Marks

Short-Answer Questions. Questions 1 – 4 are short-answer questions. Put your answers in the boxes provided. **Simplify your answers as much as possible, and show your work.** Each question is worth 3 marks, but not all questions are of equal difficulty.

[9] 1. (a) Find a function $f(x)$ that satisfies $f'(x) = 2\cos x - e^x$ and $f(0) = 0$.

Answer $2\sin x - e^x + 1$

$f(x) = 2\sin x - e^x + C$

$f(0) = 0$ gives $2(0) - 1 + C = 0$, so $C = 1$

(b) Use a linear approximation to estimate $(2.001)^4$. Write your answer in the form $n/1000$, where n is an integer.

Answer $16032/1000$

Let $f(x) = x^4$, so $f'(x) = 4x^3$. The linearization of $f(x)$ at 2 is $L(x) = f(2) + f'(2)(x-2)$

$= 2^4 + 4 \cdot 2^3 (x-2)$

$= 16 + 32(x-2)$.

So, $(2.001)^4 \approx L(2.001) = 16 + \frac{32}{1000} = \frac{16032}{1000}$

(c) If $y = (\sin x)^{\sin x}$, find y'.

Answer $(\sin x)^{\sin x}((\cos x)\ln(\sin x) + \cos x)$

$\ln y = (\sin x)\ln(\sin x)$

$\frac{y'}{y} = (\cos x)\ln(\sin x) + (\cancel{\sin x}) \cdot \frac{1}{\cancel{\sin x}} \cdot \cos x$

$y' = y((\cos x)\ln(\sin x) + \cos x)$

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[12] **2.** (a) If $f(x) = e^{(\sin x)^2}$, find $f'(x)$.

Answer $2(\sin x)(\cos x)\, e^{(\sin x)^2}$

By Chain Rule,

$$f'(x) = e^{(\sin x)^2} \cdot (2\sin x)(\cos x)$$

(b) Find the slope of the tangent line to the curve $y + x\cos y = \cos x$ at the point $(0, 1)$.

Answer $-\cos 1$

Diff. w.r.t. x:

$$y' + \cos y + x(-\sin y)(y') = -\sin x$$

Set $x = 0$, $y = 1$:

$$y' + \cos 1 + 0 = -\sin 0 \;;\; y' = -\cos 1$$

(c) If $y = \sin^{-1}(\ln x)$, find dy/dx. *Note:* Another notation for $\sin^{-1}$ is arcsin.

Answer $1/x\sqrt{1-(\ln x)^2}$

By Chain Rule,

$$y' = \frac{1}{\sqrt{1-(\ln x)^2}} \cdot \frac{1}{x}$$

(d) Let $f(x) = g(2\sin x)$, where $g'(\sqrt{2}) = \sqrt{2}$. Find $f'(\pi/4)$.

Answer 2

$$f'(x) = g'(2\sin x) \cdot 2\cos x$$

$$f'(\pi/4) = g'(2\sin \tfrac{\pi}{4}) \cdot 2\cos\tfrac{\pi}{4}$$

$$= g'(2 \cdot \tfrac{1}{\sqrt{2}}) \cdot 2 \cdot \tfrac{1}{\sqrt{2}}$$

$$= g'(\sqrt{2}) \cdot \sqrt{2} = 2$$

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[9] 3. (a) Suppose the tangent line to the curve $y = f(x)$ at $x = 1$ passes through the points $(-2, 3)$ and $(0, 5)$. Find $f(1)$ and $f'(1)$.

Answer: $f(1) = 6$, $f'(1) = 1$

The tangent line has slope $\frac{5-3}{0-(-2)} = 1 = f'(1)$.

It has equation $y - 5 = 1 \cdot (x - 0)$ or $y = 5 + x$,

so $f(1) = 5 + 1 = 6$.

(b) If $4x - 9 \le f(x) \le x^2 - 4x + 7$ for $x \ge 0$, find $\lim_{x \to 4} f(x)$.

Answer: 7

Note $\lim_{x \to 4} (4x - 9) = 16 - 9 = 7$

and $\lim_{x \to 4} (x^2 - 4x + 7) = 16 - 16 + 7 = 7$.

Hence, by the Squeeze Theorem, $\lim_{x \to 4} f(x) = 7$.

(c) If $f(1) = 3$, f is continuous on the interval $[1, 4]$, and $f'(x) \le -2$ for $1 < x < 4$, how large can $f(4)$ possibly be?

Answer: -3

Since f is differentiable in $(1, 4)$ and continuous in $[1, 4]$ by the Mean Value Theorem $\frac{f(4) - f(1)}{4-1} = f'(c)$ for some c in $(1, 4)$. Since $f'(c) \le -2$,

$f(4) - f(1) \le (3)(-2) = -6$ and $f(4) \le f(1) - 6 = 3 - 6 = -3$.

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[9] 4. (a) Let a be a constant, and define

$$f(x) = \begin{cases} x^2 + a & \text{if } x \le 1 \\ 2x - 3 & \text{if } x > 1 \end{cases}$$

Find the value of a for which f is continuous everywhere.

Answer -2

f is continuous except maybe at 1.

$f(1) = 1^2 + a = 1 + a$

$\lim_{x\to 1^-} f(x) = \lim_{x \to 1^-} (x^2 + a) = 1 + a$

$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (2x - 3) = 2 - 3 = -1$

If $a = -2$ then all of the above are equal, so $f(x)$ is continuous at 1.

(b) Suppose f and g are continuous functions such that $g(3) = 2$ and $\lim_{x\to 3} (xf(x) + g(x)) = 1$. Find $f(3)$.

Answer $-1/3$

$$1 = \lim_{x\to 3} (x f(x) + g(x)) = \lim_{x\to 3} x \lim_{x \to 3} f(x) + \lim_{x\to 3} g(x)$$

$$= 3 \lim_{x\to 3} f(x) + g(3) = 3f(3) + 2 \Rightarrow f(3) = -1/3$$

(c) Use Newton's Method to find the second approximation x_2 to $\sqrt[6]{2}$, starting with the initial approximation $x_1 = 1$.

Answer $7/6$

Let $x = \sqrt[6]{2}$.

We have $x^6 - 2 = 0$. Let $f(x) = x^6 - 2$.

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 1 - \frac{1 - 2}{6 \cdot 1^5} = 1 + \frac{1}{6} = \frac{7}{6}.$$

Full-Solution Problems. In questions 5–11, justify your answers and **show all your work.** If a box is provided, write your final answer there. **Unless otherwise indicated, simplification of numerical answers is required in these questions.**

[8] **5.** A sample of the radioactive substance Rhodium-101 decayed to 12.5% of its original amount after 10 years.

(a) What percentage of the original amount is remaining after another 5 years?

Answer $100\left(\frac{1}{8}\right)^{3/2}$

Let $y(t)$ = % remaining after t years. We have $y = Ce^{kt}$ since radioactivity is an example of exponential decay. $y(0) = Ce^0 = C$ so $C = 100$. $y(10) = 100e^{10k} = 12.5$ so $e^{10k} = \frac{12.5}{100} = \frac{1}{8}$; $10k = \ln\frac{1}{8}$, $k = \frac{1}{10}\ln\frac{1}{8}$. % left after another 5 years is $y(15) = 100e^{15\left(\frac{1}{10}\ln\frac{1}{8}\right)} = 100e^{\frac{3}{2}\ln\frac{1}{8}} = 100\left(\frac{1}{8}\right)^{3/2}$ $\left(= \frac{100}{\sqrt{512}}\right)$.

(b) What is the half-life of Rhodium-101? Remember to simplify your answer completely.

Answer $\frac{10}{3}$ years

Since $\frac{12.5}{100} = \frac{1}{8} = \left(\frac{1}{2}\right)^3$, 10 years is 3 half-lifes, so half-life is $\frac{10}{3}$ years.

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[16] **6.** Let $f(x) = 2x^{3/5} + 3x^{-2/5}$. Note that the domain of f is the set of all nonzero real numbers; for example, $f(-1) = -2 + 3 = 1$.

(a) Determine the interval(s) where $f(x)$ is increasing, and the interval(s) where $f(x)$ is decreasing.

$$f'(x) = \frac{6}{5}x^{-2/5} - \frac{6}{5}x^{-7/5} = \frac{6}{5}x^{-\frac{7}{5}}(x-1).$$

We have: f': $+$ | $-$ | $+$

f: $\nearrow$ 0 $\searrow$ 1 (CP) $\nearrow$

(note 0 is not in domain; $x^{-7/5} < 0$ for $x < 0$)

Intervals of increase & decrease are as in the diagram

(b) Find the x-coordinates of the local maxima and local minima of $f(x)$.

From above, there is just one local extrema, a local minimum at $x = 1$. (Note 0 is not in the domain, hence is not a local extrema.)

(c) Determine the interval(s) where $f(x)$ is concave up and concave down, respectively. You may use: $f''(x) = -(12/25)x^{-7/5} + (42/25)x^{-12/5} = (1/25)x^{-12/5}(-12x + 42)$.

$f''(x) = 0$ if $12x = 42$, $x = 42/12 = 7/2$

f'': $+$ | $+$ | $-$

f: $\cup$ (CU) 0 $\cup$ (CU) $7/2$ $\cap$ (CD)

(e.g., for $x < 0$, $x^{-12/5} > 0$ and $42 - 12x > 0$ so $f'' > 0$)

(d) Find the x-coordinates of all inflection points of $f(x)$.

From above, since the concavity changes at $x = \frac{7}{2}$, there is an inflection point there, and nowhere else.

parts (e) and (f) on next page...

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(e) Find all vertical asymptotes of the graph $y = f(x)$.

We have $\lim_{x \to 0^+} (2x^{3/5} + 3x^{-2/5}) = \infty$

(and $\lim_{x \to 0^-} (2x^{3/5} + 3x^{-2/5}) = \infty$),

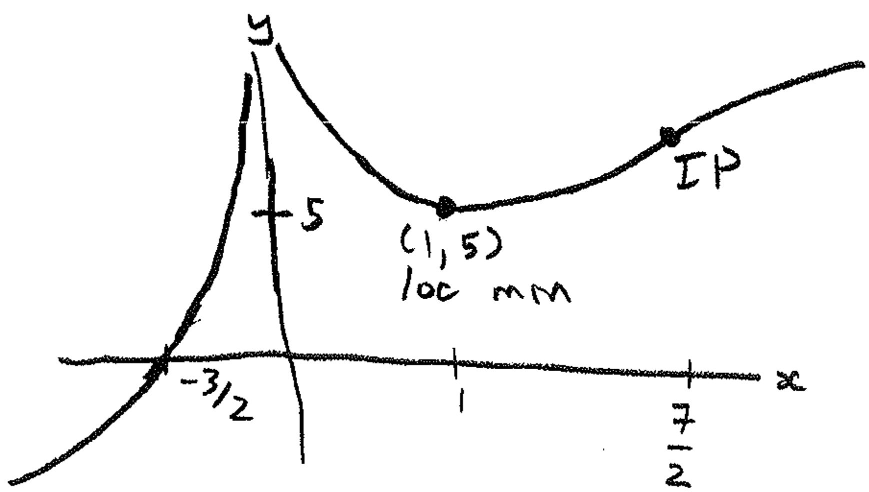
so $x = 0$ is a vertical asymptote.

(f) Sketch the graph of $y = f(x)$, exhibiting all features in parts (a) – (e) and showing the y-coordinates of any points you found in part (b) (you do not need to include the y-coordinates of inflection points). Also, indicate the coordinates of all x-intercepts.

For the x-intercept(s), we need

$2x^{3/5} + 3x^{-2/5} = 0$; $x^{-2/5}(2x+3) = 0$,

$x = -3/2$. Also, $f(1) = 2 + 3 = 5$. So,



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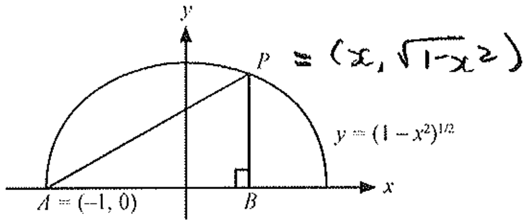
[10] 7. Find the coordinates of the point P, which lies on the curve $y = \sqrt{1-x^2}$, in the diagram below for which the area of the right triangle ABP is the maximum. Remember to justify that your answer actually gives the *maximum* area.



Answer $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$

ABP has base $1+x$, height $\sqrt{1-x^2}$, where $-1 \le x \le 1$.

So, the area is

$$A(x) = \frac{1}{2}(1+x)\sqrt{1-x^2}$$

$$A'(x) = \frac{1}{2}\left(\sqrt{1-x^2} + \frac{(1+x)(-2x)}{2\sqrt{1-x^2}}\right)$$

$A'(x) = 0$ if $\cancel{2}(1-x^2) = \cancel{2}x(1+x)$

$$(1-x)(1+x) = x(1+x)$$

We may cancel $1+x$ since we only need interior critical points, so $1-x = x$, $x = \frac{1}{2}$.

Since $A(-1) = 0$ & $A(1) = 0$ & $A(\frac{1}{2}) > 0$, $x = \frac{1}{2}$ gives the maximum area. The required point P is $\left(\frac{1}{2}, \sqrt{1-\frac{1}{4}}\right) = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$.

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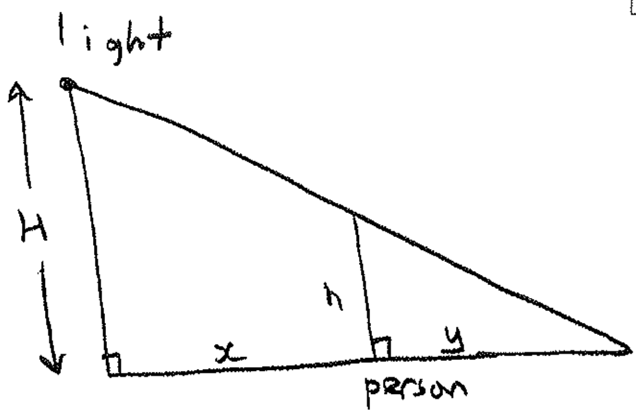
[8] **8.** A light sits atop a post H meters high, and a person of height h, where $H > h$, walks along a straight line away from the lamppost at a speed of v m/s. At what rate is the person's shadow lengthening?

Answer $\dfrac{hv}{H-h}$ m/s



Let x = distance of person from post, in m

y = length of shadow, in m

We are given $\frac{dx}{dt} = v$, where t is in s.

By similar triangles,

$$\frac{y}{h} = \frac{x+y}{H}$$

$$yH = xh + yh$$

Diff. w.r.t. t:

$$H\frac{dy}{dt} = h\frac{dx}{dt} + h\frac{dy}{dt} \quad \text{(H, h are constants)}$$

$$\frac{dy}{dt}(H-h) = h\frac{dx}{dt} = hv$$

$$\frac{dy}{dt} = \frac{hv}{H-h} \text{ m/s}$$

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[8] **9.** Using the definition of the derivative, compute $f'(x)$ if $f(x) = \sqrt{1-2x^2}$. Also, specify the domain of $f'(x)$. *No marks* will be given for the use of differentiation rules, but you may use them to check your answer.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0} \left(\frac{\sqrt{1-2(x+h)^2} - \sqrt{1-2x^2}}{h} \cdot \frac{\sqrt{1-2(x+h)^2} + \sqrt{1-2x^2}}{\sqrt{1-2(x+h)^2} + \sqrt{1-2x^2}} \right)$$

$$= \lim_{h\to 0} \frac{(1-2(x+h)^2) - (1-2x^2)}{h\left(\sqrt{1-2(x+h)^2} + \sqrt{1-2x^2}\right)}$$

The numerator simplifies to

$$(1 - 2x^2 - 4xh - 2h^2) - (1-2x^2)$$
$$= -4xh - 2h^2 = -h(4x+2h).$$

Cancelling h gives

$$f'(x) = \lim_{h\to 0} - \frac{4x+2h}{\sqrt{1-2(x+h)^2} + \sqrt{1-2x^2}}$$

$$= -\frac{4x}{2\sqrt{1-2x^2}} = -\frac{2x}{\sqrt{1-2x^2}}$$

The domain is the set of x's satisfying $1-2x^2 > 0$, i.e. $x^2 < \frac{1}{2}$, i.e. $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

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[6] **10.** Find a function $f(x)$ such that $f'(x) = x^3$ and such that the line $x + y = 0$ is tangent to the graph of $y = f(x)$.

Answer $\frac{1}{4}x^4 + \frac{3}{4}$

$f(x) = \frac{x^4}{4} + C$

The slope of $x + y = 0$ or $y = -x$ is -1, and $f'(x) = x^3 = -1$ if $x = -1$, so the point of tangency is $(-1, \frac{1}{4} + C)$.

This point lies on $y = -x$,

so $\frac{1}{4} + C = -(-1) = 1$, $C = \frac{3}{4}$

So, $f(x) = \frac{1}{4}x^4 + \frac{3}{4}$

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[5] **11.** Give a *complete* proof that for all x satisfying $-1 \le x \le 1$,

$$0 \le \cos(x) - \left(1 - \frac{x^2}{2}\right) \le \frac{1}{24}$$

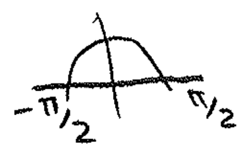
Hint: A question on Taylor polynomials has not yet appeared on this exam.

Since for $f(x) = \cos x$, $f'(x) = -\sin x$, $f'' = -\cos x$, $f''' = -\sin x$, we have $f(0)=1$, $f'(0)=0$, $f''(0)=-1$, $f'''(0)=0$ and the $\underline{\underline{3}}^{rd}$-degree Maclaurin polynomial is $T_3(x) = 1 - \frac{x^2}{2}$. Let $-1 \le x \le 1$.

Then $R_3(x) = f(x) - T_3(x) = \cos x - (1 - \frac{x^2}{2})$ equals $\frac{f^{(4)}(c)}{4!}(x-0)^4$ for some c between 0 & x. Since $f^{(4)}(x) = \cos x$, and $-1 \le x \le 1$,

$R_3(x) = \frac{\cos c}{24} x^4$, where $-1 \le c \le 1$.

Since $1 < \frac{\pi}{2}$, $0 \le \cos c \le 1$ (cos x looks like



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Also, $x^4 \le 1$.

Thus, $0 \le R_3(x) \le \frac{1}{24}$, as required.

The End