MATH 1402A **Practice Term test #3 – solution** **31 October 2013**

Total marks: XX marks

Non-programmable calculators are permitted. This test is closed book.
<u>Supply your answers on this sheet</u>, but TA's have extra paper if you need it.

PLEASE PRINT	________________ First name	________________ Last name	__________ Student number

Please show your work where appropriate!

1. Evaluate the following integrals.

a. $\int\left(\frac{1}{x}+10x^4-\frac{2}{x^3}-8+15\sqrt{x^3}-4e^x\right)dx = \int \frac{1}{x}+10x^4-2x^{-3}-8x^0+15x^{3/2}-4e^x\,dx =$

$\ldots = \ln|x| + 10\frac{x^5}{5} - 2\frac{x^{-2}}{-2} - 8\frac{x^1}{1} + 15\frac{x^{5/2}}{5/2} - 4e^x + C_1 =$

$\ldots = \ln|x| + 2x^5 + \frac{1}{x^2} - 8x + 6\sqrt{x^5} - 4e^x + C_1$

b. $\int\frac{(x+1)^3}{\sqrt{x^3}}\,dx = \int \frac{x^3+3x^2+3x+1}{x^{3/2}}\,dx = \int x^{3/2}+3x^{1/2}+3x^{-1/2}+x^{-3/2}\,dx =$

$\ldots = \frac{x^{5/2}}{5/2} + 3\frac{x^{3/2}}{3/2} + 3\frac{x^{1/2}}{1/2} + \frac{x^{-1/2}}{-1/2} + C_1 =$

$\ldots = \frac{2}{5}\sqrt{x^5} + 2\sqrt{x^3} + 6\sqrt{x} - \frac{2}{\sqrt{x}} + C_1$

c. $\int\frac{1}{x\sqrt{\ln x}}\,dx \Rightarrow \begin{array}{l} u = \ln x \\ du = \frac{1}{x}\,dx \end{array}$

$\Downarrow$

$\int \frac{1}{\sqrt{u}}\,du = \int u^{-1/2}\,du = \frac{u^{1/2}}{1/2} + C_1 = 2\sqrt{u} + C_1 =$

$\ldots = 2\sqrt{\ln x} + C_1$

d. $\int 9(3x^3+1)^7x^2\,dx \Rightarrow \begin{array}{l} u = 3x^3+1 \\ du = 9x^2\,dx \end{array}$

$\Downarrow$

$\int u^7\,du = \frac{u^8}{8} + C_1 = \frac{1}{8}(3x^3+1)^8 + C_1$

e. $\int e^{(\ln x+x^3)}\left(\frac{1}{x}+3x^2\right)dx$ (simplify this one after you've determined the antiderivative..)

$u = \ln x + x^3$
$du = (1/x + 3x^2)\,dx$

$\int e^u\,du = e^u + C_1 = e^{(\ln x + x^3)} + C_1 =$

$\ldots = e^{\ln x}\cdot e^{x^3} + C_1 = xe^{x^3} + C_1$

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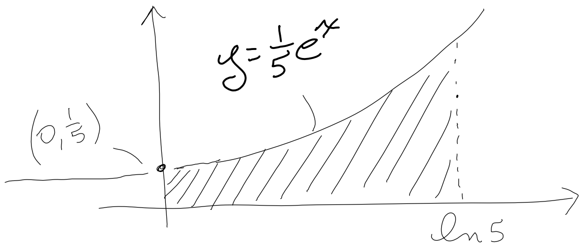
2. Determine the area under the curve $y = \frac{1}{5}e^x$ between the values $x = 0$ and $x = \ln 5$. **Note**: the curve is above the x axis in this interval.



Because the curve is above the x-axis:

$$A = \int_0^{\ln 5} f(x)\,dx, \quad f(x) = \frac{1}{5}e^x$$

$$\ldots = \int_0^{\ln 5} \frac{1}{5}e^x\,dx = \frac{1}{5}\int_0^{\ln 5} e^x\,dx = \frac{1}{5}\Big[e^x\Big]_0^{\ln 5}$$

$$\ldots = \frac{1}{5}\left[e^{\ln 5} - e^0\right] = \frac{1}{5}\left[5 - 1\right] = \boxed{4/5}$$

3. Determine the area of the region that is enclosed between the curves $y = 4x + 13$ and $y = x^2 + 5x - 7$.

- Intersection points. $y = y \Rightarrow 4x + 13 = x^2 + 5x - 7$

$$x^2 + x - 20 = 0$$

$$(x+5)(x-4) = 0 \Rightarrow x = -5 \ (= a)$$

$$x = 4 \ (= b)$$

- evaluate both functions at a point between -5 and 4, say $x = 0$: $y = 4x + 13 \Rightarrow y\big|_{x=0} = 4(0) + 13) = 13$

$$y = x^2 + 5x - 7 \Rightarrow y\Big|_{x=0} = (0)^2 + 5(0) - 7 = -7$$

$\therefore$ $f_1(x) = 4x + 13$ (top curve)

$f_2(x) = x^2 + 5x - 7$ (bottom curve)

- Set up integral and evaluate:

$$A = \int_{-5}^{4} f_1(x) - f_2(x)\,dx = \int_{-5}^{4} (4x+13) - (x^2 + 5x - 7)\,dx =$$

$$\ldots = \int_{-5}^{4} -(x^2 + x - 20)\,dx = -\left[\frac{x^3}{3} + \frac{x^2}{2} - 20x\right]_{-5}^{4} =$$

$$\ldots = -\left[\left(\frac{4^3}{3} + \frac{4^2}{2} - 20(4)\right) - \left(\frac{(-5)^3}{3} + \frac{(-5)^2}{2} - 20(-5)\right)\right] =$$

$$\ldots = -\left[\frac{-304}{6} - \frac{425}{6}\right] = \frac{729}{6} = \boxed{121\ 1/2}$$

4. Let $F(x) = \int 10x^4 - 9x^2 + 1\,dx$.

   a. Determine the most general antiderivative $F(x)$.

$$F(x) = \int 10x^4 - 9x^2 + 1\ dx = 2x^5 - 3x^3 + x + C_1$$

$$\therefore \ F(x) = 2x^5 - 3x^3 + x + C_1$$

   b. Determine the constant of integration if $F(-1) = 3$

Also: $F(-1) = 3 = 2(-1)^5 - 3(-1)^3 + (-1) + C_1 \Rightarrow C_1 = 3$

$-2 + 3 - 1 + C_1$

$$\therefore \ F(x) = 2x^5 - 3x^3 + x + 3$$

5. **Profit maximization:** you are given the following information about a firm: the wage rate (***w***) is 240, the rental rate (***r***) is 10, the price (***p***) is 120 and production output is given by the model $Q = 5K^{0.2}L^{0.6}$. Use ***any method*** to determine: $\longrightarrow$ Direct substitution or Lagrange Multipliers.

a. The values of K and L that maximize profits.	b. The resulting production output Q.	c. The profit Π.

Answer: $L = 60.75$; $K = 486$; $Q = 202.5$; $\pi = 4860$

Indeed:

$$\pi = TR - TC = pQ - wL - rK = \underbrace{120 \left(5 K^{0.2} L^{0.6}\right)}_{600} - 240L - 10K$$

$$\frac{\partial \pi}{\partial L} = 0 \Rightarrow 360 K^{0.2} L^{-0.4} - 240 = 0 \Rightarrow 6K^{0.2} L^{-0.4} = 4 \quad \text{— (1)}$$

$$\frac{\partial \pi}{\partial K} = 0 \Rightarrow 120 K^{-0.8} L^{0.6} - 10 = 0 \Rightarrow 12 K^{-0.8} L^{0.6} = 1 \quad \text{— (2)}$$

$$(1) \div (2) \Rightarrow \frac{K}{L} = 8 \Rightarrow K = 8L$$

Back-substitute in (1) or (2):

$$6(8L)^{0.2} L^{-0.4} = 4 \Rightarrow L^{-0.2} = \frac{2}{3(8^{0.2})}$$

$$\Rightarrow L = \left[\frac{3(8^{0.2})}{2}\right]^5 = \frac{3^5}{2^5} \cdot 8 = \boxed{60.75}$$

and $K = 8L = 8(60.75) = \boxed{486}$

and: $Q = 5(486)^{0.2}(60.75)^{0.6} = \boxed{202.5}$

and: $\pi = pQ - wL - rK = 120(202.5) - 240(60.75) - 10(486)$

$$\boxed{\pi = 4860}$$