Concordia University
Department of Computer Science and Software Engineering
COMP 232: Mathematics for Computer Science

Solutions to Assignment 4: Fall 2013

1. Let x be a real number. Show that

$$\lfloor 3x \rfloor = \lfloor x \rfloor + \lfloor x + \frac{1}{3} \rfloor + \lfloor x + \frac{2}{3} \rfloor$$

Soln. We give a proof by cases. Let $x = n + \epsilon$ for $n \in Z$ and $0 \leq \epsilon < 1$. We consider the following cases:

$0 \leq \epsilon < 1/3$: Then $LHS = \lfloor 3(n+\epsilon) \rfloor = 3n + \lfloor 3\epsilon \rfloor = 3n$ since $0 \leq 3\epsilon < 1$.
$RHS = \lfloor n+\epsilon \rfloor + \lfloor n+\epsilon+\frac{1}{3} \rfloor + \lfloor n+\epsilon+\frac{2}{3} \rfloor = n+n+n = 3n.$

$1/3 \leq \epsilon < 2/3$: Then $LHS = \lfloor 3(n+\epsilon) \rfloor = 3n + \lfloor 3\epsilon \rfloor = 3n+1$ since $1 \leq 3\epsilon < 2$ in this case.
$RHS = \lfloor n+\epsilon \rfloor + \lfloor n+\epsilon+\frac{1}{3} \rfloor + \lfloor n+\epsilon+\frac{2}{3} \rfloor = n+n+(n+1) = 3n+1.$

$2/3 \leq \epsilon < 1$: Then $LHS = \lfloor 3(n+\epsilon) \rfloor = 3n + \lfloor 3\epsilon \rfloor = 3n+2$ since $2 \leq 3\epsilon < 3$ in this case.
$RHS = \lfloor n+\epsilon \rfloor + \lfloor n+\epsilon+\frac{1}{3} \rfloor + \lfloor n+\epsilon+\frac{2}{3} \rfloor = n+(n+1)+(n+1) = 3n+2.$

In each case, we have established that $\lfloor 3x \rfloor = \lfloor x \rfloor + \lfloor x+\frac{1}{3} \rfloor + \lfloor x+\frac{2}{3} \rfloor$

2. The Fibonacci numbers are defined as follows: $f_0 = 0$, $f_1 = 1$, and for $n \geq 2$, $f_n = f_{n-1}+f_{n-2}$. Prove that for every positive integer n,

$$f_3 + f_6 + \cdots + f_{3n} = \frac{1}{2}(f_{3n+2} - 1)$$

Soln. We give a proof by induction.
Basis. $n = 1$. We compute the Fibonacci numbers f_3 and f_5.($f_2 = 1, f_3 = 2, f_4 = 4$, and $f_5 = 5$). We see that $f_3 = 2 = \frac{1}{2}(5-1) = \frac{1}{2}(f_5 - 1)$, thus the basis is proved.
Inductive step: Assume that $f_3 + f_6 + \cdots + f_{3n} = \frac{1}{2}(f_{3n+2} - 1)$ for some $n \geq 1$. We have to prove that $f_3 + f_6 + \cdots + f_{3n} + f_{3n+3} = \frac{1}{2}(f_{3n+5} - 1)$.

$$\begin{aligned} f_3 + f_6 + \cdots + f_{3n} + f_{3n+3} &= \frac{1}{2}(f_{3n+2} - 1) + f_{3n+3} \\ &= \frac{1}{2}(f_{3n+2} - 1 + 2f_{3n+3}) \\ &= \frac{1}{2}(f_{3n+4} + f_{3n+3} - 1) \\ &= \frac{1}{2}(f_{3n+5} - 1) \text{ as needed.} \end{aligned}$$

3. For the following relations on the set of all real numbers, state whether or not they are reflexive, symmetric, anti-symmetric, and/or transitive. Justify your answers.

   (a) $R = \{(x, y) \mid xy \geq 0\}$
   Soln. R is reflexive since for every real x, we have $x^2 \geq 0$.
   R is symmetric since $xy \geq 0 \Rightarrow yx \geq 0$.
   R is not anti-symmetric: $(1, 0) \in R$ and $(0, 1) \in R$.
   R is not transitive: $(5, 0) \in R, (0, -1) \in R$ but $(5, -1) \notin R$.

   (b) $R = \{(x, y) \mid x = 1 \text{ or } y = 1\}$
   Soln. R is not reflexive: $(2, 2) \notin R$.
   R is symmetric, since $(x, y) \in R \Rightarrow x = 1 \vee y = 1 \Rightarrow (y, x) \in R$.
   R is not anti-symmetric: $(1, 2) \in R$ and $(2, 1) \in R$.
   R is not transitive: $(2, 1) \in R$ and $(1, 2) \in R$ but $(2, 2) \notin R$.

4. Find the smallest relation containing the relation $S = \{(1, 2), (2, 1), (2, 3), (3, 4), (4, 1)\}$ that is

   (a) reflexive and transitive
   Soln. $\{1, 2, 3, 4\} \times \{1, 2, 3, 4\}$

   (b) reflexive, symmetric, and transitive
   Soln. $\{1, 2, 3, 4\} \times \{1, 2, 3, 4\}$

5. Given a relation R, is the symmetric closure of the transitive closure of R equal to the transitive closure of the symmetric closure of R? If yes, prove it. If no, then give a counter-example.

   Soln. No, they are not necessarily equal. Counter-example: Let $A = \{a, b\}$ and let R be a relation on A such that $R = \{(a, b)\}$. Then the symmetric closure of R is $\{(a, b), (b, a)\}$, and the transitive closure of the symmetric closure of R is $\{(a, a), (a, b), (b, a), (b, b)\}$. On the other hand, the transitive closure of R is $\{(a, b)\}$, and the symmetric closure of the transitive closure of R is $\{(a, b), (b, a)\}$.

6. Let $S = \{u, v, w\}$. List all equivalence relations on S. How many of these are also partial orders?
   Soln. By considering all possible relations on S, it is easily seen that there are 8 relations on S that are are reflexive and symmetric. One can then simply check which of these are also transitive. Alternatively, you can use the fact that every partition of S corresponds to an equivalence relation. Using the second method, we list all partitions of S:
   $\{\{a\}, \{b\}, \{c\}\}$ corresponds to the equivalence relation $\{(a, a), (b, b), (c, c)\}$,
   $\{\{a, b\}, \{c\}\}$ corresponds to the equivalence relation $\{(a, a), (b, b), (a, b), (b, a), (c, c)\}$,
   $\{\{a, c\}, \{b\}\}$ corresponds to the equivalence relation $\{(a, a), (c, c), (a, c), (c, a), (b, b)\}$,
   $\{\{b, c\}, \{a\}\}$ corresponds to the equivalence relation $\{(b, b), (c, c), (b, c), (c, b), (a, a)\}$, and
   $\{\{a, b, c\}\}$ to the equivalence relation $\{(a, a), (b, b), (c, c), (a, b), (a, c), (b, c), (c, b), (c, a)\ (b, a)\}$
   The only one that is also a partial order is the first one $\{(a, a), (b, b), (c, c)\}$.

7. Let R be the relation on $Z^+ \times Z^+$ such that $(a, b)R(c, d)$ if $gcd(a, b) = gcd(c, d)$.

   (a) Prove that R is an equivalence relation.
   Soln. For every pair (a, b), we have $gcd(a, b) = gcd(a, b)$, hence R is reflexive.
   If $gcd(a, b) = gcd(c, d)$ then $gcd(c, d) = gcd(a, b)$, hence R is symmetric.

If $gcd(a,b) = gcd(c,d)$ and $gcd(c,d) = gcd(e,f)$, then $gcd(a,b) = gcd(e,f)$, hence R is transitive.
R is reflexive, symmetric, and transitive, so R is an equivalence relation.

(b) What is the equivalence class of $(1,2)$?
Soln. $[(1,2)]_R = \{(a,b) \mid gcd(a,b) = 1\}$. That is, the equivalence class of $(1,2)$ is the set of all pairs of integers that are co-prime (relatively prime).

(c) Give an interpretation of the equivalence classes for R.
Soln. For every $k \in Z^+$, the equivalence class $A_k = \{(a,b) \mid gcd(a,b) = k\}$.

8. Let R be the relation on the set of all logical propositions defined as

$$aRb \text{ whenever } a \to b \equiv True$$

(a) Is R an equivalence relation? Justify your answer.
Soln. Let $a \equiv T$ and $b \equiv F$, then aRb but $\neg(bRa)$, so R is not symmetric. Therefore, R is not an equivalence relation. (Observe that $a \to a \equiv True$ for every proposition a, so R is reflexive. Also, if $a \to b \equiv T$ and $b \to c \equiv T$, it follows from the rule of inference known as resolution that $a \to c \equiv T$. Therefore, R is transitive.)

(b) Is R a partial order? Justify your answer.
Let $a \equiv F$ and $b \equiv F$ where a and b denote distinct propositions, then aRb and bRa, so R is not anti-symmetric. Therefore, R is not a partial order.

9. Let R be the relation on Z such that xRy if and only if $x - y = c$.

(a) Define R^2.

$$\begin{aligned} xR^2z &\equiv \exists y :\ xRy \wedge yRz \\ &\equiv \exists y :\ \ x - y = c \wedge y - z = c \\ &\equiv x - z = 2c \end{aligned}$$

(b) Define R^i for arbitrary $i \geq 1$.
Soln. By induction, one can show that $xR^iy \equiv x - y = ic$

(c) Define R^*, the transitive closure of R.
Soln. $R^* = \{(x,y) \mid x - y = kc \text{ for some } k \in Z^+\}$

(d) Is R an equivalence relation? Justify your answer.
Soln. No, since R is not symmetric: $(c,0) \in R$ but $(0,c) \notin R$

(e) Is R^* an equivalence relation? Justify your answer.
Soln. No, since R^* is not symmetric: $(c,0) \in R^*$ but $(0,c) \notin R^*$

10. Give proofs by induction for the following:

(a) Let R and S be relations such that $R \subseteq S$. Prove that $R^n \subseteq S^n$ for all positive integers n.
Soln. Assume $R \subseteq S$. We give a proof by induction that $R^n \subseteq S^n$ for all positive integers n.

Basis. $n = 1$. Since $R^1 = R$ and $S^1 = S$ and $R \subseteq S$, clearly $R^1 \subseteq S^1$.
Inductive step. Assume $R^n \subseteq S^n$ for some $n \geq 1$. We have to show that $R^{n+1} \subseteq S^{n+1}$.

$$\begin{aligned}
(x,z) \in R^{n+1} &\Rightarrow \exists y \ \ (x,y) \in R^n \wedge (y,z) \in R \\
&\Rightarrow \exists y \ \ (x,y) \in S^n \wedge (y,z) \in R \text{ (since } R^n \subseteq S^n\text{)} \\
&\Rightarrow \exists y \ \ (x,y) \in S^n \wedge (y,z) \in S \text{ (since } R \subseteq S\text{)} \\
&\Rightarrow (x,z) \in S^{n+1}
\end{aligned}$$

(b) Let R be a symmetric relation. Prove that R^n is symmetric for all positive integers n.
Soln. Let R be a symmetric relation. We give a proof by induction that R^n is symmetric for all positive integers n.
Basis. $n = 1$: $R^1 = R$ which is symmetric by assumption.
Inductive step. Assume R^n is symmetric for some $n \geq 1$. We have to prove that R^{n+1} is symmetric.

$$\begin{aligned}
xR^{n+1}z &\equiv \exists y: \ \ xR^n y \wedge yRz \\
&\equiv \exists y: \ \ yR^n x \wedge yRz \text{ since } R^n \text{ is symmetric} \\
&\equiv \exists y: \ \ yR^n x \wedge zRy \text{ since } R \text{ is symmetric} \\
&\equiv zR^{n+1}x
\end{aligned}$$