 second). Therefore

$$P(X \geq 1) = 1 - P(X = 0) = 1 - \exp(-3).$$

Therefore answer is (e).

4. The thickness of hockey pucks manufactured by a certain company has a normal distribution with mean 1 inch and standard deviation 0.05 inch. If pucks used in NHL must have a thickness between 0.9 and 1.1 inch, what percentage of pucks manufactured by this company can be used by the NHL ?
- (a) 100 (b) 95.44 (c) 4.56 (d) 97.72 (e) 2.28.

Solution.

$$P(0.9 < X < 1.1) = P(-2 < Z < 2) = 0.954.$$

Answer is (b).

5. A seed distributor claims 80% of its beet seeds will grow. How many seeds must be tested in order to estimate p , the proportion that will germinates, so that the maximum error of the estimate is 0.03 with 95% confidence.
- (a) 80 (b) 90 (c) 683 (d) 110 (e) 1490.

Solution.

$$n = \frac{pqz^2(\alpha/2)}{e^2} = \frac{(0.8)(0.2)(1.96^2)}{0.03^2} = 682.95 \approx 683.$$

Therefore the answer is (c).

6. Let X and Y be two independent random variables such that $E(X) = E(Y) = 4$ and $Var(X) = Var(Y) = 2$. Define $U = 3X - 2Y$. Find $E(U)$ and $Var(U)$?

- (a) $E(U) = -4, Var(U) = 4$
- (b) $E(U) = 4, Var(U) = 2$
- (c) $E(U) = 4, Var(U) = 20$
- (d) $E(U) = -4, Var(U) = 2$
- (e) $E(U) = 4, Var(U) = 26$.

Solution.

$$E(3X - 2Y) = 3E(X) - 2E(Y) = 12 - 8 = 4,$$

$$Var(3X - 2Y) = 9Var(X) + 4Var(Y) = 26.$$

Therefore the answer is (e)

7. A company claims that the average amount of deflection of a 10-foot steel plates is equal to 0.012 inches. A contractor suspected that the true mean is greater than 0.012. He measures the deflection of 10-foot steel plates x and obtains the following data :

0.0132, 0.0138, 0.0108, 0.0126, 0.0136,

0.0112, 0.0124, 0.0116, 0.0127, 0.0131

A simple computation shows that $\bar{x} = 0.0125$ and $s = 0.0010$. Compute the $p = p$ -value and give a conclusion.

- (a) $p \in (0.05, 0.1)$; Reject H_0 at $\alpha = 0.05$.
- (b) $p \in (0.1, 0.25)$; Reject H_0 at $\alpha = 0.05$.
- (c) $p \in (0.05, 0.1)$; Do not reject H_0 at $\alpha = 0.05$.
- (d) $p \in (0.1, 0.25)$; Do not reject H_0 at $\alpha = 0.05$.
- (e) $p > 0.7$; Do not reject H_0 at $\alpha = 0.05$.

Solution.

$$T = \frac{0.0125 - 0.012}{0.001/\sqrt{10}} = 1.581.$$

We have

$$P(t(9) > 1.58) \in (0.05, 0.1)$$

and we do not reject H_0 . The answer is (c).

8. An electrical system consists of 4 components. A parallel system of these components works if at least one of these components works. Assume that four components work independently. The reliability (probability of working) of each component is 0.75. What is the probability that the entire parallel system works?

- (a) 0.9926 (b) 0.9984 (c) 0.9887 (d) 0.9961 (e) 0.7500

Solution.

$$\begin{aligned} P(\text{AT LEAST ONE WORKS}) &= 1 - P(\text{NONE WORKS}) \\ &= 1 - 0.25^4 = 0.9960938 \end{aligned}$$

The answer is (d).

9. Let X_1, \dots, X_n be a random sample from a population with mean $\mu = 5$ and variance $\sigma^2 = 1.5$. Let \bar{X} be the sample mean. Find c such that

$$P\left(\frac{\bar{X} - \mu}{\sigma/\sqrt{n}}\right) = 0.90.$$

- (a) 1.645 (b) 1.96 (c) -1.96 (d) -1.28 (e) 1.28.

Solution. Since $\frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \sim N(0, 1)$ we have $P(Z > c) = 0.9$. From normal table the answer is $c = -1.28$. Answer is (d).

10. A and B are two events such that $P(A) = 0.3$, $P(B) = 0.5$ and $P(A \cup B) = 0.65$. Which of the following statements is true?

- (a) A and B are independent and mutually exclusive events
 (b) A and B are dependent and mutually exclusive events
 (c) A and B are dependent but not mutually exclusive events
 (d) A and B are independent but not mutually exclusive events
 (e) Insufficient information is provided

Solution. We have

$$P(A \cap B) = P(A) + P(B) - P(A \cup B) = 0.3 + 0.5 - 0.65 = 0.15.$$

We also have

$$P(A)P(B) = 0.15.$$

Therefore A and B are independent. $P(A \cap B) \neq 0$. Therefore A and B are not mutually exclusive. Therefore the answer is (d).

11. Transportation officials state that 90% of the population wear their seatbelts while driving. A random sample of 1000 drivers has been taken. Find the approximate probability that 888 or fewer drivers were wearing their seatbelts.

- (a) 0.888 (b) 0.104 (c) 0.113 (d) 0.141 (e) 0.258

Solution.

$$P(X \leq 888) = P\left(Z \leq \frac{888.5 - 900}{1000(0.9)(0.1)}\right) = P(Z \leq -1.2122) = 0.113.$$

The answer is (c).

12. A random sample of 167 engineering students produced the following 95% confidence interval for the proportion of students who own an iPhone : (0.344, 0.494). Identify the point estimate for estimating the true proportion of engineering students who own an iPhone.

- (a) 0.419 (b) 1.96 (c) 95 (d) 0.494 (e) 0.344

Solution.

$$(0.344 + 0.494)/2 = 0.419.$$

The answer is (a).