Math 115 Fall 2012 Midterm Solutions

1. {11 marks} Short answers

   You still need to show all the work for this question.

   (a) What is the dot product of the two vectors $\vec{u} = \begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} -1 \\ 5 \\ 2 \end{bmatrix}$?

   Solution. $\vec{u} \cdot \vec{v} = (3)(-1) + (-1)(5) + (4)(2) = 0$.

   (b) Determine the general solution for the system of linear equations whose augmented matrix in RREF is

$$\left[\begin{array}{ccccc|c} 1 & 0 & 3 & 0 & 0 & -5 \\ 0 & 0 & 0 & 1 & -4 & 2 \end{array}\right].$$

   Solution. Set parameters $x_2 = s, x_3 = t, x_5 = u$. Then $x_1 = -5 - 3t, x_4 = 2 + 4u$. So the general solution is

$$\vec{x} = \begin{bmatrix} -5-3t \\ s \\ t \\ 2+4u \\ u \end{bmatrix} = \begin{bmatrix} -5 \\ 0 \\ 0 \\ 2 \\ 0 \end{bmatrix} + s \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + u \begin{bmatrix} 0 \\ 0 \\ 0 \\ 4 \\ 1 \end{bmatrix}, \quad s, t, u \in \mathbb{R}.$$

   (c) Determine if the following set is linearly independent, and briefly explain why.

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 14 \\ 159 \end{bmatrix}, \begin{bmatrix} \sqrt{2} \\ e^2 \\ \cos 2 \end{bmatrix}, \begin{bmatrix} 1597 \\ 2584 \\ 4181 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} \right\}.$$

   Solution. This set is linearly dependent. In $\mathbb{R}^3$, any linearly independent set has at most 3 vectors, while this set has 5 vectors.

   (d) Evaluate the following expression or explain why it is not defined.

$$\begin{bmatrix} 3 & 5 \\ 2 & 4 \\ 0 & 7 \end{bmatrix}^T \begin{bmatrix} 2 & 3 \\ 9 & -1 \\ 4 & 3 \end{bmatrix}^T$$

   Solution. Taking the transposes into account, we are multiplying a 3×2 matrix with a 3×2 matrix. This is not possible since the number of columns in the first matrix does not equal to the number of rows of the second matrix.

   (e) Let $f : \mathbb{R}^3 \to \mathbb{R}^2$ be a linear mapping defined by $f(\vec{x}) = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 1 \end{bmatrix} \vec{x}$. Determine $f\left(\begin{bmatrix} 9 \\ 1 \\ 1 \end{bmatrix}\right)$.

   Solution. $f\left(\begin{bmatrix} 9 \\ 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 7 \\ 11 \end{bmatrix}$.

2. {10 marks} VECTORS, LINES AND PLANES

Let $\vec{u} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$, and $\vec{w} = \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix}$.

(a) Write $\vec{w}$ as a linear combination of $\vec{u}$ and $\vec{v}$.

Solution. $\vec{w} = 2\vec{u} + \vec{v}$.

(b) Determine $\|\vec{u} + \vec{v}\|$.

Solution. $||\vec{u} + \vec{v}|| = \left\| \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \right\| = \sqrt{2^2 + 1^2 + 1^2} = \sqrt{6}$.

(c) Let $\vec{n} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$. Notice that $\vec{n}$ is orthogonal to $\vec{u}$ and $\vec{v}$. Using this fact, show that

$$-x_2 + x_3 = 0$$

is a scalar equation of the plane that passes through the origin and contains the points $U(2, 0, 0)$ and $V(0, 1, 1)$.

Solution. Since O is on the plane, the normal of the plane must be orthogonal to both $\overrightarrow{OU}$ and $\overrightarrow{OV}$. As given in the question, $\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$ is one such vector, so an equation of this plane must be in the form $-x_2 + x_3 = c$ for some constant c. But since O is on the plane, $c = 0$. So this plane has equation $-x_2 + x_3 = 0$.

(d) Find the point on the plane $-x_2 + x_3 = 0$ that is closest to the point $(1, 2, 3)$.

Solution. Let point P be $(1, 2, 3)$. The normal vector of the plane is $\vec{n} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$. The point on the plane closest to P is

$$\overrightarrow{OP} - \mathbf{proj}_{\vec{n}}\left(\overrightarrow{OP}\right) = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 5/2 \\ 5/2 \end{bmatrix}.$$

3. {7 marks} SUBSPACES

(a) Complete the following definition of a subspace.
A non-empty set of vectors S in $\mathbb{R}^n$ is a subspace of $\mathbb{R}^n$ if for every $\vec{x}, \vec{y} \in S$, $t \in \mathbb{R}$,

i. $\vec{x} + \vec{y} \in S$.

ii. $t\vec{x} \in S$.

(b) Explain why the following set is not a subspace of $\mathbb{R}^3$:

$$S = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \mid x_1 + {x_2}^2 = x_3 \right\}.$$

Solution. The closure of addition (or scalar multiplication) is not satisfied. Here is a counterexample for the closure of scalar multiplication. We see that $\vec{x} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ is in S since $1 + 1^2 = 2$. But $2\vec{x} = \begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix}$ is not in S, since $2 + 2^2 \neq 4$.

(c) Let A be an $m \times n$ matrix. Let $T = \{\vec{x} \in \mathbb{R}^n \mid A\vec{x} = \vec{0}\}$. Using the definition of subspaces, prove that T is a subspace of $\mathbb{R}^n$.

Solution. The zero vector $\vec{0}$ satisfies $A\vec{x} = \vec{0}$, so T is not empty.
Closure of addition: Let $\vec{x}, \vec{y} \in T$. Then $A\vec{x} = \vec{0}$ and $A\vec{y} = \vec{0}$. So

$$A(\vec{x} + \vec{y}) = A\vec{x} + A\vec{y} = \vec{0} + \vec{0} = \vec{0}.$$

Therefore, $\vec{x} + \vec{y} \in T$.
Closure of scalar multiplication: Let $\vec{x} \in T$ and $t \in \mathbb{R}$. Then $A\vec{x} = \vec{0}$. So

$$A(t\vec{x}) = t(A\vec{x}) = t\vec{0} = \vec{0}.$$

Therefore, $t\vec{x} \in T$.

4. {6 marks} Linear independence and spanning sets

Consider

$$S = \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 0 \\ -3 \end{bmatrix} \right\}.$$

(a) Prove that S is a linearly independent set of vectors.

(b) For which values of k does the vector $\begin{bmatrix} k^2 \\ -1 \\ -k \\ 1 \end{bmatrix}$ lie in the **span** $\{S\}$?

(Note: It is possible to complete both parts simultaneously, but you are not required to do so.)

Solution. We will row reduce the following:

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & k^2 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 0 & -k \\ 2 & -1 & -3 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 3 & k^2+2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 3 & -k+k^2+2 \\ 0 & 0 & 0 & k^2-3k+2 \end{array}\right].$$

(a) The coefficient matrix above has rank 3, so the three vectors in S are linearly independent.

(b) The system of linear equations above has a solution if and only if $k^2 - 3k + 2 = 0$, that is, when $k = 1, 2$. So the vector is in the span of S when $k = 1, 2$.

5. {6 marks} Linear mappings

Let $\vec{v} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$. A mapping, $f : \mathbb{R}^2 \to \mathbb{R}^2$, is defined as $f(\vec{x}) = 2\vec{x} - \mathbf{proj}_{\vec{v}}(\vec{x})$.

(a) Show that the mapping f is linear. (You may use the fact that the projection mapping is linear.)

Solution. Let $\vec{x}, vecy \in \mathbb{R}^2$. Then

$$\begin{aligned} f(\vec{x}+\vec{y}) &= 2(\vec{x}+\vec{y}) - \mathbf{proj}_{\vec{v}}(\vec{x}+\vec{y}) \\ &= 2\vec{x} + 2\vec{y} - \mathbf{proj}_{\vec{v}}(\vec{x}) - \mathbf{proj}_{\vec{v}}(\vec{y}) \quad \text{since projection is linear} \\ &= (2\vec{x} - \mathbf{proj}_{\vec{v}}(\vec{x})) + (2\vec{y} - \mathbf{proj}_{\vec{v}}(\vec{y})) \\ &= f(\vec{x}) + f(\vec{y}). \end{aligned}$$

Let $\vec{x} \in \mathbb{R}^2$ and $t \in \mathbb{R}$. Then

$$f(t\vec{x}) = 2(t\vec{x}) - \mathbf{proj}_{\vec{v}}(t\vec{x}) = t(2\vec{x} - \mathbf{proj}_{\vec{v}}(\vec{x})) = tf(\vec{x}).$$

So f is linear.

Alternate solution. Since $g(\vec{x}) = 2\vec{x}$ is linear and $h(\vec{x}) = \mathbf{proj}_{\vec{v}}(\vec{x})$ is also linear, $f(\vec{x}) = g(\vec{x}) - h(\vec{x})$ is also linear, since a linear combination of linear mappings is also linear.

(b) Find the standard matrix $[f]$ for the mapping f.

Solution. We need to find $f(\vec{e}_1)$ and $f(\vec{e}_2)$.

$$\begin{aligned} f(\vec{e}_1) &= 2\begin{bmatrix} 1 \\ 0 \end{bmatrix} - \mathbf{proj}_{\vec{v}}(\vec{e}_1) = \begin{bmatrix} 2 \\ 0 \end{bmatrix} - \frac{2}{13}\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 22/13 \\ -6/13 \end{bmatrix}. \\ f(\vec{e}_2) &= 2\begin{bmatrix} 0 \\ 1 \end{bmatrix} - \mathbf{proj}_{\vec{v}}(\vec{e}_2) = \begin{bmatrix} 0 \\ 2 \end{bmatrix} - \frac{3}{13}\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -6/13 \\ 17/13 \end{bmatrix}. \end{aligned}$$

So the standard matrix is

$$\begin{bmatrix} 22/13 & -6/13 \\ -6/13 & 17/13 \end{bmatrix}.$$

6. {10 marks} TRUE OR FALSE

For the following statements, determine whether they are true or false. Give justifications for your answers by either giving an appropriate proof or a counterexample. Assume all vectors are in $\mathbb{R}^n$.

(a) If $\vec{x}$ is orthogonal to $\vec{y}$, then $||\vec{x}+\vec{y}||^2 = ||\vec{x}||^2 + ||\vec{y}||^2$.

Solution. True. If $\vec{x} \cdot \vec{y} = 0$, then

$$\begin{aligned} ||\vec{x}+\vec{y}||^2 &= (\vec{x}+\vec{y}) \cdot (\vec{x}+\vec{y}) \\ &= \vec{x} \cdot \vec{x} + 2\vec{x} \cdot \vec{y} + \vec{y} \cdot \vec{y} \\ &= ||\vec{x}||^2 + ||\vec{y}||^2. \end{aligned}$$

(b) If $\vec{x}$ is orthogonal to each of $\vec{v}_1, \vec{v}_2, \vec{v}_3$, then $\vec{x}$ is orthogonal to every vector in $\mathbf{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$.

Solution. True. We know that $\vec{x} \cdot \vec{v}_1 = \vec{x} \cdot \vec{v}_2 = \vec{x} \cdot \vec{v}_3 = 0$. Let $\vec{y} \in \mathbf{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$. Then $\vec{y} = a_1\vec{v}_1 + a_2\vec{v}_2 + a_3\vec{v}_3$ for some $a_1, a_2, a_3 \in \mathbb{R}$. So

$$\vec{x} \cdot \vec{y} = a_1(\vec{x} \cdot \vec{v}_1) + a_2(\vec{x} \cdot \vec{v}_2) + a_3(\vec{x} \cdot \vec{v}_2) = 0.$$

Hence $\vec{x}$ is orthogonal to every vector in the span.

(c) If the RREF of the augmented matrix of a system of linear equations has a row of 0's, then this system has infinitely many solutions.

Solution. False. For example, the system with augmented matrix $\left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 2 \end{array}\right]$ has RREF $\left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{array}\right]$, but this system has a unique solution.

(d) If the set $\{v_1, v_2, v_3\}$ is linearly dependent, then v_1 must be a scalar multiple of either $\vec{v}_2$ or $\vec{v}_3$.

Solution. False. For example, $\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$ is linearly dependent, but $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is not a scalar multiple of any of the remaining two vectors.

(e) If A, B are matrices where $AB = BA$, then A, B are both square matrices.

Solution. True. Since AB exists, we may assume that A is $m \times n$ and B is $n \times q$. Since BA exists, $m = q$, so B is $n \times m$. Now $AB = BA$ where AB is $m \times m$ and BA is $n \times n$, so $n = m$. So both A and B are $n \times n$ matrices.

7. {Extra credit: 2 marks} THE TOTALLY UNFAIR BONUS QUESTION

Determine the volume of the tetrahedron whose four vertices are the points $P(1, 0, 1)$, $Q(-1, 1, -2)$, $R(0, 2, -1)$, $S(0, 0, 0)$.

Solution. The volume of a tetrahedron is 1/3 times base times height. We use triangle QRS as base. For this triangle, $||\overrightarrow{SQ}|| = \sqrt{6}$ (base of the triangle), and $||\mathbf{perp}_{\overrightarrow{SQ}}\left(\overrightarrow{SR}\right)|| = \sqrt{21}/3$ (height of the triangle). So the area of the triangle is $\sqrt{14}/2$.

To find the height, we note that $\vec{n} = \begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix}$ is a normal vector to this triangle. So the height is $||\mathbf{proj}_{\vec{n}}\left(\overrightarrow{SP}\right)|| = 1/\sqrt{14}$. So the volume of the tetrahedron is $\frac{1}{3}\frac{\sqrt{14}}{2}\frac{1}{\sqrt{14}} = 1/6$.