Solutions

Question 1- Calculate the inverse of the following function:

$$f(x) = \frac{3x+2}{x-5}$$

A) $\frac{5x+2}{x-3}$ B) $\frac{3x-4}{2x-5}$ C) $\frac{5x+3}{2x-7}$ D) $\frac{3x-2}{x+4}$ E) $\frac{4x-2}{3x+1}$

$$y = \frac{3x+2}{x-5}$$

$$x = \frac{3y+2}{y-5}$$

$$x(y-5) = 3y+2$$

$$xy - 5x = 3y+2$$

$$xy - 3y = +5x+2$$

$$y(x-3) = 5x+2$$

$$y = \frac{5x+2}{x-3}$$

Question 2- Over what interval is the function $f(x) = xe^{2x}$ concave up?

A) $(\frac{1}{2}, \infty)$ B) $(-\infty, -\frac{1}{2})$ C) $(-\infty, 1)$ D) $(-1, \infty)$ E) $(0, 1)$

$$f'(x) = e^{2x} + 2xe^{2x}$$

$$= (2x+1)e^{2x}$$

$$f''(x) = 2e^{2x} + (4x+2)e^{2x}$$

$$= (4x+4)e^{2x}$$

One possible IP at $x = -1$

		-1	
$f''(x)$	$-$	0	$+$
$f(x)$	$\cap$		$\cup$

Question 3-Suppose $g(x) = \sqrt{2x^2+1}$. Find the equation of the tangent line at $x = 2$.

A) $y = \frac{4}{3}x - \frac{8}{3}$ **B)** $y = \frac{13}{3}x - \frac{4}{3}$ **C)** $y = \frac{13}{3}x - \frac{17}{3}$ **D)** $y = \frac{4}{3}x + \frac{1}{3}$ **E)** $y = \frac{13}{3}x + \frac{1}{3}$

$g(x) = (2x^2+1)^{1/2}$

$g'(x) = \frac{1}{2}(2x^2+1)^{-1/2}(4x)$

$= \dfrac{2x}{\sqrt{2x^2+1}}$

$g'(2) = \dfrac{4}{3}$

So $y = 4/3\, x + b$

Plug in $(2,3)$

$3 = 2(4/3) + b$

So $b = 1/3$

Question 4- Consider the following function:

$$f(x) = \begin{cases} x^2+6 & \text{if } x \geq 2; \\ -ax+3 & \text{if } x < 2. \end{cases}$$

For what value of the constant a is $f(x)$ continuous for all real numbers?

A) $\frac{1}{2}$ **B)** $-\frac{1}{2}$ **C)** $\frac{5}{4}$ **D)** $\frac{8}{3}$ **E)** $-\frac{7}{2}$

$\lim_{x \to 2^+} f(x) = 10$

$\lim_{x \to 2^-} f(x) = 3-2a.$

So we must have

$10 = 3-2a$

$a = \dfrac{-7}{2}$

Question 5- Solve the following equation for t.

$$2^{3t} = 5^{t-1}$$

A) $\frac{1}{\ln(5)+2\ln(3)}$ **B)** $-\frac{\ln(5)}{3\ln(2)-\ln(5)}$ **C)** $\frac{\ln(3)+4\ln(5)}{\ln(3)}$ **D)** $\frac{8\ln(5)+1}{\ln(3)}$ **E)** $-\frac{\ln(5)-2\ln(3)}{2\ln(5)}$

$$2^{3t} = 5^{t-1}$$

$$\ln(2^{3t}) = \ln(5^{t-1})$$

$$3t\ln(2) = (t-1)\ln(5)$$

$$3t\ln(2) - t\ln(5) = -\ln(5)$$

$$t = \frac{-\ln(5)}{3\ln(2)-\ln(5)}$$

Question 6- Evaluate:

$$\int_2^\infty \frac{1}{\sqrt{2x+3}}\mathrm{dx}$$

A) 1 **B)** $\frac{1}{2}$ **C)** 2 **D)** $2\ln(2)$ **E)** divergent

$$= \lim_{b\to\infty} \int_2^b (2x+3)^{-1/2}\,dx$$

$$= \lim_{b\to\infty} \left[(2b+3)^{1/2} - \sqrt{7}\right] = \infty$$

$$\boxed{\begin{aligned} &\int (2x+3)^{-1/2}\,dx \\ &\boxed{\begin{aligned} u &= 2x+3 \\ du &= 2dx \end{aligned}} \\ &= \frac{1}{2}\int u^{-1/2}\,du \\ &= \frac{1}{2}\cdot\frac{u^{1/2}}{1/2} = u^{1/2} \\ &= (2x+3)^{1/2} \end{aligned}}$$

Question 7- Suppose $f'(x) = 2x^3 + 4x^2 - 3$ and $f(1) = 3$. Find $f(0)$.

A) $\frac{17}{6}$ B) $-\frac{1}{6}$ C) $\frac{25}{6}$ D) $\frac{8}{3}$ E) $-\frac{17}{2}$

$$\int (2x^3 + 4x^2 - 3)\,dx = \frac{1}{2}x^4 + \frac{4}{3}x^3 - 3x + C$$

$$f(1) = \frac{1}{2} + \frac{4}{3} - 3 + C = 3$$

$$\Rightarrow -\frac{7}{6} + C = 3$$

$$C = \frac{25}{6}$$

So $f(0) = 0 + \frac{25}{6} = \frac{25}{6}$

Question 8- Suppose that for a certain product, the demand function is given by $D(x) = 28 - x^2$ and the supply function is given by $S(x) = x^2 + 2x + 4$. Calculate the producer surplus.

A) 24 B) $\frac{19}{3}$ C) $\frac{26}{3}$ D) 27 E) $\frac{25}{7}$

Equilibrium

$$28 - x^2 = x^2 + 2x + 4$$

$$0 = 2x^2 + 2x - 24$$

$$= 2(x^2 + x - 12)$$

$$= 2(x+4)(x-3)$$

$x = \cancel{-4}$ or 3

So $(3, 19)$ is equilibrium

$$PS = \int_0^3 [19 - (x^2 + 2x + 4)]\,dx$$

$$= \int_0^3 (15 - x^2 - 2x)\,dx$$

$$= 15x - \frac{x^3}{3} - x^2 \Big|_0^3$$

$$= 45 - 9 - 9$$

$$= 27$$

Question 9- If $f(x,y) = x\sqrt{x^2+y^2+1}$, what is $f_x(2,2)$?

A) $\frac{8}{27}$ **B)** $\frac{-1}{4}$ **C)** $\frac{3}{2}$ **D)** $\frac{1}{2}$ **E)** $\frac{13}{3}$

$$\frac{\partial f}{\partial x} = \sqrt{x^2+y^2+1} + x\left(\frac{1}{2}(x^2+y^2+1)^{-1/2}\cdot 2x\right)$$

$$= \sqrt{x^2+y^2+1} + \frac{x^2}{\sqrt{x^2+y^2+1}}$$

$$\frac{\partial f}{\partial x}(2,2) = 3 + \frac{4}{3} = \frac{13}{3}$$

Question 10- If $f(x,y) = x^2 + 6xy + 2y^2 - 3y + 4$, how many critical points does $f(x,y)$ have?

A) 0 B) 1 C) 2 D) 3 E) 4

$f_x = 2x + 6y$
$f_y = 6x + 4y - 3$ } Set both equal to 0

$-3(2x+6y=0)$
$6x+4y=3$

① $-6x - 18y = 0$
② $6x + 4y = 3$

Add 2 equations

$-14y = 3$

$y = \frac{-3}{14}$

Plug into ①

$2x - \frac{18}{14} = 0$

$x = \frac{9}{14}$

So one CP at $(9/14, -3/14)$

Long Answer Question 1 (10 points)

Recall that radioactive substances decay exponentially. Suppose I have 12 grams of a radioactive substance. After 11 years, I have 6 grams remaining.

- (4 points) Find a formula for the number of grams of the substance left after t years.
- (6 points) How many years will it take before there are only 2 grams remaining? (Do not simplify your answer.)

$$P(t) = 12b^t$$

$$P(11) = 12b^{11} = 6$$

$$b^{11} = \frac{1}{2}$$

$$b = \left(\frac{1}{2}\right)^{1/11}$$

a) So $P(t) = 12\left(\frac{1}{2}\right)^{t/11}$

b) Solve

$$2 = 12\left(\frac{1}{2}\right)^{t/11} \text{ for } t$$

$$\frac{1}{6} = \left(\frac{1}{2}\right)^{t/11}$$

$$\ln\left(\frac{1}{6}\right) = \frac{t}{11}\ln(1/2)$$

So $t = \dfrac{11\ln(1/6)}{\ln(1/2)}$

Long Answer Question 2 (12 points)

Calculate the following two indefinite integrals:

$$\int x^4 \ln(x) dx$$

Parts :

$$\boxed{\begin{array}{ll} u = \ln(x) & v = \frac{x^5}{5} \\ du = \frac{1}{x} dx & dv = x^4 dx \end{array}}$$

$$= \frac{x^5}{5} \ln(x) - \int \frac{x^4}{5} dx$$

$$= \frac{x^5}{5} \ln(x) - \frac{x^5}{25} + C$$

$$\int \frac{x}{(2x^2+1)^4} dx$$

$$\boxed{\begin{array}{l} \text{U-Substitution} \\ u = 2x^2+1 \\ du = 4x\,dx \end{array}}$$

$$= \frac{1}{4} \int \frac{4x\,dx}{(2x^2+1)^4}$$

$$= \frac{1}{4} \int u^{-4} du$$

$$\frac{1}{4} \frac{u^{-3}}{-3}$$

$$= \frac{-1}{12} u^{-3}$$

$$= \frac{-1}{12} (2x^2+1)^{-3} + C$$

Long Answer Question 3 (12 points)

A bus company will charter a bus for tours. If a group contains exactly 36 people, each person pays 60 dollars. In larger groups, every person's fare is reduced by 50 cents for each person in excess of 36 people. The cost to the bus company is 8 dollars per person. Determine the size of the group for which the bus company's profit will be greatest. What price will they charge this size group per person?

Be sure to explain why your answer is an absolute maximum.

$x =$ # of people in group

Revenue = # of people $\cdot$ price/person

$$= x \cdot \Big(60 - \underbrace{(x-36)}_{\text{\# of people over 36}} \cdot \underbrace{\tfrac{1}{2}}_{\text{50 cent reduction}}\Big)$$

$$= x\left(60 - \frac{1}{2}x + 18\right)$$

$$= -\frac{1}{2}x^2 + 78x$$

Cost $= 8x$

Profit = Revenue − Cost $= -\frac{1}{2}x^2 + 70x$

$P'(x) = -x + 70$ So $x = 70$

It's an absolute max since $P(x)$ is a CD parabola

Long Answer Question 4 (14 points)

Consider the two functions:

$f(x) = x^2 - 2x$ and $f(x) = 4 - x^2$

(a) **(2 points)** Find the intersection points of the graphs of the two functions.

(b) **(6 points)** On the next page, graph these functions, and shade the region between the graphs of f and g for $x \in [0, 3]$.

(c) **(6 points)** Find the area of the shaded region.

a) $x^2 - 2x = 4 - x^2$

$0 = 2x^2 - 2x - 4 = 2(x^2 - x - 2) = 2(x-2)(x+1)$

I.P.s at $x = -1, 2$

b) $x^2 - 2x = x(x-2)$ CU parabola, local min at $x=1$

$4 - x^2 = (2+x)(2-x)$ CD parabola, local max at $x=0$

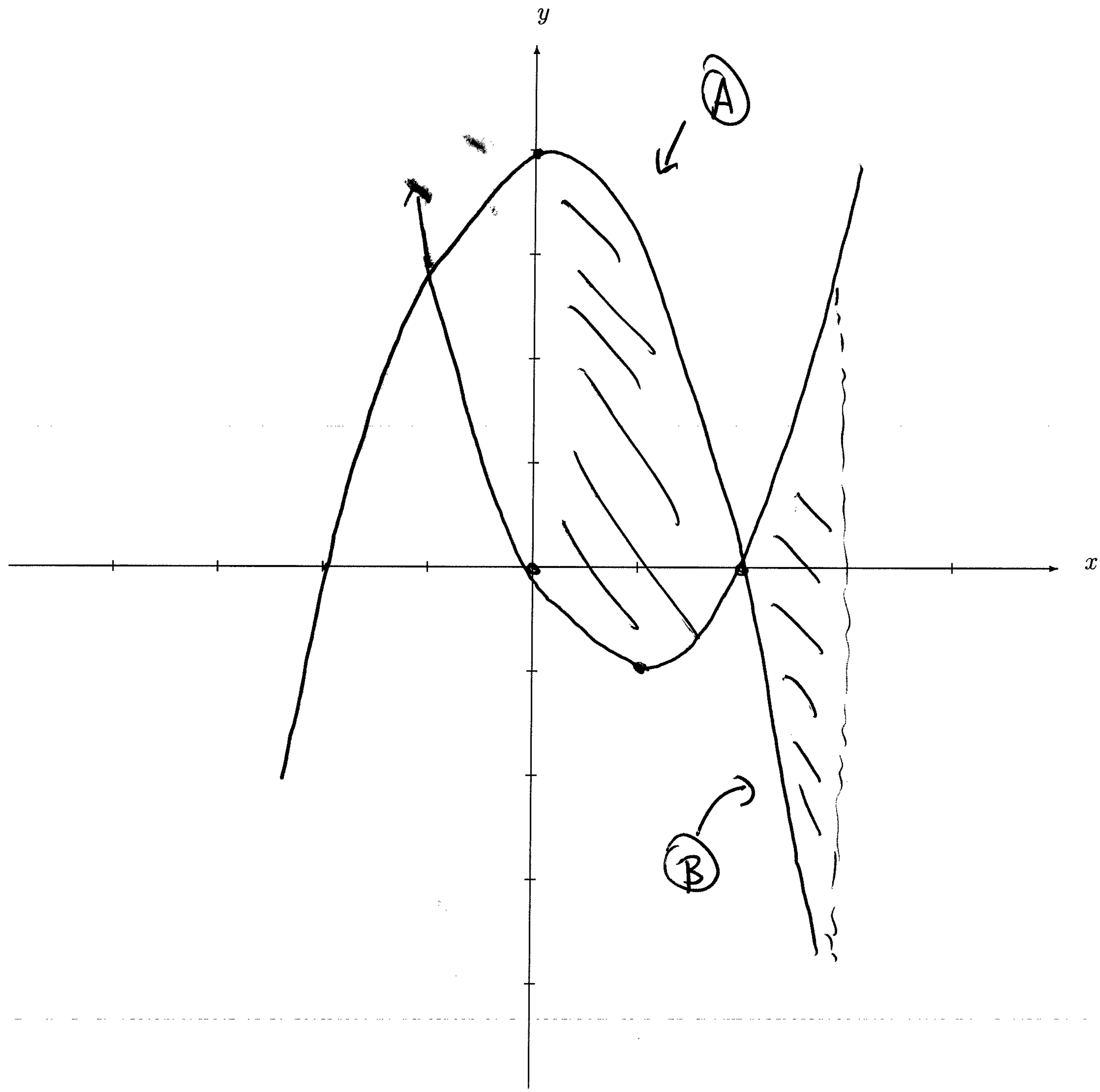
c) Area of (A) $= \int_0^2 [(4-x^2) - (x^2-2x)]\,dx = \int_0^2 (4+2x-2x^2)\,dx$

$= 20/3$

Area of (B) $= \int_2^3 [(x^2-2x) - (4-x^2)]\,dx = \int_2^3 (2x^2-2x-4)\,dx$

$= 11/3$

Total $= 31/3$



y
x
A
B

1.

Long Answer Question 5 (12 points)

Consider the function of two variables

$$f(x,y) = xy - x^3 + y^3 + 2$$

(a) **(3 points)** Calculate the first-order partial derivatives.

(b) **(3 points)** Find all critical points.

(c) **(6 points)** Identify what type of critical points they are (local max, local min or saddle point).

$f_x = y - 3x^2 = 0 \Rightarrow y = 3x^2$

$f_y = x + 3y^2 = 0$

plug into second equation

$x + 3(3x^2)^2 = 0$

$x + 27x^4 = 0$

$x(1 + 27x^3) = 0$

So $x = 0$ or $-\frac{1}{3}$

If $x = 0$, $y = 0$

If $x = -\frac{1}{3}$, $y = \frac{1}{3}$

So 2 CPs at $(0,0)$ & $\left(-\frac{1}{3}, \frac{1}{3}\right)$

$f_{xx} = -6x$

$f_{yy} = 6y$

$f_{xy} = 1$

$D = -36xy - 1$

At $(0,0)$, $D = -1$

So saddle point

At $\left(-\frac{1}{3}, \frac{1}{3}\right)$, $D = 3$

and $f_{xx} > 0$. So local min.