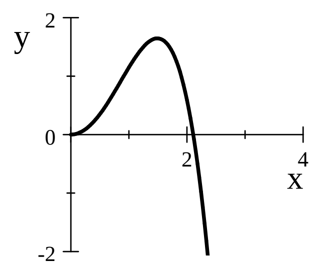
Sample final Test

1. (a) Solve the equations:

   i. $\log_2(x+3) - \log_2(2x+1) = 1$,

   ii. $\ln(x+1) + \ln(x+2) = 2\ln 3 + 3\ln 2$

   (b) For the graph $y = f(x)$:

   sketch and label the graphs of:

   i. $y = f(x+1)$

   ii. $y = 2f(x+1)$

   iii. $y = 2f(x+1) - 1$

   (c) Express $F(x) = \cos^2 x$ in the form $(f \circ g)(x)$ and then calculate:

   i. $G(x) = (f \circ f)(x)$ and

   ii. $H(x) = (g \circ f)(x)$

2. Without L'Hospital's Rule calculate the limits:

   (a) $\lim_{x \to 2} \frac{x^2 - 3x + 2}{x - 2}$

   (b) $\lim_{x \to 1^+} \frac{|x-1|}{x-1}$, $\lim_{x \to 1^-} \frac{|x-1|}{x-1}$ and , $\lim_{x \to 1} \frac{|x-1|}{x-1}$

3. Use the L'Hospital's Rule calculate the limits:

   (a) $\lim_{x \to 0} \frac{\sin x}{x}$

   (b) $\lim_{x \to 0} \left(\frac{1}{\ln(x+1)} - \frac{1}{x} \right)$

   (c) $\lim_{x \to 0} x^{\frac{1}{x}}$

4. Write equations of the (vertical and horizontal) asymptotes for $y = \frac{x^2+2x-3}{x^2-3x+2}$ and draw a rough sketch including the asymptotes. Also identify all the $x-$ intercepts and the values of x where the graph is discontinuous.

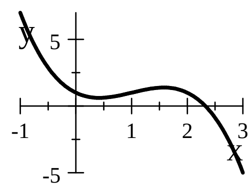
5. The displacement of a particle moving in a straight line is given by $s(t) = t^2 - 8t + 18$ where s is measured in meters and t in seconds. Calculate:

(a) the average velocities over the time intervals $[4, 6]$ and $[6, 8]$ and

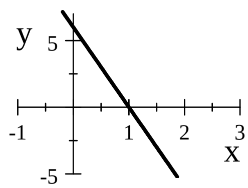
(b) the instantaneous velocity when $t = 6$.

(c) From the above answers (without using the second derivative) explain why is the accelleration $a(6) > 0$.

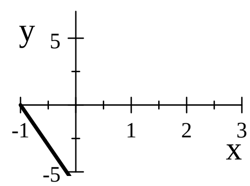
6. Below is the graph of a $y = f(x)$ and four graphs of other functions.



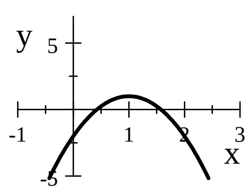
Graph of $\mathbf{y = f(x)}$



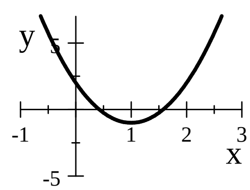
#1: **Graphs of :** $\mathbf{y = f'(x)}$ or $\mathbf{y = f''(x)}$



#2: **Graphs of :** $\mathbf{y = f'(x)}$ or $\mathbf{y = f''(x)}$



#3: **Graphs of :** $\mathbf{y = f'(x)}$ or $\mathbf{y = f''(x)}$



#4: **Graphs of :** $\mathbf{y = f'(x)}$ or $\mathbf{y = f''(x)}$

All graphs are drawn on the same scale. Identify the one which is the graph of $y = f'(x)$ and the one of $y = f''(x)$. Explain.

7. (a) State the definition of the derivative $f'(x)$.

   (b) Use the above definition to calculate $f'(x)$ for $f(x) = \sqrt{x}$

8. Use the rules for differentiation to calculate $f'(x)$ for (do not simplify your answers):

(a) $f(x) = \frac{\sin x}{x^2+1}$

(b) $f(x) = x \ln \frac{1}{x}$ (hint simplify first)

(c) $f(x) = \sqrt{\sin x + 1}$

(d) $f(x) = \sin x^x \sqrt{(3x+1)^3}$ (hint: use logarithmic differentiation)

9. The curve $\mathfrak{C}$ is given implicitely by: $y^2 - x^3(2-x) = 28$.

   (a) verify that $\mathbf{A}(3,-1) \in \mathfrak{C}$

   (b) Write an equation of the tangent line to $\mathfrak{C}$ at $\mathbf{A}$.

10. Calculate: $f'''(x)$ if $f(x) = \frac{1+2x}{1-x}$.

11. Show that $\cos x = x$ has exactly one real solution in $(a, a+1)$ where a is an integer. Identify this interval. (Hint: To show that root exists use the IVT and for no more you may use MVT).

12. Calculate the shortest distance between the hyperbola $x^2 - y^2 = 1$ ant the point $\mathbf{A}(0,2)$.

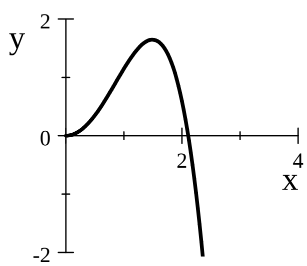
Solutions for the sample final test

1. (a) For equation:

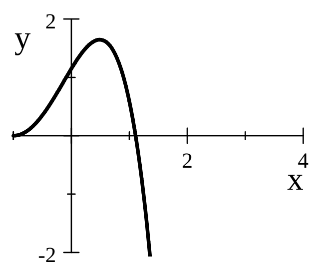
   i. $\log_2(x+3) - \log_2(2x+1) = 1$ we get:
   $\log_2 \frac{x+3}{2x+1} = \log_2 2$
   $\frac{x+3}{2x+1} = 2$, with: $x = \frac{1}{3}$.
   Verify:
   $LHS = \log_2\left(\frac{1}{3}+3\right) - \log_2\left(2\times\frac{1}{3}+1\right) =$
   $\log_2\left(\frac{10}{3} \div \frac{5}{3}\right) = \log_2 2 = 1 = RHS$

   ii. $\ln(x+1) + \ln(x+2) = 2\ln 3 + 3\ln 2$ we get:
   $\ln(x+1)(x+2) = \ln\left(2^3 \times 3^2\right)$
   $(x+1)(x+2) = 72$, with: $x = -10, x = 7$
   Verify: $x = -10:$
   $LHS = \ln(-9) + ...$ is not defined
   Verify: $x = 7:$
   $LHS = \ln(7+1) + \ln(7+2) = \ln 8 + \ln 9 =$
   $\ln 3^2 + \ln 2^3 = 2\ln 3 + 3\ln 2 = RHS$
   Therefore the only solution is $x = 7$.

   (b) For the graph $y = f(x)$ we get the graphs of: $y = 2f(x+1) - 1$ by moving 1 to the left, strech vertically twice and then move 1 down:

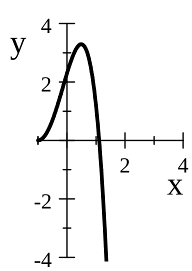


$\mathbf{y = f(x)}$



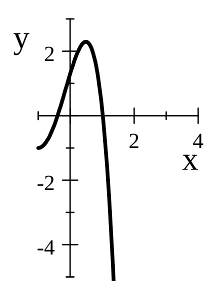
$\mathbf{y = f(x+1)}$

$\rightarrow$



$\mathbf{y = 2f(x+1)}$

$\rightarrow$



$\mathbf{y = 2f(x+1) - 1}$

(c) For $F(x) = \cos^2 x$ we get:

$$\begin{array}{ccccc}
x & & x & & x \\
\downarrow & & \downarrow & & \downarrow \\
\boxed{g(x) = \cos x} & & \boxed{f(x) = x^2} & & \boxed{f(x) = x^2} \\
\downarrow & & \downarrow & & \downarrow \\
u = \cos x & \rightarrow & u = x^2 & \rightarrow & u = x^2 \\
\downarrow & & \downarrow & & \downarrow \\
\boxed{f(u) = u^2} & & \boxed{f(u) = u^2} & & \boxed{g(u) = \cos u} \\
\downarrow & & \downarrow & & \downarrow \\
y = F(x) = \cos^2 x & & y = G(x) = x^4 & & y = H(x) = \cos\left(x^2\right)
\end{array}$$

2. Calculate the following limits:

   (a) $\lim\limits_{x\to 2} \frac{x^2-3x+2}{x-2} = \lim\limits_{x\to 2} \frac{(x-1)(x-2)}{x-2} = \lim\limits_{x\to 2} (x-1) = 1$

   (b) Since for $x > 1 : |x-1| = x-1 : \lim\limits_{x\to 1^+} \frac{|x-1|}{x-1} = 1$.

   Since for $x < 1 : |x-1| = -(x-1) : \lim\limits_{x\to 1^-} \frac{|x-1|}{x-1} = -1$.

   Therefore $\lim\limits_{x\to 1} \frac{|x-1|}{x-1}$ does not exist.

3. Use the L'Hospital's Rule to calculate:

   (a) $\lim\limits_{x\to 0} \frac{\sin x}{x} = \lim\limits_{x\to 0} \cos x = 1$ since the limit is "$\frac{0}{0}$".

   (b) $\lim\limits_{x\to 0} \left(\frac{1}{\ln(x+1)} - \frac{1}{x}\right) = \lim\limits_{x\to 0} \frac{x-\ln(x+1)}{x\ln(x+1)}$ and

   $\lim\limits_{x\to 0} \frac{x-\ln(x+1)}{x\ln(x+1)} = \lim\limits_{x\to 0} \frac{1-\frac{1}{x+1}}{\ln(x+1)+\frac{x}{x+1}} = \lim\limits_{x\to 0} \frac{x}{(x+1)\ln(x+1)+x} =$

   $\lim\limits_{x\to 0} \frac{1}{\ln(x+1)+2} = \frac{1}{2}$ since the limits are "$\frac{0}{0}$".

   (c) $\lim\limits_{x\to 0} (x+1)^{\frac{1}{x}} = e^{\lim\limits_{x\to 0} \frac{\ln(x+1)}{x}}$ and

   $\lim\limits_{x\to 0} \frac{\ln(x+1)}{x} = \lim\limits_{x\to 0} \frac{1}{x+1} = 1$ since the limit is "$\frac{0}{0}$". Therefore

   $\lim\limits_{x\to 0} (x+1)^{\frac{1}{x}} = e^{\lim\limits_{x\to 0} \frac{\ln(x+1)}{x}} = e^1 = e.$

4. The graph of: $y = \frac{x^2+2x-3}{x^2-3x+2} = \frac{(x+3)(x-1)}{(x-2)(x-1)} = \frac{x+3}{x-2}$. Therefore:

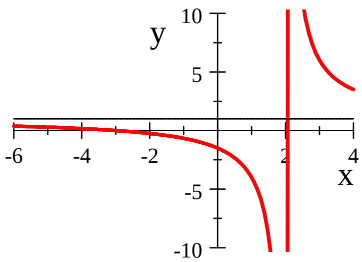
   (a) $\lim\limits_{x\to 2} \left|\frac{x+3}{x-2}\right| = \infty$ gives a vertical asymptote $x = 2$.

   (b) $\lim\limits_{x\to \infty} \frac{x+3}{x-2} = 1$ gives a horizontal asymptote $y = 1$.

   (c) and the graph is discontinuous at $x = 1$.

   (d) There is only one $x-$intercept: $x = -3$.

   (e) The garph is then:

i. 

5. Since $s(t) = t^2 - 8t + 18$ we get:

   (a) the average velocitiy over the time interval $s(8) = 18$

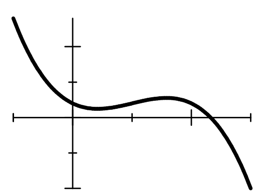
      i. $[4,6] : \frac{s(6)-s(4)}{4-2} = \frac{6-2}{2} = 2$ m/sec and

      ii. $[6,8] : \frac{s(8)-s(6)}{8-6} = \frac{18-6}{2} = 6$ m/sec.

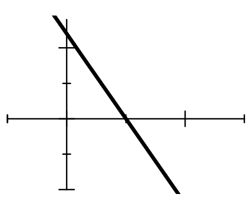
   (b) The instantaneous velocity when $t = 6$ is: $s'(t) = 2t - 8$ and $s'(6) = 4$ m/sec.

   (c) From the above answers it is clear that the velocity is increasing all the time. Therefore the accelleration $a(6) > 0$.

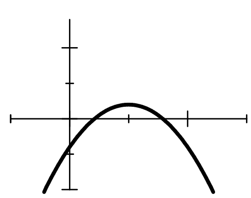
6. Since $f(x)$ is decreasing in $(-\infty, a) \cup (b, \infty)$ the derivaitive is there negative. That points to graph #3. From this graph it follows that the deraivative is increasing in (b, ∞) where $b > 0$. Therefore the second (its) derivative is there positive. That points to graph #1.



Graph of $\mathbf{y = f(x)}$



:#1:**Graphs is of $\mathbf{y = f''(x)}$**



#3: **Graphs is of : $\mathbf{y = f'(x)}$**

7. (a) The definition of the derivative is: $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$.

   (b) From the above definition for $f(x) = \sqrt{x}$ we get
   $f'(x) = \lim_{h\to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h} = \lim_{h\to 0} \frac{\left(\sqrt{x+h}-\sqrt{x}\right)\left(\sqrt{x+h}+\sqrt{x}\right)}{h\left(\sqrt{x+h}+\sqrt{x}\right)} =$
   $\lim_{h\to 0} \frac{h}{h\left(\sqrt{x+h}+\sqrt{x}\right)} = \lim_{h\to 0} \frac{1}{\sqrt{x+h}+\sqrt{x}} = \frac{1}{2\sqrt{x}}$ for $x > 0$.

8. Use the rules for differentiation to calculate $f'(x)$ for:

   (a) $f(x) = \frac{\sin x}{x^2+1} \Rightarrow f'(x) = \frac{\left(x^2+1\right)\cos x - 2x\sin x}{(x^2+1)^2}$

   (b) $f(x) = x \ln \frac{1}{x} = -x\ln x \Rightarrow f'(x) = -\ln x - 1$

   (c) $f(x) = \sqrt{\sin x + 1} \Rightarrow f'(x) = \frac{\cos x}{2\sqrt{\sin x+1}}$

   (d) $f(x) = \sin x^x \sqrt{(3x+1)^3} \Rightarrow \ln f(x) = x \ln\sin x + \frac{3}{2}\ln(3x+1)$
   giving:
   $\frac{f'(x)}{f(x)} = \ln\sin x + \frac{x\cos x}{\sin x} + \frac{9}{2(3x+1)}$. Therefore:
   $f'(x) = \sin x^x \sqrt{(3x+1)^3}\left[\ln\sin x + \frac{x\cos x}{\sin x} + \frac{9}{2(3x+1)}\right]$

9. The curve $\mathfrak{C}$ is given implicitely by: $y^2 - x^3(2-x) = 28$.

   (a) Then for $\mathbf{A}(3,-1)$ we get on the $LHS = 1 - 27(-1) = 28 = RHS$.

   (b) From: $\frac{d}{dx}\left(y^2 - x^3(2-x) = 28\right)$ we get:
   $2yy' - 6x^2 + 4x^3 = 0$, giving:
   $y' = -x^2 \frac{-3+2x}{y}]_{(3,-1)} = 27$.
   The tangent line's equation is: $y = 27(x-3) - 1 = 27x - 82$

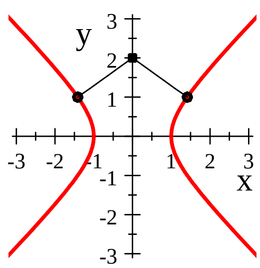
10. For $f(x) = \frac{1+2x}{1-x}$, then
    $f'(x) = \frac{3}{(x-1)^2}$, $f''(x) = \frac{-6}{(x-1)^3}$ and $f'''(x) = \frac{18}{(x-1)^4}$.

11. We use $f(x) = x - \cos x$. The function $f(x)$ is continuous everywhere.

    (a) Since $f(0) = -1 < 0$ and $f(1) \approx 0.46 > 0$ from IVT it follows that there is a solution in $(0,1)$.

    (b) Also: $f'(x) = 1 + \sin x \geq 0$ therefore the function is always increasing, it can never go back to zero. Therefore there is only one solution.

12. We shall calculate the shortest distance between $\mathbf{A}(0,2)$ and the hyperbola $x^2 - y^2 = 1$



    as follows:

    (a) x is the $x-$ coordinate of a point on the hyperbola. The $y-$ coordinate is then: $y = \sqrt{x^2-1}$ since the closest point (we look for) is above $x-$ axis. This gives a point $\mathbf{X}\left(x, \sqrt{x^2-1}\right)$.

    (b) The function to be minimized is the distance:
    $\overline{\mathbf{XA}} : f(x) = \sqrt{x^2 + \left(\sqrt{x^2-1} - 2\right)^2}$
    or just the square of the distance $\overline{\mathbf{XA}}^2 : g(x) = x^2 + \left(\sqrt{x^2-1} - 2\right)^2$.

    (c) $g'(x) = 4x\frac{}{\sqrt{x^2-1}} = 0$. Since $x \neq 0$ we get: $\sqrt{x^2-1} - 1 = 0$, giving: $x = \pm\sqrt{2}$. This point cannot give a local maximum, as it does not exist.

    (d) The minimum distance is then: $f\left(\sqrt{2}\right) = \sqrt{3} \approx 1.7$.