

Non-programmable calculators are permitted. This test is closed book.

Supply your answers on this sheet, but TA's have extra paper if you need it.

PLEASE PRINT

First name

Last name

Student number

Please show your work where appropriate!

1. Find the minimum Total Cost, given the following information:

TC is given by $TC = 128L + 32K$, and is subject to the constraint $Q = Q_0 = 256 = 32K^{1/4} L^{3/4}$.

USE ANY METHOD. For ex., using method 3 (Lagrange):

$$\Lambda = TC + \lambda (Q_0 - AK^{\alpha}L^{\beta}) = 128L + 32K + \lambda (256 - 32K^{1/4}L^{3/4})$$

$$\Lambda_L = \frac{\partial \Lambda}{\partial L} = 0 \Rightarrow 128 - 32\lambda \left(\frac{3}{4}\right) K^{1/4} L^{-1/4} = 0 \Rightarrow \lambda K^{1/4} L^{-1/4} = 16/3 \text{ --- (1)}$$

$$\Lambda_K = \frac{\partial \Lambda}{\partial K} = 0 \Rightarrow 32 - 32\lambda \left(\frac{1}{4}\right) K^{-3/4} L^{3/4} = 0 \Rightarrow \lambda K^{-3/4} L^{3/4} = 4 \text{ --- (2)}$$

$$\Lambda_{\lambda} = \frac{\partial \Lambda}{\partial \lambda} = 0 \Rightarrow 256 - 32K^{1/4}L^{3/4} = 0 \Rightarrow K = \frac{4096}{L^3} \text{ --- CONSTRAINT}$$

$$(1) \div (2) \Rightarrow \frac{K}{L} = \frac{4}{3} \Rightarrow K = \frac{4}{3}L$$

Substitute " $\frac{4}{3}L$ " for "K" (i.e. result from hybrid) into constraint:
 $\therefore \frac{4}{3}L = \frac{4096}{L^3} \Rightarrow L^4 = 3 \cdot (1024) \Rightarrow L^4 = 3072$
 $L \approx 7.44$

Then: $K = \frac{4}{3} (3072)^{1/4} \approx \frac{4}{3} (7.44) = 9.93$

And: $TC = 128(7.44) + 32(9.93) \approx 1271$

....

The other methods yield the same result.

$$\underbrace{f(K,L) = 128L + 32K}_{\text{objective } f} \quad \text{and} \quad \underbrace{256 - 32K^{1/4}L^{3/4} = g(K,L) = 0}_{\text{constraint}}$$

$$f_K = 32, f_L = 128$$

$$g_K = -8K^{-3/4}L^{3/4}$$

$$g_L = -24K^{1/4}L^{-1/4}$$

$$\frac{f_L}{f_K} = \frac{g_L}{g_K} \Rightarrow \frac{128}{32} = \frac{-24K^{1/4}L^{-1/4}}{-8K^{-3/4}L^{3/4}} \Rightarrow K = \frac{4}{3}L$$

Substitute $\frac{4}{3}L$ for K in constraint:

$$256 - 32\left(\frac{4}{3}L\right)^{1/4}L^{3/4} = 0$$

$$L = 8\left(\frac{3}{4}\right)^{1/4} \approx 7.44, \text{ etc...}$$

2. Let $x^3 + y^3 = 4xy$. Determine dy/dx at the point $(2, 2)$.

Method 1: $\frac{d}{dx}(x^3 + y^3) = \frac{d}{dx}(4xy) = 4 \frac{d}{dx}(xy)$

$$3x^2 + 3y^2 y' = 4(xy' + y)$$

$$3x^2 + 3y^2 y' = 4xy' + 4y$$

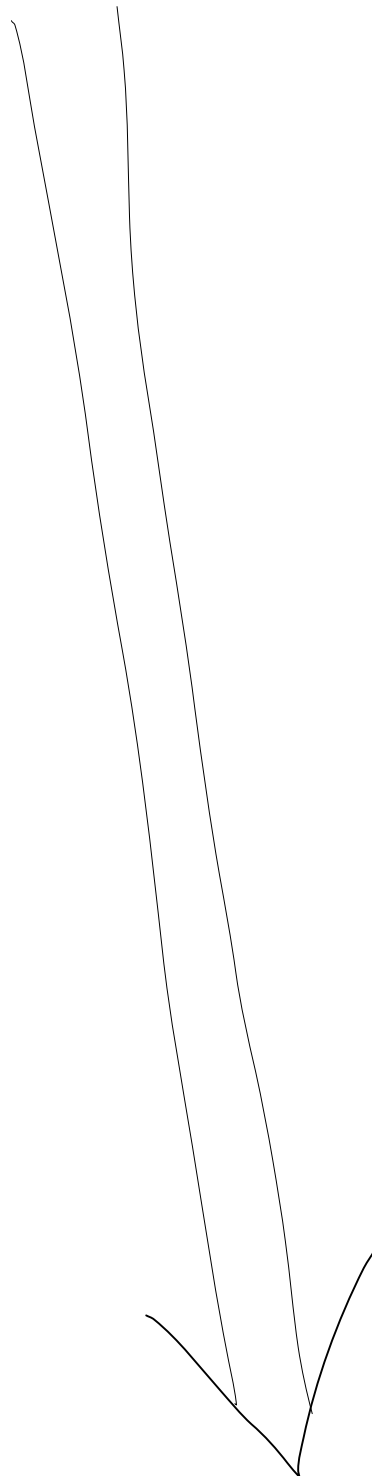
$$y'(3y^2 - 4x) = 4y - 3x^2$$

$$y' = \frac{4y - 3x^2}{3y^2 - 4x} \Rightarrow y' \Big|_{(2,2)} = \frac{4(2) - 3(2)^2}{3(2)^2 - 4(2)} = -1$$

Method 2: $x^3 + y^3 = 4xy \Rightarrow x^3 + y^3 - 4xy = f(x, y) = 0$

$$f_x = 3x^2 - 4y \quad ; \quad f_y = 3y^2 - 4x$$

$$\frac{dy}{dx} = y' = -\frac{f_x}{f_y} = -\frac{3x^2 - 4y}{3y^2 - 4x} \Rightarrow y' \Big|_{(2,2)} = -\frac{3(2)^2 - 4(2)}{3(2)^2 - 4(2)} = -1$$



3. Let: $z = f(x, y) = x^3 - 2xy^2$ and $y = g(x) = 2x - 5$. Find dz/dx , the total derivative, **using the 2 methods in reviewed in class** (i.e. direct substitution or the formula for the total derivative)

Method 1: Direct Substitution (z is a function of both x and y , but y is itself a function of x . The variable z can therefore be expressed as a sole function of x and the derivative dz/dx determined.

$$z = f(x, y) = x^3 - 2xy^2 \quad \text{and} \quad y = g(x) = 2x - 5$$

$$\therefore z = h(x) = x^3 - 2x(2x - 5)^2 = x^3 - 2x(4x^2 - 20x + 25) = x^3 - 8x^3 + 40x^2 - 50x =$$

$$\therefore z = h(x) = -7x^3 + 40x^2 - 50x \quad \text{and} \quad \therefore \frac{dz}{dx} = -21x^2 + 80x - 50$$

Method 2: Total derivative formula: use the following formula to compute dz/dx :

$$\frac{dz}{dx} = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} \frac{dy}{dx} \quad (\text{The first and second terms are called the direct and indirect effects respectively.})$$

respectively.

$$z = f(x, y) = x^3 - 2xy^2 \quad \text{and} \quad y = g(x) = 2x - 5$$

$$\text{Use: } \frac{dz}{dx} = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} \frac{dy}{dx}, \text{ with: } \frac{\partial z}{\partial x} = 3x^2 - 2y^2, \quad \frac{\partial z}{\partial y} = -4xy, \quad \text{and } \frac{dy}{dx} = 2$$

$$\therefore \frac{dz}{dx} = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} \frac{dy}{dx} = 3x^2 - 2y^2 - 8xy$$

$$\therefore \frac{dz}{dx} = 3x^2 - 2y^2 - 8xy = 3x^2 - 2(2x - 5)^2 - 8x(2x - 5) =$$

$$\dots = 3x^2 - 2(4x^2 - 20x + 25) - 16x^2 + 40x = 3x^2 - 8x^2 + 40x - 50 - 16x^2 + 40x$$

$$\therefore \frac{dz}{dx} = -21x^2 + 80x - 50$$

4. Let $z = f(x, y) = 2x^2 + 2xy - y^2 + 10x - 7y + 20$. Use the differential dz to estimate the change in z (i.e. Δz) when moving from $(x, y) = (2, 2)$ to $(2.05, 1.9)$. Compare the result with the actual value of Δz .

We consider $(2, 2)$ to be the “reference point” (x_0, y_0) , with $dx = 0.05$ and $dy = -0.1$. The total differential is given by:

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$$

And we write:

$$\left. \begin{array}{l} \frac{\partial z}{\partial x} = 4x + 2y + 10 \\ \frac{\partial z}{\partial y} = 2x - 2y - 7 \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{l} \left. \frac{\partial z}{\partial x} \right|_{(x,y)=(2,2)} = 4(2) + 2(2) + 10 = 22 \\ \left. \frac{\partial z}{\partial y} \right|_{(x,y)=(2,2)} = 2(2) - 2(2) - 7 = -7 \end{array} \right.$$

Such that:

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy \Rightarrow dz = 22(0.05) + (-7)(-0.1) = 1.8$$

The **actual** change in z is given by:

$$\Delta z = f(2.05, 1.9) - f(2, 2), \text{ with:}$$

$$f(2.05, 1.9) = 2(2.05)^2 + 2(2.05)(1.9) - (1.9)^2 + 10(2.05) - 7(1.9) + 20 = 39.785$$

and

$$f(2, 2) = 2(2)^2 + 2(2)(2) - (2)^2 + 10(2) - 7(2) + 20 = 38$$

Such that: $\Delta z = f(2.05, 1.9) - f(2, 2) = 39.785 - 38 = 1.785 \dots$ so dz is not that *bad* an estimate...

5. Using the same function as in the previous exercise, determine the (x_0, y_0, z_0) coordinates of any stationary point and determine whether they are a minimum, maximum or saddle point.

$$z = f(x, y) = 2x^2 + 2xy - y^2 + 10x - 7y + 20$$

$$\left(\frac{\partial z}{\partial x} = 0\right) \Leftrightarrow (4x + 2y + 10 = 0) \qquad \left(\frac{\partial z}{\partial y} = 0\right) \Leftrightarrow (2x - 2y - 7 = 0)$$

$$\begin{cases} 4x + 2y + 10 = 0 \\ 2x - 2y - 7 = 0 \end{cases} \Leftrightarrow \begin{cases} x = -\frac{1}{2} \\ y = -4 \end{cases} \quad \text{Stationary point at : } (x_0, y_0) = \left(-\frac{1}{2}, -4\right) \text{, max., min., saddle ??}$$

$$\frac{\partial^2 z}{\partial x^2} = 4 \quad \text{and} \quad \frac{\partial^2 z}{\partial y^2} = -2 \quad \text{Second derivatives of opposite sign... saddle point..}$$

$$z_0 = f(x_0, y_0) = 2\left(-\frac{1}{2}\right)^2 + 2\left(-\frac{1}{2}\right)(-4) - (-4)^2 + 10\left(-\frac{1}{2}\right) - 7(-4) + 20 = 31.5$$

$(x_0, y_0, z_0) = \left(-\frac{1}{2}, -4, 31.5\right)$ is a saddle point

6. Let $Q = f(K, L) = 4K^{\frac{1}{4}} L^{\frac{1}{2}}$ be the rule for a production function, where K corresponds to *input capital*, L corresponds to *input labour* and Q corresponds to *output production*.

Assuming $Q = Q_0 = 8$:

Determine MRS using the ANY METHOD

If $Q=Q_0=8$, then the isoquant of the Production function is a curve describing the relationship between K and L . dK/dL is determined directly from this equation. Alternatively, we can use the formula $dK/dL = -f_L/f_K$, where f_L and f_K correspond to dQ/dL and dQ/dK respectively. MRS is the absolute value of dK/dL .

Method 1:

$$Q = f(K, L) = 4K^{\frac{1}{4}} L^{\frac{1}{2}} \quad \text{and} \quad Q = Q_0 = 8 = 4K^{\frac{1}{4}} L^{\frac{1}{2}}$$

$$\left(8 = 4K^{\frac{1}{4}} L^{\frac{1}{2}}\right) \Leftrightarrow \left(K^{\frac{1}{4}} L^{\frac{1}{2}} = 2\right) \Leftrightarrow \left(K^{\frac{1}{4}} = 2L^{-\frac{1}{2}}\right) \Leftrightarrow \left(K = 16L^{-2}\right) \Leftrightarrow \left(K = \frac{16}{L^2}\right)$$

Hence :

$$\frac{dK}{dL} = -\frac{32}{L^3} \text{ and } MRS = \frac{32}{L^3}$$

Method 2: $dK/dL = -f_L/f_K$

$$Q = f(K, L) = 4K^{\frac{1}{4}} L^{\frac{1}{2}}$$

$$f_L = 2K^{\frac{1}{4}} L^{-\frac{1}{2}} \quad \text{and} \quad f_K = K^{-\frac{3}{4}} L^{\frac{1}{2}}$$

$$\therefore \frac{dK}{dL} = -\frac{f_L}{f_K} = -\frac{2K^{\frac{1}{4}} L^{-\frac{1}{2}}}{K^{-\frac{3}{4}} L^{\frac{1}{2}}} = -2\frac{K}{L} \quad \text{but : } Q = 8 \rightarrow K = \frac{16}{L^2}$$

$$\text{Hence:} \quad \frac{dK}{dL} = -2\frac{K}{L} = -2\frac{\left(\frac{16}{L^2}\right)}{L} = -\frac{32}{L^3} \text{ and } MRS = \frac{32}{L^3}$$

7. Let: $z = f(x, y) = 3x^2 + 4y^2$, subject to the following constraint: $2x - 6 - y = 0$

Determine its coordinates (x_0, y_0, z_0) of the stationary point (it's actually a minimum). **USE Lagrange Multipliers.**

Actually, here are all 3 methods:

METHOD 1: Direct substitution

$$z = f(x, y) = 3x^2 + 4y^2 \text{ and } 2x - 6 - y = 0 \Leftrightarrow y = g(x) = 2x - 6$$

$$\therefore z = f(x, y) = f(x, g(x)) = h(x) = 3x^2 + 4(2x - 6)^2 = 3x^2 + 4(4x^2 - 24x + 36) = 19x^2 - 96x + 156$$

$$\frac{dz}{dx} = 0 \Leftrightarrow 38x - 96 \Leftrightarrow x = \frac{48}{19} \approx 2.53$$

$$y = 2x - 6 \Rightarrow y|_{x=\frac{48}{19}} = 2\left(\frac{48}{19}\right) - 6 = -\frac{18}{19} \approx -0.95$$

$$z = 2x^2 + 4y^2 \Rightarrow z_0 = z|_{(x,y)=(x_0,y_0)} = z|_{(x,y)=(2.53,-0.95)} = 2\left(\frac{48}{19}\right)^2 + 4\left(-\frac{18}{19}\right)^2 \approx 16.4$$

$\therefore (x_0, y_0, z_0) = (2.53, -0.95, 16.4)$ is the optimum

METHOD 2: Setting dy/dx from $z=f(x, y)$ equal to $dy/dx (=2)$ gotten from the constraint.

$z = f(x, y) = 3x^2 + 4y^2 \Rightarrow \frac{dy}{dx} = -\frac{f_x}{f_y}$ must equal the slope of the constraint, which is equal to 2...

$$f_x = \frac{\partial z}{\partial x} = 6x \text{ and } f_y = \frac{\partial z}{\partial y} = 8y \Rightarrow \therefore -\frac{f_x}{f_y} = 2 \Leftrightarrow -\frac{6x}{8y} = 2 \Leftrightarrow x = -\frac{8}{3}y \quad (1)$$

$$\text{But : } y = 2x - 6 \quad (2)$$

2 equations, 2 unknowns :

$$\left\{ \begin{array}{l} x = -\frac{8}{3}y \\ y = 2x - 6 \end{array} \right\} \Rightarrow y = 2\left(-\frac{8}{3}y\right) - 6 \Leftrightarrow y = -\frac{16}{3}y - 6 \Leftrightarrow y = -\frac{18}{19} \Rightarrow \therefore x = -\frac{8}{3}y = -\frac{8}{3}\left(-\frac{18}{19}\right) = \frac{48}{19}$$

$$\text{Finally : } z_0 = z|_{(x,y)=(x_0,y_0)} = z|_{(x,y)=(2.53,-0.95)} = 2\left(\frac{48}{19}\right)^2 + 4\left(-\frac{18}{19}\right)^2 \approx 16.4$$

$\therefore (x_0, y_0, z_0) = (2.53, -0.95, 16.4)$ is the optimum

METHOD 3: Using the Lagrangian!

$$z = f(x, y) = 3x^2 + 4y^2 \text{ and } y = 2x - 6 \Leftrightarrow 2x - 6 - y = 0 (\equiv g(x, y) = 0)$$

LANGRANGIAN :

$$V = f(x, y) + \lambda(g(x, y)) = 3x^2 + 4y^2 + \lambda(2x - 6 - y) = 3x^2 + 4y^2 + 2\lambda x - 6\lambda - y\lambda$$

$V = h(x, y, \lambda)$, so we need 3 partial derivatives set to zero :

$$\frac{\partial V}{\partial \lambda} = 0 \Leftrightarrow 2x - 6 - y = 0 \Leftrightarrow y = 2x - 6 \quad (1)$$

$$\frac{\partial V}{\partial x} = 0 \Leftrightarrow 6x + 2\lambda = 0 \quad (2)$$

$$\frac{\partial V}{\partial y} = 0 \Leftrightarrow 8y - \lambda = 0 \quad (3)$$

Use (2) and (3) to eliminate λ :

$$\left. \begin{array}{l} (2) \text{ becomes : } 2\lambda = -6x \\ (3) \text{ becomes : } 2\lambda = 16y \end{array} \right\} \Leftrightarrow x = -\frac{8}{3}y \quad (4)$$

Use (1) and (4) and proceed as in method 2...

8. Given the following information, set up the Lagrangian expression needed to maximize Utility:

$$U = X^2Y^3 \text{ and } B_o = 120 = 4X + 3Y$$

DO **NOT** SOLVE THE SYSTEM, simply write out the Lagrangian expression!

$$\text{Objective function : } U = f(X, Y) = X^2Y^3$$

$$\text{CONSTRAINT : } B_o = 120 = 4X + 3Y \Leftrightarrow 4X + 3Y - 120 = 0 (\equiv g(X, Y) = 0)$$

LAGRANGIAN :

$$V = f(X, Y) + \lambda(g(X, Y))$$

$$V = X^2Y^3 + \lambda(4X + 3Y - 120) = X^2Y^3 + 4\lambda X + 3\lambda Y - 120\lambda$$